QM II: Übungszettel 1
(Abgabetermin: 23.04.2014)

Problem 1 - Matrix groups, eigenvalues and diagonalization (25 pts)

(a) The set of unitary 2×2 matrices is given by $U(2) = \left\{ U \in M(2, \mathbb{C}) \,|\, U^\dagger U = \left(U^T\right)^* U = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Show that $U(2)$ is a group under matrix multiplication but not a vector space (neither real nor complex). Is the group Abelian?
Hint: A group is a set S of elements with an operation $\circ$ such that the group is closed under this operation (that is to say that for all $s_1, s_2 \in S : s_1 \circ s_2 \in S$), the operation is associative (that is for all $s_1, s_2, s_3 \in S : s_1 \circ (s_2 \circ s_3) = (s_1 \circ s_2) \circ s_3$) and there exists the identity and an inverse element to each element of the group. A group is Abelian if the operation is commutative (that is for all $s_1, s_2 \in S : s_1 \circ s_2 = s_2 \circ s_1$).

(b) Show that the Pauli matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

form a basis of the real vector space of Hermitian complex 2×2 matrices. Compute the commutators of the Pauli matrices with themselves and with each other. Find the complex eigenvalues and eigenvectors of the Pauli matrices. Is there a connection between the commutation relations and the eigenvectors if one compares two Pauli matrices?
Hint: For a degenerate complex 2×2 matrix (that is, a matrix with two identical eigenvalues), every vector of $\mathbb{C}^2$ is an eigenvector.

(c) Diagonalize the matrix

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\varphi \in \,]0, \pi[$. Interpret this matrix as an operation on the points of $\mathbb{R}^3$. Expalin how this interpretation is connected to the eigenvectors of the matrix.
Hint: Consider the action of the matrix on the canonical basis of $\mathbb{R}^3$. Some of the eigenvalues are complex. What does this imply for the action of the matrix on the real vector space $\mathbb{R}^3$?

(d) Compute the matrix function $f : \mathbb{R} \to M(3, \mathbb{R})$, $f(\varphi) = e^{\varphi \mathcal{R}}$, where $\mathcal{R}$ is defined by

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find a connection between $f(\varphi)$ and $R(\varphi)$ from part (c).
Hint: The exponential of a matrix A is defined by the infinite sum $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$. Moreover, $A^0 \equiv \mathbb{1}$ is true in general. Show that $\mathcal{R}^5 = \mathcal{R}^1$. You might want to consider the elements of the resulting matrix as a taylor series of a function.

Problem 2 - Commuting Operators (15 pts)

A quantum mechanical operator has a 3×3 matrix representation as follows,

$$\mathcal{O} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Find the normalized eigenvectors of $\mathcal{O}$ and the corresponding eigenvalues. Is there any degeneracy?

(b) Give a physical example where all this is relevant.

(c) Let $\tilde{\mathcal{O}}$ be another arbitrary operator. Suppose the simultaneous eigenkets of $\mathcal{O}$ and $\tilde{\mathcal{O}}$ $\{|\mathcal{O}', \tilde{\mathcal{O}}'\rangle\}$ form a *complete orthonormal* basis . Can we always conclude that

$$[\mathcal{O}, \tilde{\mathcal{O}}] = 0 \quad ?$$

If your answer is no, give a counter-example, otherwise prove the assertion.

(d) Suppose now that the operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$ commute with their commutator, i.e., $[\tilde{\mathcal{O}}, [\mathcal{O}, \tilde{\mathcal{O}}]] = [\mathcal{O}, [\mathcal{O}, \tilde{\mathcal{O}}]] = 0$. Show that

(i) $[\mathcal{O}, \tilde{\mathcal{O}}^n] = n\tilde{\mathcal{O}}^{n-1}[\mathcal{O}, \tilde{\mathcal{O}}]$

(ii) $[\mathcal{O}^n, \tilde{\mathcal{O}}] = n\mathcal{O}^{n-1}[\mathcal{O}, \tilde{\mathcal{O}}]$

Problem 3 - The group SU(2): Spin operators (10 pts)

Consider a particle with spin $S = 1/2$. We introduce the matrices S_x, S_y, and S_z written in the basis eigenvectors of S_z,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Show that the algebra is closed under the commutation operator.
Hint: An algebra is a vector space (here that of Hermitian 2×2 matrices) equipped with a bilinear product (here the commutator).

(b) Find the eigenvalues and eigenvectors of the operator $\mathcal{S}$ defined by

$$\mathcal{S} = S_x + S_y$$

(c) Assume that $|\tilde{s}\rangle$ denotes the eigenvector of $\mathcal{S}$ associated to the maximal eigenvalue, and that the particle is in state $|\tilde{s}\rangle$. If we measure the spin in the z-direction, what are the possible values and their probabilities?