

QM II: Übungszettel 10 (Abgabetermin: 25.06.2014)

Permutation symmetry (18 pts)

- (a) Show that the spin part of the permutation operator for two electrons, $\hat{\mathbf{P}}_{12} = \hat{\mathbf{P}}_{12}^{\vec{x}} \hat{\mathbf{P}}_{12}^{\text{spin}}$, can be written as

$$\hat{\mathbf{P}}_{12}^{\text{spin}} = \frac{1}{2} \left(\mathbb{1} + \frac{4}{\hbar} \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \right).$$

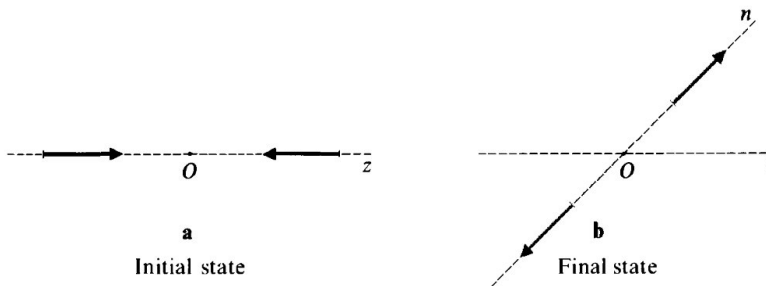
- (b) Consider the permutation operators, $\hat{\mathbf{P}}_{ijk}$, for three identical particles. Show that they form a group under multiplication.

Hint: You need to show that one of the operators corresponds to identity, that the product of two permutations is again a permutation, and that each permutation has its inverse.

- (c) Show that permutation operators are unitary. Are they also Hermitian? Why (not)?

Scattering of particles (32 pts)

Consider the scattering of two particles with the same mass m and initial opposite momenta as depicted in the figure:



The initial state can be denoted by $|-p\vec{e}_z, p\vec{e}_z\rangle$. The probability of finding the two particles in the state $|-p\vec{n}, p\vec{n}\rangle$ at time t is given by

$$P(\vec{n}) = \langle -p\vec{n}, p\vec{n} | \hat{\mathbf{U}}(t, t_0) | -p\vec{e}_z, p\vec{e}_z \rangle,$$

where the vector \vec{n} can be described by its polar angles θ and ϕ . The two particles interact through a potential $V(r)$ which only depends on the distance r between them.

- Assume the particles to have no spin and to be distinguishable. Show that $P(\vec{n})$ does not depend on ϕ . Calculate the probability to find any one of the particles (without specifying which one) with momentum $p\vec{n}$ and the other one with $-p\vec{n}$ in terms of $P(\vec{n})$. What happens to this probability when θ is changed to $\pi - \theta$?
- Now assume the particles to be identical and with spin s . The initial states are then $|-p\vec{e}_z m_s\rangle$ and $|p\vec{e}_z m_{s'}\rangle$ where m_s is the quantum number for $\hat{\mathbf{S}}_z$ and $m_s \neq m_{s'}$. Calculate the probability of finding one particle with momentum $p\vec{n}$ and spin m_s and the other one with $-p\vec{n}$ and $m_{s'}$ in terms of $P(\vec{n})$. If the spins are not measured, what is the probability of finding one particle with $p\vec{n}$ and the other one with $-p\vec{n}$? What happens to these probabilities when θ is changed to $\pi - \theta$?
- Repeat the calculation for $m_s = m_{s'}$. Examine in particular the case $\theta = \pi/2$ and distinguish whether the particles are bosons or fermions. What happens to the probabilities when θ is changed to $\pi - \theta$?
- Finally, assume that the initial spin state of the particles is not known such that each of the particles has the same probability of being in one of the $2s + 1$ orthogonal spin states. Show that the probability for scattering in the \vec{n} direction is given by

$$\mathcal{P}_{\vec{n}} = |P(\vec{n})|^2 + |P(-\vec{n})|^2 + \frac{\epsilon}{2s + 1} (P^*(\vec{n})P(-\vec{n}) + c.c.),$$

where $\epsilon = 1$ for bosons and $\epsilon = -1$ for fermions.