

QM II: Übungszettel 11

(Abgabetermin: 02.07.2014)

Second quantization (22 pts)

- (a) Show that for an operator $\hat{\mathbf{a}}$ that, with its adjoint, obeys the anticommutation relation $[\hat{\mathbf{a}}, \hat{\mathbf{a}}^+]_+ = 1$, the operator $\hat{\mathbf{n}} = \hat{\mathbf{a}}^+ \hat{\mathbf{a}}$ has eigenstates with eigenvalues 0 and 1.

Now consider a Hamiltonian for N identical spin-less particles which is of the form

$$\hat{\mathbf{H}}_N = \sum_{i=1}^N \frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} V(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|).$$

- (b) Show that the one-particle part in second quantization becomes

$$\sum_{i=1}^N \frac{\hat{\mathbf{p}}_i^2}{2m} = \int d^3k \frac{\hbar^2 \vec{k}^2}{2m} \hat{\mathbf{a}}_k^+ \hat{\mathbf{a}}_k.$$

Hint: Use plane waves as your single particle basis functions.

- (c) Show that the two-particle part in second quantization becomes

$$\frac{1}{2} \sum_{i \neq j} V(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|) = \frac{1}{2} \int d^3k \int d^3k' \int d^3q V(\vec{q}) \hat{\mathbf{a}}_{k+\vec{q}}^+ \hat{\mathbf{a}}_{k'-\vec{q}}^+ \hat{\mathbf{a}}_{k'} \hat{\mathbf{a}}_k.$$

Hint: It is useful to replace \vec{r}_1, \vec{r}_2 by center of mass and relative coordinates.

Permutations and Symmetries of Wavefunctions (28 pts)

- (a) Can two non-identical spin-1 particles with no orbital angular momenta (that is $\ell_{1,2} = 0$) form $j = 0$, $j = 1$ and $j = 2$? Explain your answer. Suppose now that the two spin-1 particles are identical. What restrictions do you get on j ?
- (b) Use the symmetrization postulate for fermions to derive the Pauli exclusion principle. Is the Pauli exclusion principle applicable to bosons? Write down the appropriate equations to justify your answer.
- (c) Show explicitly that the Slater determinant for two fermions is antisymmetric.
- (d) Finally, show that the Slater determinant is a zero-order approximation to the Schrödinger equation of a system of N identical fermions.

Hint: Consider a system of N identical non-interacting fermions.