QM II: Übungszettel 12
(Abgabetermin: 09.07.2014)

Born approximation of the scattering amplitude (20 pts)

(a) Show that, for a spherically symmetric potential, $V(\vec{x}') = V(|\vec{x}'|)$, the first-order Born amplitude,

$$f^{(1)}(\vec{k}', \vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' e^{i(\vec{k}'-\vec{k})\cdot\vec{x}'} V(\vec{x}'),$$

becomes

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr\, r V(r) sin(qr),$$

where $q = |\vec{k} - \vec{k}'|$ and θ the angle between $\vec{k}$ and $\vec{k}'$.

(b) Simplify this expression further for small k. Interpret your result. Now consider large q – what do you expect for $f^{(1)}(\theta)$ in this case? In general, do you expect the first-order Born approximation to be valid for small k or for large k (or both)? Justify your answer.

(c) Calculate the integral for a finite square potential well.

(d) Calculate the integral for the Yukawa potential from nuclear physics,

$$V(r) = V_0\frac{e^{-\mu r}}{\mu r}.$$

Disclaimer:
Since we have not yet fully covered scattering theory in the lecture,
you are strongly encouraged to consult textbooks for help on the following two problems.

Expansion in partial waves (10 pts.)

For spherically symmetric scattering potentials, the Hamiltonian $\hat{\mathbf{H}}$ commutes with $\hat{\mathbf{L}}^2$ and $\hat{\mathbf{L}}_z$. It is therefore useful to introduce an eigenbasis $|E, l, m\rangle$ of $\hat{\mathbf{H}}_0$ which reflects this symmetry. It is obtained for a spherically symmetric potential of strength zero by making a separation ansatz involving radial functions $R_l(r)$ and spherical harmonics $Y_{lm}(\theta, \phi)$.

(a) Show that for a spherically symmetric potential of strength zero the Schrödinger equation for the radial function reads

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 1 - \frac{l(l+1)}{r^2}\right) R_l(r) = 0.$$

You may recognize Bessel's equations. Explain that this implies that the eigenbasis is given by

$$\langle\vec{x}|E, l, m\rangle = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}} j_l(kr) Y_{lm}(\theta, \phi).$$

(b) Show that plane waves $|\vec{k}\rangle$ can be expanded into the spherical wave basis with expansion coefficients

$$\langle\vec{k}|E, l, m\rangle = \frac{\hbar}{mk}\delta\left(E - \hbar^2k^2/2m\right) Y_{lm}(\theta_k, \phi_k).$$

Methods of partial waves (20 pts.)

(a) Show that for a spherically symmetric scattering potential, $\hat{\mathbf{V}}$, the transition operator, defined by

$$\hat{\mathbf{T}} = \hat{\mathbf{V}} + \hat{\mathbf{V}} \frac{1}{E - \hat{\mathbf{H}}_0 + i\epsilon} \hat{\mathbf{T}} \,,$$

is diagonal in l and m, that is $\langle E', l', m' | \hat{\mathbf{T}} | E, l, m \rangle = T_l(E) \delta_{l'l} \delta_{m'm}$.
Hint: The Wigner-Eckart theorem might come in handy.
Use diagonality of $\hat{\mathbf{T}}$ in l, m together with the partial wave expansion of the previous problem to show that the scattering amplitude,

$$f(\vec{k}', \vec{k}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} (2\pi)^3 \langle \vec{k}' | \hat{\mathbf{T}} | \vec{k} \rangle \,,$$

can be written as

$$\begin{aligned} f(\vec{k}', \vec{k}) &= -\frac{4\pi^2}{k} \sum_{l,m} T_l(E = \frac{\hbar^2 k^2}{2m}) Y_{lm}(\theta_{k'}, \phi_{k'}) Y^*_{lm}(\theta_k, \phi_k) \\ &= \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta) \,, \end{aligned}$$

where $f_l(\theta) = -\pi T_l(E)/k$.
For the second step, it is useful to choose $\vec{k}$ parallel to $\vec{z}$ such that $Y_{lm}(\theta_k, \phi_k) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}$ and $Y_{l0}(\theta_{k'}, \phi_{k'}) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$ with θ the angle between $\vec{k}'$ and $\vec{k}$.

(b) Calculate the differential cross section, using the expansion into partial waves, for scattering off a hard sphere with radius R:

$$V = \begin{cases} \infty & r < R \\ 0 & r \geq R \end{cases} .$$

Hint: Use the boundary conditions for the radial function to obtain $T_l(E)$.

Bonus problem (optional, 50 pts) Abgabe bis 16.7.2014 möglich

Consider a potential

$$V = \begin{cases} 0 & r \geq R \\ V_0 & r < R \end{cases} ,$$

where V_0 may be positive or negative.

(a) Using the method of partial waves, show that for $|V_0| \ll E = \hbar^2 k^2 / 2m$ and $kR \ll 1$ the differential cross section is isotropic and the total cross section (obtained by integrating the differential cross section over $d\Omega = d\theta d\phi$) is given by

$$\sigma_{tot} = \frac{16\pi}{9} \frac{m^2 V_0^2 R^6}{\hbar^4} .$$

What is the physical interpretation of this result?

(b) Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B \cos\theta \,.$$

Obtain an approximate expression for B/A.