QM II: Übungszettel 3
(Abgabetermin: 07.05.2014)

Problem 1 - Propagator for a charged particle (15 pts)

Calculate the propagator for a charged particle in three dimensions subject to a uniform electric field.
Hint: The interaction with the field is described by $\hat{\mathbf{H}}_{int} = -\vec{E} \cdot \hat{\vec{\mathbf{x}}}$. *In 1D, the result reads*

$$K(x_b, t_b; x_a, t_a) = \sqrt{\frac{m}{2\pi i \hbar T}} \exp\left[\frac{i}{\hbar}\left(\frac{m(x_b - x_a)^2}{2(t_b - t_a)} - \frac{1}{2}E(t_b - t_a)(x_b + x_a) - \frac{E^2(t_b - t_a)^3}{24m}\right)\right]$$

Problem 2 - Euler angles (5 pts)

In spectroscopy, one often needs to transform between the body-fixed frame (attached e.g. to the molecule of interest) and the lab frame defined by the polarization axis of a laser field or the direction of a magnetic field. To this end, one rotates one frame into the other using Euler angles. Show that a general rotation as given in the lecture can also be expressed in rotations with respect to the lab frame (unprimed) axes,

$$\hat{\mathbf{D}}_{z'}(\gamma)\hat{\mathbf{D}}_{y'}(\beta)\hat{\mathbf{D}}_z(\alpha) = \hat{\mathbf{D}}_z(\alpha)\hat{\mathbf{D}}_y(\beta)\hat{\mathbf{D}}_z(\gamma)\,.$$

Hint: Show first that $\hat{\mathbf{D}}_{y'}(\beta) = \hat{\mathbf{D}}_z(\alpha)\hat{\mathbf{D}}_y(\beta)\hat{\mathbf{D}}_z^{-1}(\alpha)$ *and analogously* $\hat{\mathbf{D}}_{z'}(\gamma) = \hat{\mathbf{D}}_{y'}(\beta)\hat{\mathbf{D}}_z(\gamma)\hat{\mathbf{D}}_{y'}^{-1}(\beta)$. *What is the geometrical interpretation of these operations?*

Problem 3 - Spin coupled to a harmonic oscillator (30 pts)

Consider a spin with energy levels split by Δ that interacts with a harmonic oscillator of frequency ω,

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_S + \hat{\mathbf{H}}_{HO} + \hat{\mathbf{H}}_{int} = \frac{\hbar}{2}\omega_S\hat{\boldsymbol{\sigma}}_z + \hbar\omega_0\left(\hat{\mathbf{n}} + \frac{1}{2}\right) - i\hbar\Omega_0\left(\hat{\mathbf{a}}\hat{\boldsymbol{\sigma}}_+ - \hat{\mathbf{a}}^+\hat{\boldsymbol{\sigma}}_-\right)\,,$$

where $\hat{\boldsymbol{\sigma}}_\pm = \frac{1}{2}\left(\hat{\boldsymbol{\sigma}}_x \pm i\hat{\boldsymbol{\sigma}}_y\right)$ and $\hat{\mathbf{n}} = \hat{\mathbf{a}}^+\hat{\mathbf{a}}$.

(a) Determine the eigenvalues and eigenvectors of the total system for the case that the interaction is negligible. Sketch the energy levels. You may assume $\omega_0 \gg \omega_S$.
Hint: Consider the product basis of spin eigenstates $|g/e\rangle$ *and harmonic oscillator eigenstates* $|n\rangle$. *It is convenient to introduce the so-called detuning,* $\delta = \omega_0 - \omega_S$.

(b) Show that for the interaction term only the matrix elements $\langle e, n-1|\hat{\mathbf{H}}_{int}|g, n\rangle$ are non-zero. This observation gives rise to a block-diagonal form of the total Hamiltonian. Give an explicit expression for the 2x2 blocks $\hat{\mathbf{H}}_n$.
Hint: It is useful to introduce $\Omega_n = \Omega_0\sqrt{n}$.

(c) Calculate the eigenvalues and eigenvectors of the total Hamiltonian by diagonalizing $\hat{\mathbf{H}}_n$.
Hint: It is useful to introduce a 'mixing' angle: $\tan\theta_n = \Omega_n/\delta$.

(d) Assume the spin to be initially in its ground state and the harmonic oscillator in state $|n\rangle$. What is the probability to find, at time $t > 0$, the spin in the excited state? Discuss the special case $\delta = 0$ (resonance).

(e) Assume the spin to be initially in its ground state but the harmonic oscillator in a coherent superposition state characterized by the complex number α, $|\varphi(t=0)\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$. Calculate the expectation value of $\hat{\mathbf{n}}$ for this state. What is the probability to find, at time $t > 0$, the spin in the excited state? Discuss again the special case $\delta = 0$. Sketch the time-dependence of this probability for some small $\langle \hat{\mathbf{n}} \rangle$. Interpret your observation.

(f) Finally, assume the spin to be initially in its excited state and the harmonic oscillator in the vacuum state $|0\rangle$. What is the probability to find, at time $t > 0$, the spin in the excited state? Discuss the special case $\delta = 0$. *This phenomenon is called vacuum Rabi oscillations and can only be understood quantum mechanically. It was first observed experimentally in the mid-1990s.*