QM II: Übungszettel 4
(Abgabetermin: 14.05.2014)

Problem 1 - Rotation and Angular Momentum (15 pts)

Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis.

(a) Find the probability for the new state to be found in $m = 0, \pm 1, \pm 2$.
Hint: The explicit expression of the spherical harmonics for $l = 0,\ 1,$ *and* 2 *may be useful.*

(b) Is the probability to find the system with *total* orbital angular-momentum $l = 2$ conserved after rotation? Justify your answer.

Problem 2 - Representation of the Rotation Operator (15 pts)

We consider a system with $j = 1$.

(a) Explicitly write

$$\langle j = 1, m' | J_y | j = 1, m\rangle$$

in 3×3 matrix form.

(b) Show that for $j = 1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2 (1 - \cos\beta)$$

(c) Using question (b), show that

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}(\sin\beta) & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}(\sin\beta) & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

Problem 3 - Rotation of Spin-1/2 Systems (20 pts)

Consider a finite rotation by an angle φ about the z-axis. If the (normalized) state vector of a spin-1/2 system before rotation is given by $|\psi\rangle$, the same ket after rotation is given by

$$|\psi'\rangle = \hat{\mathbf{D}}_z(\varphi)|\psi\rangle$$

(a) Show that

$$\hat{\mathbf{D}}_z(\varphi) = \exp\left(-\frac{i}{\hbar}\hat{\mathbf{S}}_z\,\varphi\right)$$

Hint: First consider an infinitesimal rotation about the z *axis.*

(b) Is $|\psi'\rangle = \hat{\mathbf{D}}_z(\varphi)|\psi\rangle$ still normalized after the rotation? Why?

(c) Calculate the expectation values $\langle\hat{\mathbf{S}}_x\rangle$, $\langle\hat{\mathbf{S}}_y\rangle$, and $\langle\hat{\mathbf{S}}_z\rangle$ before and after the rotation.

(d) Consider a rotation by an angle $\varphi = 2\pi$ about the z-axis. Do you retrieve the initial unrotated system? About which angle needs the system to be rotated in order to retrieve the initial unrotated state?

(e) Consider now a rotation about the y-axis, applied to $|\psi\rangle$. Determine the state vector $|\tilde{\psi}\rangle$ after a rotation by $\varphi_y = 2\pi$.