QM II: Übungszettel 9
(Abgabetermin: 18.06.2014)

Interaction picture (7 pts)

(a) Show that a time-independent observable $\hat{\mathbf{A}}$ obeys the following equation of motion in the interaction picture:

$$\frac{d\hat{\tilde{\mathbf{A}}}}{dt} = \left[\hat{\tilde{\mathbf{A}}}, \hat{\mathbf{H}}_0\right]_- \, .$$

(b) Consider a two-level system that is driven by an external field, that is $H_{12} = H_{21}^* = Ve^{i\omega t}$. Solve the equation of motion in the interaction picture, assuming that initially the two-level system is in its ground state. Discuss the time evolution of the level populations. How does the maximum population ever found in the upper level depend on ω?

(c) Now do the same using perturbation theory to lowest order. Compare your solution to that of (b) for small V. Distinguish between ω being close to $(E_2 - E_1)/\hbar$ and far from it.

Time-dependent perturbation theory I: Dynamic Stark shifts (19 pts)

Consider a hydrogen-like atom that interacts with an electric field, that is polarized along the z-axis

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + \hat{\mathbf{W}}(\kappa, t)\, , \quad \hat{\mathbf{W}}(\kappa, t) = \hat{\mathbf{W}} e^{-\kappa|t|}\cos(\omega_L t)\, , \quad \hat{\mathbf{W}} = -ez\mathcal{E}_L\, ,$$

with $\mathcal{E}_L$ and ω_L the electric field amplitude and frequency, respectively; $\kappa > 0$ is an infinitesimal damping parameter that ensures zero field for $t \to \pm\infty$. Assume that initially the atom is prepared in an eigenstate of $\hat{\mathbf{H}}_0$, $|\psi(t=-\infty)\rangle = |n\rangle$

(a) The excitation is assumed to be off-resonant, i.e., ω does not equal any energy difference $E_m^{(0)} - E_n^{(0)}$. Show that in this case the leading order contribution to $\langle n|\tilde{\psi}(t)\rangle$ is due to $\hat{\tilde{\mathbf{U}}}_2$.

(b) Show that up to second order

$$c_n(t) = \langle n|\tilde{\psi}(t)\rangle = 1 - \frac{i}{4\hbar}\sum_{m,\pm}\frac{\langle n|\hat{\mathbf{W}}|m\rangle\langle m|\hat{\mathbf{W}}|n\rangle e^{2\kappa t}}{2\kappa(E_n^{(0)} - E_m^{(0)} \pm \hbar\omega + i\hbar\kappa)}$$

(c) Consider $\frac{\partial}{\partial t}\ln c_n(t)$ up to second order in $\hat{\mathbf{W}}$ to deduce that $c_n(t) = \exp[-i\Delta_n^S t/\hbar]$, where the dynamic Stark shift of level n is given by

$$\Delta_n^S = \frac{1}{4}\sum_{m,\pm}\frac{\langle n|\hat{\mathbf{W}}|m\rangle\langle m|\hat{\mathbf{W}}|n\rangle}{E_n^{(0)} - E_m^{(0)} \pm \hbar\omega + i\hbar\kappa}$$

(d) Calculate $\langle n|\psi(t)\rangle$ and discuss the time evolution of the perturbed level.

Note: Time-dependent Stark shifts become important when non-resonant transitions in atoms or molecules are excited by strong laser pulses, see for example Rybak et al., Phys. Rev. Lett. 107, 273001 (2011).

Time-dependent perturbation theory II: Weak field coherent control (24 pts)

A cesium atom is excited by a weak femtosecond pulse from the ground $6s$ state to $8s$ state in a non-resonant two-photon transition. Denote these states by $|g\rangle$ and $|f\rangle$ and their energy difference by $\hbar\omega_0$. The Hamiltonian in the unperturbed eigenbasis $\{|m\rangle\}$ (skipping the index (0) for brevity) is given by

$$\langle m|\hat{\mathbf{H}}|m'\rangle = E_m^{(0)}\delta_{m,m'} + E(t)\langle m|\hat{\mathbf{W}}|m'\rangle\, .$$

(a) Calculate the final state population $a_f(t)$ in second order perturbation theory.
Hint: You need to consider virtual transitions via intermediate levels, in this specific case these are the np levels of cesium (your indices m and m′ then run over g, f and np). Use atomic selection rules to justify this statement.

(b) Since all intermediate levels are very far from resonance, their contributions will add coherently only for very short times. You may therefore approximate in $a_f(t)$

$$\sum_n W_{fn} W_{ng} \exp[iE_n^{(0)}(t_2 - t_1)/\hbar] \approx \begin{cases} \langle f|\hat{\mathbf{W}}^2|g\rangle & if \quad |t_1 - t_2| < 1/\bar{\omega} \\ 0 & if \quad |t_1 - t_2| \geq 1/\bar{\omega} \end{cases},$$

where $\hbar\bar{\omega}$ is an average energy. Show that this approximation yields for the two-photon transition probability

$$P_{g\to f}^{2P} = \frac{1}{\hbar^4} \left| \frac{\langle f|\hat{\mathbf{W}}^2|g\rangle}{\bar{\omega}} \right|^2 \left| \int_{-\infty}^{\infty} \epsilon^2(t) \exp(i\omega_0 t) dt \right|^2 .$$

(c) Show that the integral in $P_{g\to f}^{2P}$ can be rewritten

$$S_2 = \int_{-\infty}^{\infty} A(\omega_0/2 + \Omega) A(\omega_0/2 - \Omega) \exp[i\{\Phi(\omega_0/2 + \Omega) + \Phi(\omega_0/2 - \Omega)\}] d\Omega ,$$

where $\epsilon(\omega) = A(\omega)\exp(i\Phi)$ is the Fourier transform of the laser pulse. What happens for $\Phi = 0$, what for antisymmetric spectral phase, i.e., $\Phi(\omega_0/2 + \Omega) = -\Phi(\omega_0/2 - \Omega)$?
This observation is at the base of the experiment reported in Nature 396, 239 (1998).