QM II: Übungszettel 10
(Abgabetermin: 24.06.2015)

Permutation symmetry (18 pts)

(a) Show that the spin part of the permutation operator for two electrons, $\hat{\mathbf{P}}_{12} = \hat{\mathbf{P}}_{12}^{\vec{x}} \hat{\mathbf{P}}_{12}^{\text{spin}}$, can be written as

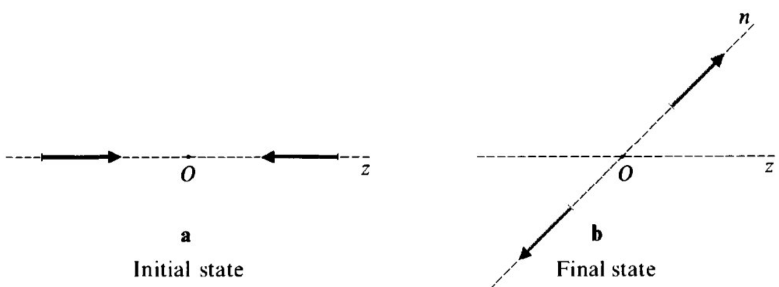
$$\hat{\mathbf{P}}_{12}^{\text{spin}} = \frac{1}{2}\left(\mathbb{1} + \frac{4}{\hbar}\hat{\vec{\mathbf{S}}}_1 \cdot \hat{\vec{\mathbf{S}}}_2\right) .$$

(b) Consider the permutation operators, $\hat{\mathbf{P}}_{ijk}$, for three identical particles. Show that they form a group under multiplication.
Hint: You need to show that one of the operators corresponds to identity, that the product of two permutations is again a permutation, and that each permutation has its inverse.

(c) Show that permutation operators are unitary. Are they also Hermitian? Why (not)?

Scattering of particles (32 pts)

Consider the scattering of two particles with the same mass m and initial opposite momenta as depicted in the figure:



The initial state can be denoted by $|-p\vec{e}_z, p\vec{e}_z\rangle$. The probability of finding the two particles in the state $|-p\vec{n}, p\vec{n}\rangle$ at time t is given by

$$P(\vec{n}) = \langle -p\vec{n}, p\vec{n}|\, \hat{\mathbf{U}}(t, t_0)\, |-p\vec{e}_z, p\vec{e}_z\rangle \ ,$$

where the vector $\vec{n}$ can be described by its polar angles θ and ϕ. The two particles interact through a potential $V(r)$ which only depends on the distance r between them.

(a) Assume the particles to have no spin and to be distinguishable. Show that $P(\vec{n})$ does not depend on ϕ. Calculate the probability to find any one of the particles (without specifying which one) with momentum $p\vec{n}$ and the other one with $-p\vec{n}$ in terms of $P(\vec{n})$. What happens to this probability when θ is changed to $\pi - \theta$?

(b) Now assume the particles to be identical and with spin s. The initial states are then $|-p\vec{e}_z m_s\rangle$ and $|p\vec{e}_z m_{s'}\rangle$ where m_s is the quantum number for $\hat{\mathbf{S}}_z$ and $m_s \neq m_{s'}$. Calculate the probability of finding one particle with momentum $p\vec{n}$ and spin m_s and the other one with $-p\vec{n}$ and $m_{s'}$ in terms of $P(\vec{n})$. If the spins are not measured, what is the probability of finding one particle with $p\vec{n}$ and the other one with $-p\vec{n}$? What happens to these probabilities when θ is changed to $\pi - \theta$?

(c) Repeat the calculation for $m_s = m_{s'}$. Examine in particular the case $\theta = \pi/2$ and distinguish whether the particles are bosons or fermions. What happens to the probabilities when θ is changed to $\pi - \theta$?

(d) Finally, assume that the initial spin state of the particles is not known such that each of the particles has the same probability of being in one of the $2s+1$ orthogonal spin states. Show that the probability for scattering in the $\vec{n}$ direction is given by

$$\mathcal{P}_{\vec{n}} = |P(\vec{n})|^2 + |P(-\vec{n})|^2 + \frac{\epsilon}{2s+1}\left(P^*(\vec{n})P(-\vec{n}) + c.c.\right) ,$$

where $\epsilon = 1$ for bosons and $\epsilon = -1$ for fermions.