QM II: Übungszettel 11
(Abgabetermin: 01.07.2015)

Second quantization (22 pts)

(a) Show that for an operator $\hat{\mathbf{a}}$ that, with its adjoint, obeys the anticommutation relation $[\hat{\mathbf{a}}, \hat{\mathbf{a}}^+]_+ = 1$, the operator $\hat{\mathbf{n}} = \hat{\mathbf{a}}^+\hat{\mathbf{a}}$ has eigenstates with eigenvalues 0 and 1.

Now consider a Hamiltonian for N identical spin-less particles which is of the form

$$\hat{\mathbf{H}}_N = \sum_{i=1}^{N} \frac{\hat{\vec{\mathbf{p}}}_i^2}{2m} + \frac{1}{2}\sum_{i\neq j} V(|\hat{\vec{\mathbf{r}}}_i - \hat{\vec{\mathbf{r}}}_j|)\,.$$

(b) Show that the one-particle part in second quantization becomes

$$\sum_{i=1}^{N} \frac{\hat{\vec{\mathbf{p}}}_i^2}{2m} = \int d^3k \frac{\hbar^2\vec{k}^2}{2m} \hat{\mathbf{a}}^+_{\vec{k}}\, \hat{\mathbf{a}}_{\vec{k}}\,.$$

Hint: Use plane waves as your single particle basis functions.

(c) Show that the two-particle part in second quantization becomes

$$\frac{1}{2}\sum_{i\neq j} V(|\hat{\vec{\mathbf{r}}}_i - \hat{\vec{\mathbf{r}}}_j|) = \frac{1}{2}\int d^3k \int d^3k' \int d^3q\, V(\vec{q})\, \hat{\mathbf{a}}^+_{\vec{k}+\vec{q}} \hat{\mathbf{a}}^+_{\vec{k}'-\vec{q}} \hat{\mathbf{a}}_{\vec{k}'}\, \hat{\mathbf{a}}_{\vec{k}}\,.$$

Hint: It is useful to replace $\vec{r}_1$, $\vec{r}_2$ by center of mass and relative coordinates.

Permutations and Symmetries of Wavefunctions (28 pts)

(a) Can two non-identical spin-1 particles with no orbital angular momenta (that is $\ell_{1,2} = 0$) form $j = 0$, $j = 1$ and $j = 2$? Explain your answer. Suppose now that the two spin-1 particles are identical. What restrictions do you get on j?

(b) Use the symmetrization postulate for fermions to derive the Pauli exclusion principle. Is the Pauli exclusion principle applicable to bosons? Write down the appropriate equations to justify your answer.

(c) Show explicitly that the Slater determinant for two fermions is antisymmetric.

(d) Finally, show that the Slater determinant is a zero-order approximation to the Schrödinger equation of a system of N identical fermions.
Hint: Consider a system of N identical non-interacting fermions.