QM II: Übungszettel 12
(Abgabetermin: 08.07.2015)

Born approximation of the scattering amplitude (20 pts)

(a) Show that, for a spherically symmetric potential, $V(\vec{x}') = V(|\vec{x}'|)$, the first-order Born amplitude,

$$f^{(1)}(\vec{k}', \vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' e^{i(\vec{k}'-\vec{k})\cdot\vec{x}'} V(\vec{x}')\,,$$

becomes

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr\, rV(r)sin(qr)\,,$$

where $q = |\vec{k} - \vec{k}'|$ and θ the angle between $\vec{k}$ and $\vec{k}'$.

(b) Simplify this expression further for small k. Interpret your result. Now consider large q – what do you expect for $f^{(1)}(\theta)$ in this case? In general, do you expect the first-order Born approximation to be valid for small k or for large k (or both)? Justify your answer.

(c) Calculate the integral for a finite square potential well.

(d) Calculate the integral for the Yukawa potential from nuclear physics,

$$V(r) = V_0\frac{e^{-\mu r}}{\mu r}\,.$$

Expansion in partial waves (10 pts.)

For spherically symmetric scattering potentials, the Hamiltonian $\hat{\mathbf{H}}$ commutes with $\hat{\mathbf{L}}^2$ and $\hat{\mathbf{L}}_z$. It is therefore useful to introduce an eigenbasis $|E, l, m\rangle$ of $\hat{\mathbf{H}}_0$ which reflects this symmetry. It is obtained for a spherically symmetric potential of strength zero by making a separation ansatz involving radial functions $R_l(r)$ and spherical harmonics $Y_{lm}(\theta, \phi)$.

(a) Show that for a spherically symmetric potential of strength zero the Schrödinger equation for the radial function reads

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 1 - \frac{l(l+1)}{r^2}\right) R_l(r) = 0\,.$$

You may recognize Bessel's equations. Explain that this implies that the eigenbasis is given by

$$\langle\vec{x}|E, l, m\rangle = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}j_l(kr)Y_{lm}(\theta, \phi)\,.$$

(b) Show that plane waves $|\vec{k}\rangle$ can be expanded into the spherical wave basis with expansion coefficients

$$\langle\vec{k}|E, l, m\rangle = \frac{\hbar}{mk}\delta\left(E - \hbar^2k^2/2m\right) Y_{lm}(\theta_k, \phi_k)\,.$$

Methods of partial waves (20 pts.)

(a) Show that for a spherically symmetric scattering potential, $\hat{\mathbf{V}}$, the transition operator, defined by

$$\hat{\mathbf{T}} = \hat{\mathbf{V}} + \hat{\mathbf{V}}\frac{1}{E - \hat{\mathbf{H}}_0 + i\epsilon}\hat{\mathbf{T}}\,,$$

is diagonal in l and m, that is $\langle E', l', m'|\hat{\mathbf{T}}|E, l, m\rangle = T_l(E)\delta_{l'l}\delta_{m'm}$.
Hint: The Wigner-Eckart theorem might come in handy.

Use diagonality of $\hat{\mathbf{T}}$ in l, m together with the partial wave expansion of the previous problem to show that the scattering amplitude,

$$f(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle\vec{k}'|\hat{\mathbf{T}}|\vec{k}\rangle\,,$$

can be written as

$$\begin{aligned} f(\vec{k}',\vec{k}) &= -\frac{4\pi^2}{k}\sum_{l,m} T_l\Big(E=\frac{\hbar^2k^2}{2m}\Big)Y_{lm}(\theta_{k'},\phi_{k'})Y^*_{lm}(\theta_k,\phi_k) \\ &= \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)\,, \end{aligned}$$

where $f_l(\theta) = -\pi T_l(E)/k$.

For the second step, it is useful to choose $\vec{k}$ parallel to $\vec{z}$ such that $Y_{lm}(\theta_k,\phi_k) = \sqrt{\frac{2l+1}{4\pi}}\delta_{m0}$ and $Y_{l0}(\theta_{k'},\phi_{k'}) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$ with θ the angle between $\vec{k}'$ and $\vec{k}$.

(b) Calculate the differential cross section, using the expansion into partial waves, for scattering off a hard sphere with radius R:

$$V = \begin{cases}\infty & r<R \\ 0 & r\geq R\end{cases}.$$

Hint: Use the boundary conditions for the radial function to obtain $T_l(E)$.

Bonus problem (optional, 50 pts) Abgabe bis 15.7.2015 möglich

Consider a potential

$$V = \begin{cases}0 & r\geq R \\ V_0 & r<R\end{cases},$$

where V_0 may be positive or negative.

(a) Using the method of partial waves, show that for $|V_0| \ll E = \hbar^2k^2/2m$ and $kR \ll 1$ the differential cross section is isotropic and the total cross section (obtained by integrating the differential cross section over $d\Omega = d\theta d\phi$) is given by

$$\sigma_{tot} = \frac{16\pi}{9}\frac{m^2V_0^2R^6}{\hbar^4}\,.$$

What is the physical interpretation of this result?

(b) Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta\,.$$

Obtain an approximate expression for B/A.