QM II: Übungszettel 3
(Abgabetermin: 06.05.2015)

Problem 1 - Baker Campbell Hausdorff formula (15 pts)

Proof the following formula,

$$e^{i\lambda\hat{\mathbf{G}}}\hat{\mathbf{A}}e^{-i\lambda\hat{\mathbf{G}}} = \hat{\mathbf{A}} + i\lambda[\hat{\mathbf{G}},\hat{\mathbf{A}}] + \frac{i^2\lambda^2}{2!}[\hat{\mathbf{G}},[\hat{\mathbf{G}},\hat{\mathbf{A}}]] + \ldots + \frac{i^n\lambda^n}{n!}[\hat{\mathbf{G}},[\hat{\mathbf{G}},[\hat{\mathbf{G}},[\ldots[\hat{\mathbf{G}},\hat{\mathbf{A}}]\ldots]]]] + \ldots$$

where λ is a real parameter and $\hat{\mathbf{G}}$ a Hermitian operator.

Problem 2 - Euler angles (5 pts)

In spectroscopy, one often needs to transform between the body-fixed frame (attached e.g. to the molecule of interest) and the lab frame defined by the polarization axis of a laser field or the direction of a magnetic field. To this end, one rotates one frame into the other using Euler angles. Show that a general rotation as given in the lecture can also be expressed in rotations with respect to the lab frame (unprimed) axes,

$$\hat{\mathbf{D}}_{z'}(\gamma)\hat{\mathbf{D}}_{y'}(\beta)\hat{\mathbf{D}}_z(\alpha) = \hat{\mathbf{D}}_z(\alpha)\hat{\mathbf{D}}_y(\beta)\hat{\mathbf{D}}_z(\gamma)\,.$$

Hint: Show first that $\hat{\mathbf{D}}_{y'}(\beta) = \hat{\mathbf{D}}_z(\alpha)\hat{\mathbf{D}}_y(\beta)\hat{\mathbf{D}}_z^{-1}(\alpha)$ and analogously $\hat{\mathbf{D}}_{z'}(\gamma) = \hat{\mathbf{D}}_{y'}(\beta)\hat{\mathbf{D}}_z(\gamma)\hat{\mathbf{D}}_{y'}^{-1}(\beta)$. What is the geometrical interpretation of these operations?

Problem 3 - Spin coupled to a harmonic oscillator (30 pts)

Consider a spin with energy levels split by Δ that interacts with a harmonic oscillator of frequency ω,

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_S + \hat{\mathbf{H}}_{HO} + \hat{\mathbf{H}}_{int} = \frac{\hbar}{2}\omega_S\hat{\boldsymbol{\sigma}}_z + \hbar\omega_0\left(\hat{\mathbf{n}} + \frac{1}{2}\right) - i\hbar\Omega_0\left(\hat{\mathbf{a}}\hat{\boldsymbol{\sigma}}_+ - \hat{\mathbf{a}}^+\hat{\boldsymbol{\sigma}}_-\right)\,,$$

where $\hat{\boldsymbol{\sigma}}_\pm = \frac{1}{2}\left(\hat{\boldsymbol{\sigma}}_x \pm i\hat{\boldsymbol{\sigma}}_y\right)$ and $\hat{\mathbf{n}} = \hat{\mathbf{a}}^+\hat{\mathbf{a}}$.

(a) Determine the eigenvalues and eigenvectors of the total system for the case that the interaction is negligible. Sketch the energy levels. You may assume $\omega_0 \gg \omega_S$.
Hint: Consider the product basis of spin eigenstates $|g/e\rangle$ and harmonic oscillator eigenstates $|n\rangle$. It is convenient to introduce the so-called detuning, $\delta = \omega_0 - \omega_S$.

(b) Show that for the interaction term only the matrix elements $\langle e, n-1|\hat{\mathbf{H}}_{int}|g, n\rangle$ are non-zero. This observation gives rise to a block-diagonal form of the total Hamiltonian. Give an explicit expression for the 2x2 blocks $\hat{\mathbf{H}}_n$.
Hint: It is useful to introduce $\Omega_n = \Omega_0\sqrt{n}$.

(c) Calculate the eigenvalues and eigenvectors of the total Hamiltonian by diagonalizing $\hat{\mathbf{H}}_n$.
Hint: It is useful to introduce a 'mixing' angle: $\tan\theta_n = \Omega_n/\delta$.

(d) Assume the spin to be initially in its ground state and the harmonic oscillator in state $|n\rangle$. What is the probability to find, at time $t > 0$, the spin in the excited state? Discuss the special case $\delta = 0$ (resonance).

(e) Assume the spin to be initially in its ground state but the harmonic oscillator in a coherent superposition state characterized by the complex number α, $|\varphi(t=0)\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$. Calculate the expectation value of $\hat{\mathbf{n}}$ for this state. What is the probability to find, at time $t > 0$, the spin in the excited state? Discuss again the special case $\delta = 0$. Sketch the time-dependence of this probability for some small $\langle\hat{\mathbf{n}}\rangle$. Interpret your observation.

(f) Finally, assume the spin to be initially in its excited state and the harmonic oscillator in the vacuum state $|0\rangle$. What is the probability to find, at time $t > 0$, the spin in the excited state? Discuss the special case $\delta = 0$. *This phenomenon is called vacuum Rabi oscillations and can only be understood quantum mechanically. It was first observed experimentally in the mid-1990s.*