

QM II: Übungszettel 4 (Abgabetermin: 13.05.2015)

Problem 1 - Rotation and Angular Momentum (15 pts)

Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y -axis.

- (a) Find the probability for the new state to be found in $m = 0, \pm 1, \pm 2$.
Hint: The explicit expression of the spherical harmonics for $l = 0, 1$, and 2 may be useful.
- (b) Is the probability to find the system with *total* orbital angular-momentum $l = 2$ conserved after rotation? Justify your answer.

Problem 2 - Representation of the Rotation Operator (15 pts)

We consider a system with $j = 1$.

- (a) Explicitly write

$$\langle j = 1, m' | J_y | j = 1, m \rangle$$

in 3×3 matrix form.

- (b) Show that for $j = 1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i \left(\frac{J_y}{\hbar} \right) \sin \beta - \left(\frac{J_y}{\hbar} \right)^2 (1 - \cos \beta)$$

- (c) Using question (b), show that

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1 + \cos \beta) & -\frac{1}{\sqrt{2}}(\sin \beta) & \frac{1}{2}(1 - \cos \beta) \\ \frac{1}{\sqrt{2}} \sin \beta & \cos \beta & -\frac{1}{\sqrt{2}} \sin \beta \\ \frac{1}{2}(1 - \cos \beta) & \frac{1}{\sqrt{2}}(\sin \beta) & \frac{1}{2}(1 + \cos \beta) \end{pmatrix}$$

Problem 3 - Rotation of Spin-1/2 Systems (20 pts)

Consider a finite rotation by an angle φ about the z -axis. If the (normalized) state vector of a spin-1/2 system before rotation is given by $|\psi\rangle$, the same ket after rotation is given by

$$|\psi'\rangle = \hat{\mathbf{D}}_z(\varphi)|\psi\rangle$$

- (a) Show that

$$\hat{\mathbf{D}}_z(\varphi) = \exp\left(-\frac{i}{\hbar} \hat{\mathbf{S}}_z \varphi\right)$$

Hint: First consider an infinitesimal rotation about the z axis.

- (b) Is $|\psi'\rangle = \hat{\mathbf{D}}_z(\varphi)|\psi\rangle$ still normalized after the rotation? Why?
- (c) Calculate the expectation values $\langle \hat{\mathbf{S}}_x \rangle$, $\langle \hat{\mathbf{S}}_y \rangle$, and $\langle \hat{\mathbf{S}}_z \rangle$ before and after the rotation.
- (d) Consider a rotation by an angle $\varphi = 2\pi$ about the z -axis. Do you retrieve the initial unrotated system? About which angle needs the system to be rotated in order to retrieve the initial unrotated state?
- (e) Consider now a rotation about the y -axis, applied to $|\psi\rangle$. Determine the state vector $|\tilde{\psi}\rangle$ after a rotation by $\varphi_y = 2\pi$.