QM II: Übungszettel 5
(Abgabetermin: 20.05.2014)

Problem 1 - Rank 1 Tensor Operators (8 pts)

(a) Show that the components of a spherical tensor of rank $k = 1$, that is, a vector, can be written as

$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm i V_y}{\sqrt{2}}, \; V_0^{(1)} = V_z \, .$$

Hint: It might be helpful to use the commutation relations $[\hat{\mathbf{J}}_\pm, \hat{\mathbf{T}}_q^{(k)}]$, $[\hat{\mathbf{T}}_j, \hat{\mathbf{J}}_i]$ *and* $[\hat{\mathbf{J}}_z, \hat{\mathbf{T}}_q^{(k)}]$, *where* $\hat{\mathbf{T}}_q^{(k)}$ *refers to the qth component of a tensor operator of rank k, and i, j are the Cartesian components.*

(b) Using the expression,

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}(\sin\beta) & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}(\sin\beta) & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

obtained in Übungszettel Nr. 4, problem *2.c*, evaluate

$$V_q^{(1)} = \sum_{q'} d_{q,q'}^{(1)}(\beta) V_{q'}^{(1)} \, .$$

(c) Show that the result obtained in (b) is just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

Problem 2 - Rank 2 Tensor Operators (28 pts)

(a) Construct a spherical tensor of rank $k = 1$ out of two different vectors $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$. Explicitly write $T_{\pm 1,0}^{(1)}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$. Determine the corresponding Clebsch-Gordan coefficients.

(b) Construct a spherical tensor of rank $k = 2$ out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T_{\pm 2,\pm,1,0}^{(2)}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$. Determine the corresponding Clebsch-Gordan coefficients.

(c) Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank $k = 2$.

(d) The expectation value,

$$Q \equiv e\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle \, ,$$

is known as the *quadrupole moment*. Evaluate

$$R = e\langle \alpha, j, m'|(x^2 - y^2)|\alpha, j, m = j\rangle \, ,$$

where $m' = j, j-1, j-2, \ldots$, in terms of Q and the appropriate Clebsch-Gordan coefficients.
Hint: It might be helpful to use the Wigner-Eckart Theorem.

Problem 3 - Spin Correlations (14 pts)

Consider a system made up of two spin 1/2 particles. Observer A measures the spin components of one of the particles (s_{1z}, s_{1x} and so on), while Observer B measures the spin components of the other particle. Suppose the system is known to be in a spin-singlet state, that is, $S_{total} = 0$.

(a) What is the probability for observer A to obtain $s_{1z} = \hbar/2$ when observer B makes no measurements?

(b) What is the probability for observer A to obtain $s_{1x} = \hbar/2$ when observer B makes no measurements?

(c) Observer B determines the spin of the particle -2- to be in the $s_{2z} = +\hbar/2$ state with certainty. What can we then conclude about the outcome of observer A's measurement

   (i) if A measures s_{1z} ?,

   (ii) if A measures s_{1x} ? Justify your answer.