QM II: Übungszettel 6
(Abgabetermin: 27.05.2014)

Problem 1 – Parity-odd and parity-even observables (10pts)

(a) A state vector $|\Psi\rangle$ is a simultaneous eigenstate of two Hermitian operators $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ which anticommute,

$$\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right]_+ = 0\,.$$

What can you say about the eigenvalues of $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ for the state $|\Psi\rangle$?

(b) Find all possible simultaneous eigenvalues of $|\vec{p}\rangle \otimes |\pi\rangle$, where $|\pi\rangle$ and $|\vec{p}\rangle$ refer to the eigenvectors of the parity and momentum operators, $\hat{\mathbf{\Pi}}$ and $\hat{\vec{\mathbf{p}}}$, respectively.

(c) Show that the orbital angular momentum eigenfunctions $Y_{lm}(\theta,\phi)$ are also eigenfunctions of the parity operator.
Hint: $\vec{x} \to -\vec{x}$ implies for spherical coordinates $r \to r$, $\theta \to \pi - \theta$, $\phi \to \phi + \pi$.

Problem 2 – Parity violation in atoms (25pts)

Because of weak interactions there is a parity-violating potential between the atomic electron and the nucleus,

$$\hat{\mathbf{H}}_w = \lambda\, \hat{\vec{\mathbf{S}}} \cdot \left[\hat{\vec{\mathbf{p}}}\, \delta(\hat{\vec{\mathbf{x}}}) + \delta(\hat{\vec{\mathbf{x}}})\, \hat{\vec{\mathbf{p}}}\right] \quad \text{with} \quad \lambda = \frac{G_F Q_w}{2\sqrt{2}mc\hbar}\,,$$

where $G_F \approx 3 \times 10^{-12} mc^2 (\hbar/mc)^3$ is Fermi's constant and $Q_w = -N + (1 - 4\sin^2\theta_w)Z$ the dimensionless weak nuclear charge (N is the number of neutrons and θ_w the Weinberg mixing angle with $\sin^2\theta_w \approx 0.23$). $\hat{\vec{\mathbf{S}}}$ is the electron spin, $\hat{\vec{\mathbf{p}}}$ the electron momentum and $\hat{\vec{\mathbf{x}}}$ the distance between electron and nucleus.

(a) Show that $\hat{\mathbf{H}}_w$ violates parity, i.e., show that $\hat{\mathbf{H}}_w$ and $\hat{\mathbf{\Pi}}$ do not commute.

(b) Is the product $\hat{\vec{\mathbf{S}}} \cdot \hat{\vec{\mathbf{p}}}$ a scalar or pseudoscalar quantity? Justify your answer.

(c) Due to $\hat{\mathbf{H}}_w$, the ground state of an alkali atom, usually characterized by $|n,l,j,m\rangle$ actually contains tiny contributions from other eigenstates:

$$|n,l,j,m\rangle \to |n,l,j,m\rangle + \sum_{n',l',j',m'} C_{n',l',j',m'}\, |n',l',j',m'\rangle$$

On the basis of symmetry considerations alone, what can you say about the (n',l',j',m') that give rise to nonvanishing contributions?

(d) Suppose the radial wavefunctions and energy levels are all known. Indicate how you may calculate $C_{n',l',j',m'}$. Do you get any further restrictions on (n',l',j',m')?
Hint: You can use your answer from question (b) and the Wigner-Eckart theorem.

(e) The mixing amplitudes between states $|n,l,j,m\rangle$ and $|n',l',j',m'\rangle$ turn out to be imaginary. Why?
Hint: Consider the behavior of $\hat{\mathbf{H}}_w$ under time-reversal symmetry.

Problem 3 – Time-reversal symmetry (15pts)

(a) Let $\varphi(\vec{p}\,')$ be the momentum-space wavefunction of state $|\varphi\rangle$, $\varphi(\vec{p}\,') = \langle \vec{p}\,'|\varphi\rangle$. Is the momentum-space wavefunction for the time-reversed state $\hat{\mathbf{\Theta}}\,|\varphi\rangle$ given by $\varphi(\vec{p}\,')$, $\varphi(-\vec{p}\,')$, $\varphi^*(\vec{p}\,')$, or $\varphi^*(-\vec{p}\,')$? Why?

(b) What is the time-reversed state corresponding to $\hat{\mathbf{D}}(R)\,|j,m\rangle$?

(c) Using the properties of time reversal and rotation, show that

$$D^{(j)}_{m',m}(R) = (-1)^{m-m'} D^{(j)}_{-m',-m}(R)\,.$$

(d) Prove that $\hat{\mathbf{\Theta}}\,|jm\rangle = i^{2m}\,|j,-m\rangle$.