

QM II: Übungszettel 9 (Abgabetermin: 17.06.2015)

Interaction picture (7 pts)

- (a) Show that a time-independent observable $\hat{\mathbf{A}}$ obeys the following equation of motion in the interaction picture:

$$\frac{d\hat{\mathbf{A}}}{dt} = [\hat{\mathbf{A}}, \hat{\mathbf{H}}_0]_- .$$

- (b) Consider a two-level system that is driven by an external field, that is $H_{12} = H_{21}^* = V e^{i\omega t}$. Solve the equation of motion in the interaction picture, assuming that initially the two-level system is in its ground state. Discuss the time evolution of the level populations. How does the maximum population ever found in the upper level depend on ω ?
- (c) Now do the same using perturbation theory to lowest order. Compare your solution to that of (b) for small V . Distinguish between ω being close to $(E_2 - E_1)/\hbar$ and far from it.

Time-dependent perturbation theory I: Dynamic Stark shifts (19 pts)

Consider a hydrogen-like atom that interacts with an electric field, that is polarized along the z -axis

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + \hat{\mathbf{W}}(\kappa, t), \quad \hat{\mathbf{W}}(\kappa, t) = \hat{\mathbf{W}} e^{-\kappa|t|} \cos(\omega_L t), \quad \hat{\mathbf{W}} = -ez\mathcal{E}_L,$$

with \mathcal{E}_L and ω_L the electric field amplitude and frequency, respectively; $\kappa > 0$ is an infinitesimal damping parameter that ensures zero field for $t \rightarrow \pm\infty$. Assume that initially the atom is prepared in an eigenstate of $\hat{\mathbf{H}}_0$, $|\psi(t = -\infty)\rangle = |n\rangle$

- (a) The excitation is assumed to be off-resonant, i.e., ω does not equal any energy difference $E_m^{(0)} - E_n^{(0)}$. Show that in this case the leading order contribution to $\langle n|\tilde{\psi}(t)\rangle$ is due to $\hat{\mathbf{U}}_2$.
- (b) Show that up to second order

$$c_n(t) = \langle n|\tilde{\psi}(t)\rangle = 1 - \frac{i}{4\hbar} \sum_{m,\pm} \frac{\langle n|\hat{\mathbf{W}}|m\rangle \langle m|\hat{\mathbf{W}}|n\rangle e^{2\kappa t}}{2\kappa(E_n^{(0)} - E_m^{(0)} \pm \hbar\omega + i\hbar\kappa)}$$

- (c) Consider $\frac{\partial}{\partial t} \ln c_n(t)$ up to second order in $\hat{\mathbf{W}}$ to deduce that $c_n(t) = \exp[-i\Delta_n^S t/\hbar]$, where the dynamic Stark shift of level n is given by

$$\Delta_n^S = \frac{1}{4} \sum_{m,\pm} \frac{\langle n|\hat{\mathbf{W}}|m\rangle \langle m|\hat{\mathbf{W}}|n\rangle}{E_n^{(0)} - E_m^{(0)} \pm \hbar\omega + i\hbar\kappa}$$

- (d) Calculate $\langle n|\psi(t)\rangle$ and discuss the time evolution of the perturbed level.

Note: Time-dependent Stark shifts become important when non-resonant transitions in atoms or molecules are excited by strong laser pulses, see for example Rybak et al., Phys. Rev. Lett. 107, 273001 (2011).

Time-dependent perturbation theory II: Weak field coherent control (24 pts)

A cesium atom is excited by a weak femtosecond pulse from the ground $6s$ state to $8s$ state in a non-resonant two-photon transition. Denote these states by $|g\rangle$ and $|f\rangle$ and their energy difference by $\hbar\omega_0$. The Hamiltonian in the unperturbed eigenbasis $\{|m\rangle\}$ (skipping the index (0) for brevity) is given by

$$\langle m|\hat{\mathbf{H}}|m'\rangle = E_m^{(0)}\delta_{m,m'} + E(t)\langle m|\hat{\mathbf{W}}|m'\rangle .$$

- (a) Calculate the final state population $a_f(t)$ in second order perturbation theory.
Hint: You need to consider virtual transitions via intermediate levels, in this specific case these are the np levels of cesium (your indices m and m' then run over g, f and np). Use atomic selection rules to justify this statement.
- (b) Since all intermediate levels are very far from resonance, their contributions will add coherently only for very short times. You may therefore approximate in $a_f(t)$

$$\sum_n W_{fn} W_{ng} \exp[iE_n^{(0)}(t_2 - t_1)/\hbar] \approx \begin{cases} \langle f | \hat{\mathbf{W}}^2 | g \rangle & \text{if } |t_1 - t_2| < 1/\bar{\omega} \\ 0 & \text{if } |t_1 - t_2| \geq 1/\bar{\omega} \end{cases},$$

where $\hbar\bar{\omega}$ is an average energy. Show that this approximation yields for the two-photon transition probability

$$P_{g \rightarrow f}^{2P} = \frac{1}{\hbar^4} \left| \frac{\langle f | \hat{\mathbf{W}}^2 | g \rangle}{\bar{\omega}} \right|^2 \left| \int_{-\infty}^{\infty} \epsilon^2(t) \exp(i\omega_0 t) dt \right|^2.$$

- (c) Show that the integral in $P_{g \rightarrow f}^{2P}$ can be rewritten

$$S_2 = \int_{-\infty}^{\infty} A(\omega_0/2 + \Omega) A(\omega_0/2 - \Omega) \exp[i\{\Phi(\omega_0/2 + \Omega) + \Phi(\omega_0/2 - \Omega)\}] d\Omega,$$

where $\epsilon(\omega) = A(\omega) \exp(i\Phi)$ is the Fourier transform of the laser pulse. What happens for $\Phi = 0$, what for antisymmetric spectral phase, i.e., $\Phi(\omega_0/2 + \Omega) = -\Phi(\omega_0/2 - \Omega)$?

This observation is at the base of the experiment reported in Nature 396, 239 (1998).