Analysis of some common theoretical and empirical relationships between fall velocity of a sediment particle as a function of particle size and water temperature and development of new empirical nonlinear regression equations

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Introduction

- The fall velocity of sediment particles is one of the most important particle characteristics in sediment transport studies
- The fall velocity is directly related to the relative flow conditions existing between the sediment particle and the motion of the water.
- It depends in a certain form on the size, shape, and the surface roughness of the particle and the viscosity of the fluid

Introduction

- Although the fundamental law describing this fall velocity, i.e. Stoke's law, has been known for quite some time, many scientists have been working in this field since then to come up with more precise descriptions of the sedimentation process and providing empirical relations for fall velocity.
- In present study, eight related equations describing the fall velocity v_s of a particle in a fluid have been studied and compared to each other

Literature Review

Year	Researcher	Description
1851-now	Many researchers	Estimating Fall velocity of sediment particles. Starting with Stokes (1852)
2008	Zhiyao *et al.*	established a new relationship between the Reynolds number (Re) and a dimensionless particle parameter and developed a simple formula for predicting the fall velocity of natural sediment particles
2009	Sadat *et al.*	examined and re-evaluated 22 fall velocity relationships that had been published by 17 researchers during the period 1933-2007. They developed a new formula and verified it with two sets of laboratory data

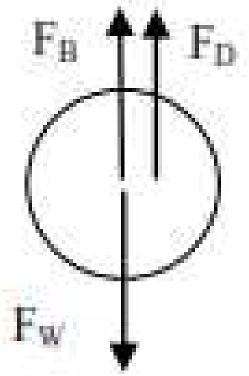
Theory of Fall velocity calculation

STOKES' LAW

- The fall velocity is derived by balancing drag (F_D), buoyancy (F_B) and gravity (F_W) forces that act on the particle

$$F_D = F_W - F_B$$

$$F_D = \frac{C_D v_s^2 \rho A}{2} \qquad F_W - F_B = \frac{4}{3}\pi r^3(\rho_s - \rho)g$$



$$v_s = \sqrt{\frac{8g(\rho_s-\rho)\pi r^3}{3C_D \rho A}} \xrightarrow{\text{spherical particle}} v_s = \sqrt{\frac{4g(\rho_s-\rho)d_s}{3C_D \rho}}$$

Theory of Fall velocity calculation

STOKES' LAW

- Once the drag coefficient has been determined, the fall velocity can be calculated. Stokes derived an expression for the drag force F_D on a small spherical particle

$$F_D = 6\mu\pi r v_s \quad \longrightarrow \quad C_D = \frac{24}{Re}$$

$$v_s = g\,(G_s - 1)d_s^2\;/18\mu \qquad\qquad G_s = \frac{\rho_s}{\rho} \sim 2.65$$

- Regarding the formula viscosity plays important role in calculating fall velocity. the viscosity changes with water temperature.

Theory of Fall velocity calculation

STOKES' LAW

- The viscosity of water is a function of the temperature

$$\nu = \frac{1.792 \times 10^{-6}}{1 + 0.0337T + 0.000221T^2}$$

T(°C)	ρ(kg/m³)	μ(N-s/m²)	ν(m²/s)
0	999.8	$1.781 * 10^{-3}$	$1.785 * 10^{-6}$
10	999.7	$1.307 * 10^{-3}$	$1.306 * 10^{-6}$
20	998.2	$1.002 * 10^{-3}$	$1.003 * 10^{-6}$
30	995.7	$0.798 * 10^{-3}$	$0.800 * 10^{-6}$
40	992.2	$0.653 * 10^{-3}$	$0.658 * 10^{-6}$

Empirical equations

- Stokes law is valid only for a small range of particle sizes and sub-laminar flow (Re<<1).

- When Re is greater than 1, one must rely on one of the many empirical formulae established over more than a century by the various researchers referenced in the introduction.

Author	Equation
Stokes (1851)	$v_s = g(G_s\text{-}1)d_s^{2} / 18\ \nu \quad Re << 1$
Rubby (1933)	$v_s = F\ [d_s\ g(G_s\text{-}1)]^{0.5}$ $F = [\ 2/3 + (36\nu^2/\ g(G_s\text{-}1)\ d_s^{3})\ ]^{\ 0.5} – [36\nu^2/\ g(G_s\text{-}1)\ d_s^{3}]$ $d_s\ > 0.02$ cm
Zanke (1977)	$v_s = (10\ \nu\ /\ d_s)\ [(1+ 0.01\ g(G_s\text{-}1)\ d_s^{3}/\ \nu^2)^{0.5} – 1]$ $0.1\text{mm} \leq d_s \leq 1\text{mm}$
Cheng (1984)	$v_s = (\ \nu\ /\ d_s\ )\ [\ (25+1.2D_*^{\ 2})^{0.5}\ \ - 5\ ]^{1.5}$ $D_* = d_s\ [g(G_s\text{-}1)\ /\ \nu^2]^{1/3}$
Van Rijn (1989)	$v_s = g(G_s\text{-}1)d_s^{2} / 18\ \nu$ $d_s\ < 0.01$ cm $v_s = 1.1\ (g(G_s\text{-}1)\ d_s)0.5$ $d_s\ \geq 0.1$ cm $v_s = (10\ \nu\ /\ d_s)\ [\ (\ 1 + 0.01\ D_*^{\ 3})^{0.5} – 1\ ]$ $0.01 \leq\ d_s < 0.1$ cm
Zhang (1989)	$v_s = [(13.95\ \nu/\ d_s)^2 + 1.09g(G_s\text{-}1)\ d_s]^{0.5} – 13.95\ \nu/\ d_s$
Julien (1995)	$v_s = (8\ \nu\ /\ d_s)\ [(1 + (0.222\ g(G_s\text{-}1)\ d_s^{3})\ /\ 16\nu^2)^{0.5} – 1]$
Soulsbey (1999)	$v_s = (10.36\ \nu\ /\ d_s)\ [(1 + (0.156\ g(G_s\text{-}1)\ d_s^{3}\ )\ /\ 16\nu^2)^{0.5} – 1]$

Empirical equations

- Once the fall velocity v_s has been calculated for all particles diameters d_s for an individual water temperature, the mean fall velocity for each d_s obtained with the eight relationships is computed.
- To account for the often large differences in the theoretical predictions by some of the formulae, outlier data is determined by a Boxplot method and subsequently eliminated from data series.

Data Analysis

Boxplot outlier test

- The boxplot contains a central line (median) and extends from Q_1 to Q_3. Cutoff points, known as fences, lie at 1.5 (Q_3-Q_1) below the lower quartile and above the upper quartile define the lower and upper limit of fences, LIF and UIF, respectively.

$$LIF = Q_1 - 1.5\,IQR\,,\quad UIF = Q_3 + 1.5\,IQR$$

- In the present study, when using ordinary least squares regression, 37 data points (equal to 5% of the total data) have been eliminated by the outlier test

Data Analysis

LS and WLS regression methods

- The goal is to fit the theoretical predictions of the formulae for the fall velocities v (=y) as a function of the particle diameter d (=x) by more generally usable simple polynomials of order two.

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$$

- This equation can be written in matrix notation as

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$$

- where X is an N x 3 predictor matrix whose three columns consist of $(1, x_i^2, x_i^3)$ $(i = 1,..,N)$, β is the vector of unknowns and ε is a random error vector, assumed to be normally distributed

Data Analysis

LS and WLS regression methods

- The general linear model, is calculated the unknown parameters β by least squares (LS) approach.
- One of the common assumptions of LS method is the standard deviation of the error term is constant over all values of the predictor or explanatory variables.
- Therefore, weighted least squares can often be used to maximize the efficiency of parameter estimation.
- The WLS and LS fitting models have been programmed in the R® statistical environment.

Data Analysis

LS and WLS regression methods

- For the selection of the optimal polynomial model the R^2 and AIC (Akaike's information criterion) are used.

$$AIC = 2k - 2\ln L$$

where k is the number of parameters in the model, and L is the maximized value of the model likelihood function.

- Once the polynomial coefficients have been determined by the two least squares methods, 90% confidence intervals for the predictors y_i^{pred} are computed by

$$CI = y_i^{pred} \pm t_{0.05,n-(k-1)} \left(\frac{s}{\sqrt{n}}\right)$$

Data Analysis

LS and WLS regression methods

- It is not possible to fit the whole the diameter range by one polynomial curve, the regressions were carried out for three separate diameter categories .

$$0.005cm \leq ds \leq 0.01cm,$$

$$0.01cm < ds \leq 0.1cm$$

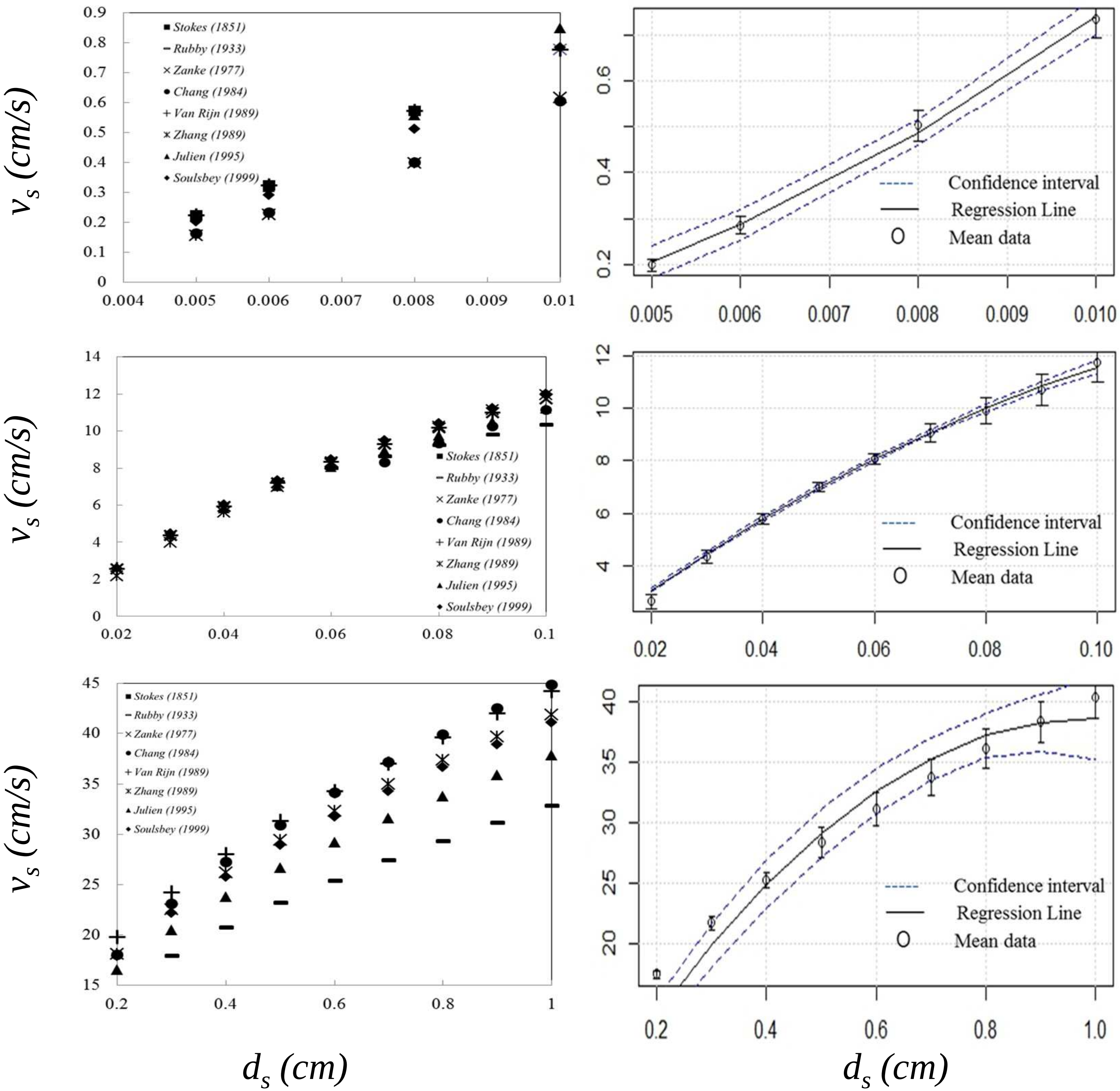
$$0.1cm < ds \leq 1cm$$

- Moreover, since the fall velocity depends on the water temperature, all regressions are done for the reference temperature of 20^{o}C. After that, the regressed velocities are linearly corrected for other temperatures.

Results and discussion

- For 20^{o}C water temperature, the fall velocities as a function of the particle diameter are calculated by the eight fall relations.

- wherefore the specific formula restrictions as noted in the table have been respected, so that for some diameters the fall velocity could not be calculated by all eight relations



Stokes (1851)
Rubby (1933)
Zanke (1977)
Chang (1984)
Van Rijn (1989)
Zhang (1989)
Julien (1995)
Soulsbey (1999)
Confidence interval
Regression Line
Mean data
v_s (cm/s)
d_s (cm)

Results and discussion

- Statistical results of LS and WLS polynomial regression model for fall velocities for T=20°C .

Diameter interval	Method	Equation $v_s= b_1 d_s + b_2 d_s^2$	R^2	AIC	sd(b_1)	sd (b_2)
$0.005cm \le d_s \le 0.01cm$	LS	$v_s= 8.32\ d_s + 6583\ d_s^2$	0.99	- 20.68	3.74	443.0
	WLS	$\mathbf{v_s= 8.53\ d_s + 6544\ d_s^2}$	0.99	-20.93	3.63	416.7
$0.01cm < d_s \le 0.1cm$	LS	$v_s= 158\ d_s - 415.9\ d_s^2$	0.99	-1.84	3.70	45.55
	WLS	$\mathbf{v_s= 163\ d_s - 473.5\ d_s^2}$	0.99	-8.74	2.21	31.17
$0.1cm < d_s \le 1cm$	LS	$\mathbf{v_s= 78.0\ d_s - 39.36\ d_s^2}$	0.99	40.91	3.98	4.90
	WLS	$v_s= 84.8\ d_s - 47.45\ d_s^2$	0.99	45.71	4.41	5.84

Results and discussion

- Fall velocity correction coefficient Δv as a function of the particle diameter for different temperatures.

d_s(cm)	T=0^{o}C	T=10^{o}C	T=30^{o}C	T=40^{o}C	d_s(cm)	T=0^{o}C	T=10^{o}C	T=30^{o}C	T=40^{o}C
0.005	-0.09	-0.05	0.05	0.09	0.09	-0.75	-0.31	0.22	0.37
0.006	-0.12	-0.07	0.07	0.15	0.1	-0.61	-0.25	0.17	0.3
0.008	-0.22	-0.11	0.1	0.23	0.2	-0.39	-0.16	0.11	0.19
0.01	-0.3	-0.16	0.17	0.33	0.3	-0.3	-0.12	0.08	0.15
0.02	-1.01	-0.44	0.36	0.65	0.4	-0.25	-0.1	0.07	0.12
0.03	-1.06	-0.46	0.36	0.65	0.5	-0.21	-0.09	0.06	0.11
0.04	-1.06	-0.45	0.34	0.6	0.6	-0.19	-0.08	0.05	0.09
0.05	-1	-0.42	0.31	0.54	0.7	-0.17	-0.07	0.05	0.09
0.06	-0.94	-0.39	0.28	0.49	0.8	-0.16	-0.06	0.05	0.08
0.07	-0.87	-0.36	0.25	0.44	0.9	-0.15	-0.06	0.04	0.07
0.08	-0.81	-0.33	0.23	0.41	1	-0.14	-0.06	0.04	0.07

Conclusions

- In this study eight of the most important relations developed over a period of more than a century for the fall velocity for a range of particle sizes have been evaluated.
- A mean fall velocity from these proposed relationships is computed and these have been used, after elimination of outliers by a boxplot method, to develop new, but simple, second order polynomial equations for $v_s(d_s)$.

Conclusions

- The WLS and LS methods are programed in R. For both methods, very good adjustments of the "observed" mean velocities by the polynomial regressions are obtained, as measured by R^2 of 0.99, but more distinctly, by low values of the AIC.
- We advocate using these regression equations in future applications of sediment transport, as they truly represent a distillation of the many historical, sometimes confusing, empirical relationships between settling velocity and particle size

Thank you for your attention

In fact, the general linear model (10) for the unknown parameters β is solved by a least-squares approach (Draper and Smith, 1998). However, because of the heteroscedasticity, ordinary least squares is not valid, so that the maximum likelihood estimation (MLE) method must be applied (DeGroot and Schervish, 2002). In MLE the probability density function $f(\boldsymbol{\beta},\boldsymbol{Y})$, i.e. the likelihood function $L(\boldsymbol{\beta},\boldsymbol{Y})$, is maximized or, more conveniently, its logarithm is $\ln f(\boldsymbol{\beta},\boldsymbol{Y}) = \ln L(\boldsymbol{\beta},\boldsymbol{Y})$ is minimized. For the estimation problem (10) and the statistical assumptions $\ln L(\boldsymbol{\beta},\boldsymbol{Y})$ can be written as (Beck and Arnold, 1977):

$$\ln L(\boldsymbol{\beta},\boldsymbol{Y}) = \ln f(\boldsymbol{\beta},\boldsymbol{Y}) = -\tfrac{1}{2}[N\, ln(2\pi) + \ln|\psi| + S_{ML}] \qquad (11)$$

where S_{ML} is the function to be minimized by the linear model:

$$S_{ML} = (\boldsymbol{Y} - \boldsymbol{X\beta})^T\, \boldsymbol{\psi}^{-1}\, (\boldsymbol{Y} - \boldsymbol{X\beta}) \qquad (12)$$

As the first two terms in Eq. (11) are constant, its minimization is equivalent to minimizing Eq. (12) which results in the general heteroscedastic ML least squares estimator:

$$\boldsymbol{b}_{MLE} = (\boldsymbol{X}^T\, \boldsymbol{\psi}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\, \boldsymbol{\psi}^{-1}\boldsymbol{Y} \qquad (13)$$

or, with $\boldsymbol{\psi} = \boldsymbol{V}\sigma^2$, in the so-called weighted least squares estimator:

$$\boldsymbol{b}_{WLS} = (\boldsymbol{X}^T\, \boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\, \boldsymbol{V}^{-1}\boldsymbol{Y} \qquad (14)$$

wherefore the elements of the diagonal matrix $\boldsymbol{V}^{-1}$ are associated with the weights w_i of the observations.

For the case that the weights w_i are equal (=1), Eq. (14) becomes the ordinary least squares estimator:

$$\boldsymbol{b}_{LS} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\, \boldsymbol{Y} \qquad (15)$$

Both weighted (WLS) and ordinary (LS) least squares fitting will be applied to the means $\bar{x}_i$ of the fall velocities predicted by the various Stokes formulae (usually seven or eight) in Table 1. For WLS the weights w_i are set to $w_i = 1/s_i^2$, where s_i^2 are standardized variances of the mean velocities, estimated by $s_i^2 = \frac{\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2}{n-1}/\bar{x}_i^2$, and n is the number of formulae used to compute the mean $\bar{x}_i$ of a velocity.