# Elektromagnetische Feldtheorie I (EFT I) / Electromagnetic Field Theory I (EFT I)

## 7th Lecture / 7. Vorlesung


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# ES Fields – Coulomb Integral / ES Felder – Coulomb-Integral

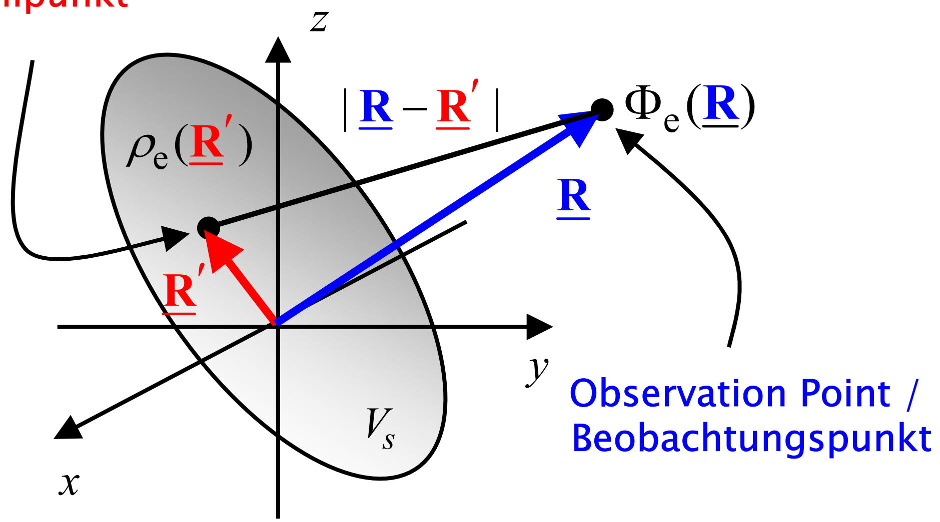
## Poisson and Laplace Equation / Poisson- und Laplace-Gleichung

$$\Delta\Phi_e(\underline{\mathbf{R}}) = \begin{cases} -\dfrac{\rho_e(\underline{\mathbf{R}})}{\varepsilon_0} & \text{for / für } \rho_e(\underline{\mathbf{R}}) \neq 0 \quad \text{Poisson Equation / Poisson-Gleichung} \\ 0 & \text{for / für } \rho_e(\underline{\mathbf{R}}) = 0 \quad \text{Laplace Equation / Laplace-Gleichung} \end{cases}$$

$\Delta = \nabla^2 = \nabla \bullet \nabla$ : Laplace Operator / Laplace-Operator

Limited Source Volume /
Begrenztes Quellvolumen

$$\rho_e(\underline{\mathbf{R}}) \begin{cases} \neq 0 & \underline{\mathbf{R}} \in V_s \\ 0 & \underline{\mathbf{R}} \notin V_s \end{cases}$$



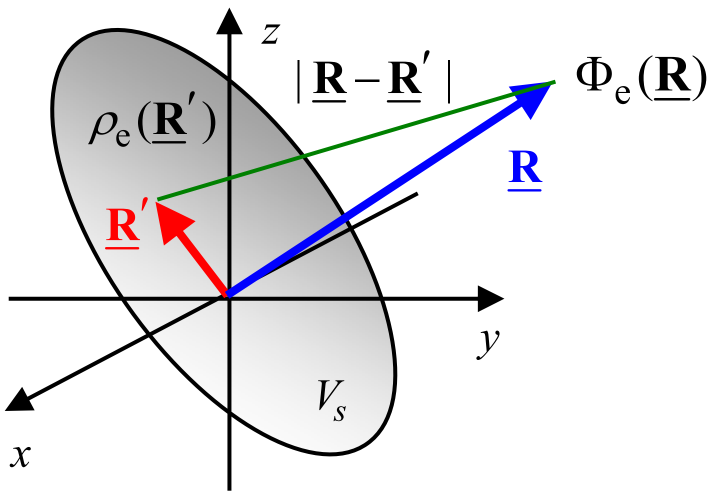

## Coulomb Integral / Coulomb-Integral:

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint_{V_s} \frac{\rho_e(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} d^3\underline{\mathbf{R}}'$$

$\rho_e(\underline{\mathbf{R}}')$ : known / bekannt

$\Phi_e(\underline{\mathbf{R}})$ : unknown / unbekannt

# ES Fields – Coulomb Integral / ES Felder – Coulomb-Integral (...)




Coulomb Integral / Coulomb-Integral:

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V_s} \frac{\rho_\mathrm{e}(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} d^3\underline{\mathbf{R}}'$$

$\rho_\mathrm{e}(\underline{\mathbf{R}}')$ : known / bekannt

$\Phi_\mathrm{e}(\underline{\mathbf{R}})$ : unknown / unbekannt

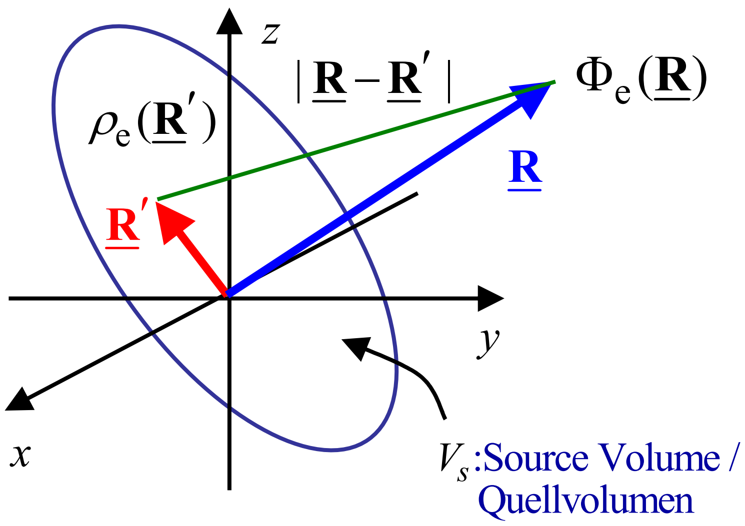
$$\Delta\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\Delta\iiint_{V_s} \frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}\rho_\mathrm{e}(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'$$

$$= \frac{1}{4\pi\varepsilon_0}\iiint_{V_s} \underbrace{\left[\Delta\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}\right]}_{=-4\pi\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}\rho_\mathrm{e}(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'$$

with $\Delta\dfrac{1}{4\pi\left|\underline{\mathbf{R}}-\underline{\mathbf{R}}'\right|} = -\delta\left(\underline{\mathbf{R}}-\underline{\mathbf{R}}'\right)$

$$= -\frac{1}{4\pi\varepsilon_0}\underbrace{\iiint_{V_s} 4\pi\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}')\rho_\mathrm{e}(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'}_{=\rho_\mathrm{e}(\underline{\mathbf{R}})} = -\frac{1}{\varepsilon_0}\rho_\mathrm{e}(\underline{\mathbf{R}})$$

# ES Fields – Green's Function / ES Felder – Greensche Funktion




$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V_s} \frac{\rho_e(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} d^3\underline{\mathbf{R}}'$$

$$= \frac{1}{\varepsilon_0}\iiint_{V_s} \underbrace{\frac{1}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}}_{=G_e^{ES}(\underline{\mathbf{R}}-\underline{\mathbf{R}}')} \rho_e(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'$$

## Electrostatic Green's Function / Elektrostatische Greensche Funktion

$$G_e^{ES}(\underline{\mathbf{R}}-\underline{\mathbf{R}}') = \frac{1}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} \qquad \text{for / für } \underline{\mathbf{R}} \neq \underline{\mathbf{R}}'$$

with $\Delta G_e^{ES}(\underline{\mathbf{R}}-\underline{\mathbf{R}}') = -\delta\left(\underline{\mathbf{R}}-\underline{\mathbf{R}}'\right)$

Normalized Potential of a Point Charge / Normiertes Potential einer Punktladung

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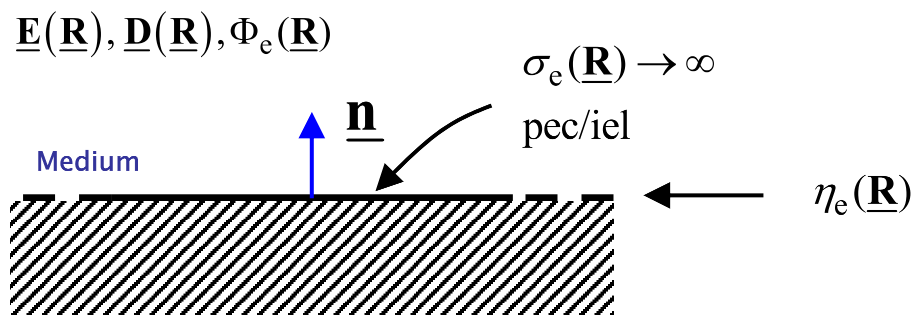
## Electrostatic Potential of an Electrostatic Point Charge / Elektrostatisches Potential einer elektrostatischen Punktladung

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} \qquad \text{for / für} \quad \rho_e(\underline{\mathbf{R}}) = Q_e\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+)$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Boundary Conditions / Randbedingungen

$\underline{\mathbf{R}} \in S_{\mathrm{B(oundary)}}$




$$\Phi_e\left(\underline{\mathbf{R}}\right) = \Phi_{e0} = \text{const.} \quad (\Phi_{e0} = 0\ \mathrm{V})$$

$$\frac{\partial}{\partial n}\Phi_e\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0\varepsilon_r}\eta_e(\underline{\mathbf{R}})$$

Neumann Boundary Conditions for $\Phi_e$ /
Neumann-Randbedingung für $\Phi_e$

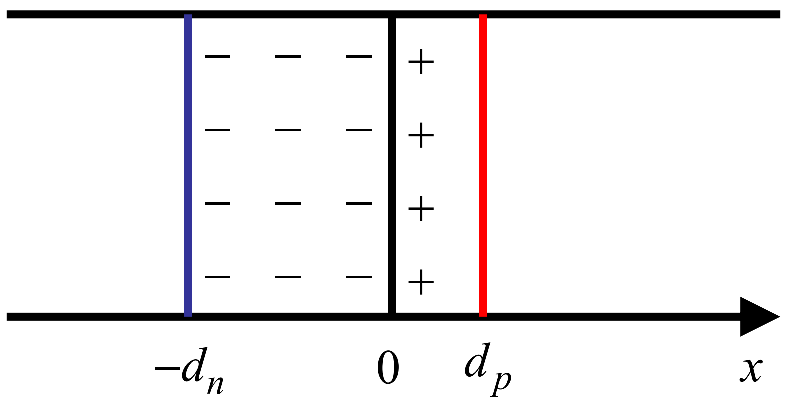
Dirichlet Boundary Conditions for $\Phi_e$ /
Dirichlet-Randbedingung für $\Phi_e$

# Electrostatic (ES) Fields – Poisson and Laplace Equation / Elektrostatische (ES) Felder – Poisson- und Laplace-Gleichung (3)

## Laplace Operator in Cartesian Coordinates / Laplace-Operator in Kartesischen Koordinaten

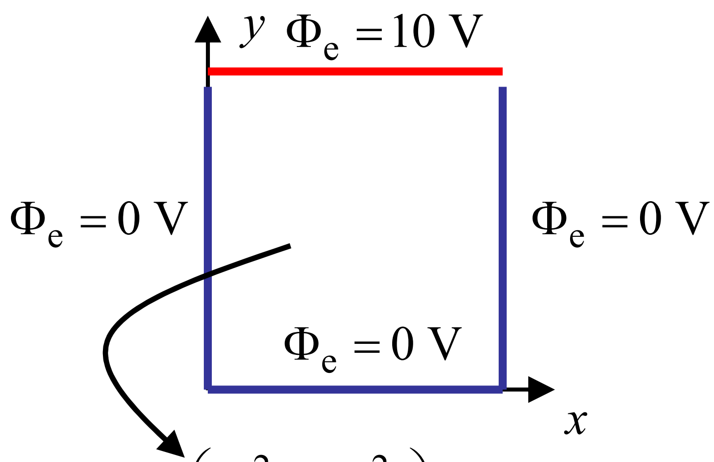
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi_\mathrm{e}(x,y,z)=\begin{cases}-\dfrac{\rho_\mathrm{e}(x,y,z)}{\varepsilon_0} & \text{for / für } \rho_\mathrm{e}(x,y,z)\neq 0 \quad \text{Poisson Equation / Poisson-Gleichung}\\ 0 & \text{for / für } \rho_\mathrm{e}(x,y,z)=0 \quad \text{Laplace Equation / Laplace-Gleichung}\end{cases}$$

## Example: pn Junction – pn Diode / Beispiel: pn-Übergang – pn Diode




$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi_\mathrm{e}(x)=\frac{e}{\varepsilon}\begin{cases}-n_e & \text{for / für } -d_n\le x\le 0\\ n_e & \text{for / für } 0\le x\le d_p\end{cases}$$

## Example: / Beispiel:




$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_\mathrm{e}(x,y)=0$$

➡ Separation of Variables / Separation der Variablen !

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

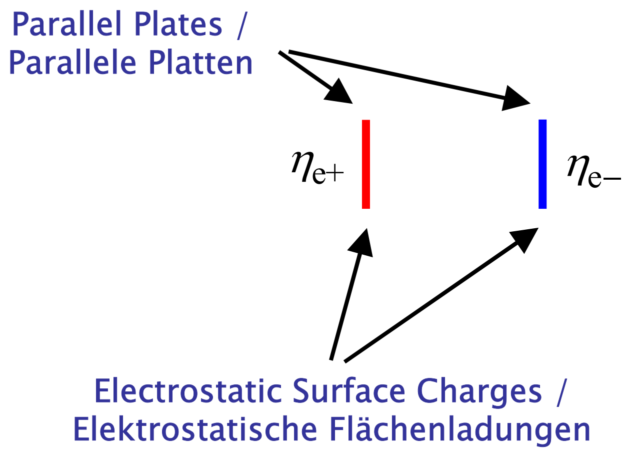
## Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono–, Di– und Quadrupol

Application: Numerical Solution of Unbounded Static Problems / Anwendung: Numerische Lösung von unbegrenzten statischen Problemen

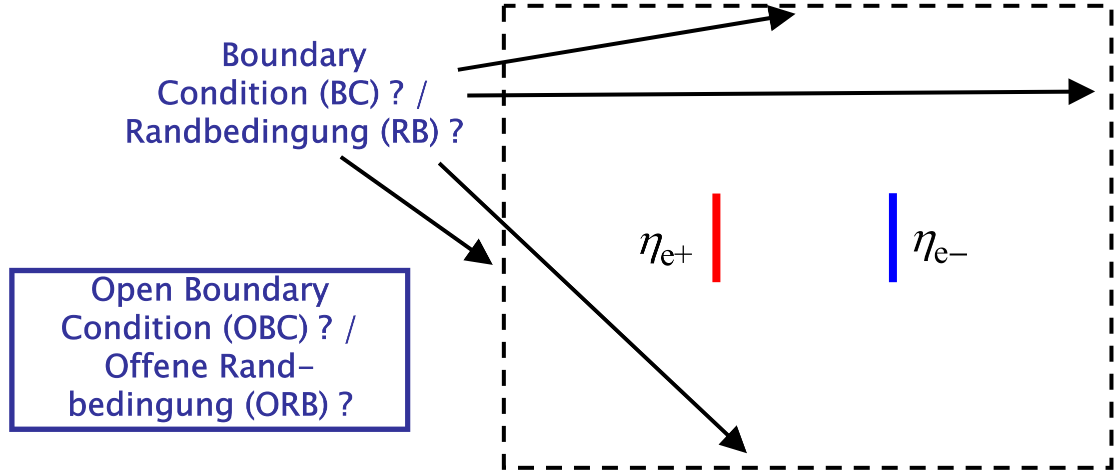
$$\Delta\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon_0}$$

Problem: Parallel Plate Capacitor in an Unbounded Region / Problem: Paralleler Plattenkondensator in einem unbegrenzten Gebiet

Outline of the Problem / Entwurf des Problems




Numerical Solution: We need to Specify Boundary Conditions at the Boundaries of the Simulation Area which is always bounded. / Numerische Lösung: Wir müssen für die Ränder des numerischen Simulationsgebietes, welches immer begrenzt ist, Randbedingungen spezifizieren.

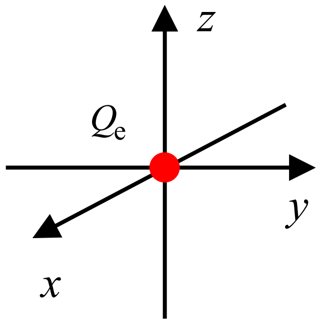


# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono–, Di– und Quadrupol

### Monopole / Monopol

One Point Charge / Eine Punktladung



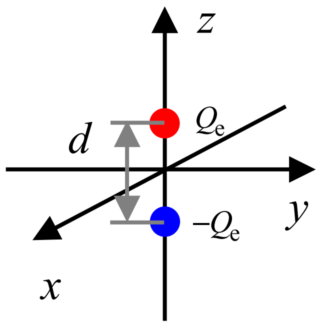

$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$$

with / mit

$$\underline{\mathbf{R}}_+ = \underline{\mathbf{0}}$$

### Dipole / Dipol

Two Point Charges / Zwei Punktladungen



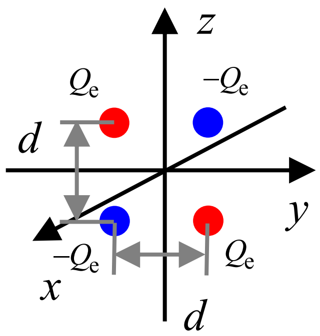

$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with / mit

$$\underline{\mathbf{R}}_+ = \frac{d}{2}\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\frac{d}{2}\underline{\mathbf{e}}_z$$

### Quadrupole / Quadrupol

Four Point Charges / Vier Punktladungen




$$\rho_e(\underline{\mathbf{R}}) = \sum_{i=1}^{4} Q_e^{(i)} \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}^{(i)})$$

with / mit

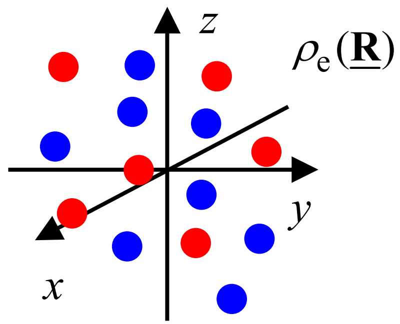
$$Q_e^{(1)} = -Q_e, \; \underline{\mathbf{R}}^{(1)} = \frac{d}{2}\underline{\mathbf{e}}_y + \frac{d}{2}\underline{\mathbf{e}}_z \qquad Q_e^{(2)} = Q_e, \; \underline{\mathbf{R}}^{(2)} = -\frac{d}{2}\underline{\mathbf{e}}_y + \frac{d}{2}\underline{\mathbf{e}}_z$$

$$Q_e^{(3)} = -Q_e, \; \underline{\mathbf{R}}^{(3)} = -\frac{d}{2}\underline{\mathbf{e}}_y - \frac{d}{2}\underline{\mathbf{e}}_z \qquad Q_e^{(4)} = Q_e, \; \underline{\mathbf{R}}^{(4)} = \frac{d}{2}\underline{\mathbf{e}}_y - \frac{d}{2}\underline{\mathbf{e}}_z$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

Arbitrary Point Charge /
Beliebige Punktsladungsverteilungen




$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_\mathrm{e}(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'$$

Expansion of $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in a Taylor Series for $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ yields: /
Entwicklung von $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in eine Taylor-Reihe für $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ ergibt:

$$\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} = \frac{1}{R} + \frac{1}{R^3}\underline{\mathbf{R}} \cdot \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^5}\underline{\mathbf{R}} \cdot \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \cdot \underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right] \cdot \underline{\mathbf{R}} + \mathcal{HOT}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

$$\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^3}\underline{\mathbf{R}}\bullet\underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^5}\underline{\mathbf{R}}\bullet\left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}'\bullet\underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right]\bullet\underline{\mathbf{R}} + \mathcal{HOT} \right\} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2}\underline{\hat{\mathbf{R}}}\bullet\underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^3}\underline{\hat{\mathbf{R}}}\bullet\left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}'\bullet\underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right]\bullet\underline{\hat{\mathbf{R}}} + \mathcal{HOT} \right\} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \hat{\underline{\mathbf{R}}} \bullet \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^3} \hat{\underline{\mathbf{R}}} \bullet \left[ 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\underline{\mathbf{I}}} \right] \bullet \hat{\underline{\mathbf{R}}} + \mathcal{HOT} \right\} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

$$\begin{aligned}
\hat{\underline{\mathbf{R}}} \bullet \left[ 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\underline{\mathbf{I}}} \right] \bullet \hat{\underline{\mathbf{R}}} &= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \hat{\underline{\mathbf{R}}} \bullet \underbrace{\underline{\underline{\mathbf{I}}} \bullet \hat{\underline{\mathbf{R}}}}_{=\hat{\underline{\mathbf{R}}}} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underbrace{\hat{\underline{\mathbf{R}}} \bullet \hat{\underline{\mathbf{R}}}}_{=1} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underbrace{\underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\underline{\mathbf{I}}}} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\underline{\mathbf{I}}} \\
&= \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left( 3\hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right)
\end{aligned}$$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \underline{\mathbf{R}}' \bullet \hat{\underline{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3} \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left( 3\hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right) + \mathcal{HOT} \right\} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

$$\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2}\underline{\mathbf{R}}'\bullet\underline{\hat{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3}\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left(3\underline{\hat{\mathbf{R}}}\,\underline{\hat{\mathbf{R}}} - \underline{\underline{\mathbf{I}}}\right) + \mathcal{HOT} \right\} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

$$= \frac{1}{4\pi\varepsilon_0} \left\{ \frac{1}{R} \underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=Q_{\mathrm{e}}} \right.$$

$$+ \frac{1}{R^2} \underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{p}}_{\mathrm{e}}} \bullet \underline{\hat{\mathbf{R}}}$$

$$\left. + \frac{1}{2}\frac{1}{R^3} \left[ \underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\underline{\mathbf{q}}}_{\mathrm{e}}} : \left(3\underline{\hat{\mathbf{R}}}\,\underline{\hat{\mathbf{R}}} - \underline{\underline{\mathbf{I}}}\right) \right] \right\} + \mathcal{HOT}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\Bigg\{ \frac{1}{R} \underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=Q_e}$$

$$+\frac{1}{R^2} \underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{p}}_e} \bullet \underline{\hat{\mathbf{R}}}$$

$$+\frac{1}{2}\frac{1}{R^3}\left[ \underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\underline{\mathbf{q}}}_e} : \left(3\underline{\hat{\mathbf{R}}}\,\underline{\hat{\mathbf{R}}} - \underline{\underline{\mathbf{I}}}\right)\right]\Bigg\} + \mathcal{HOT}$$

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{1}{R} Q_e + \frac{1}{R^2}\underline{\mathbf{p}}_e \bullet \underline{\hat{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3}\underline{\underline{\mathbf{q}}}_e : \left[3\underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underline{\underline{\mathbf{I}}}\right] + \mathcal{HOT}\right\}$$

$\mathcal{HOT}$ : Higher Order Terms / Terme höherer Ordnung

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

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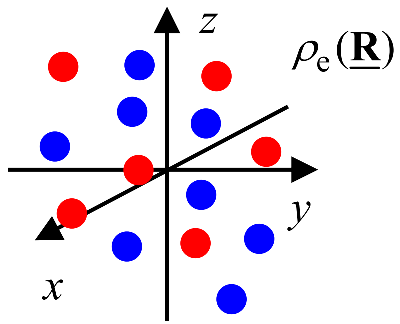
Arbitrary Point Charge / Beliebige Punktsladungsverteilungen




$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_e(\underline{\mathbf{R}}') \mathrm{d}^3\underline{\mathbf{R}}'$$

Expansion of $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in a Taylor Series for $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ yields:

Entwicklung von $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in eine Taylor-Reihe für $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ ergibt:

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{1}{R} Q_e + \frac{1}{R^2} \underline{\mathbf{p}}_e \bullet \hat{\underline{\mathbf{R}}} + \frac{1}{2} \frac{1}{R^3} \underline{\underline{\mathbf{q}}}_e : \left[ 3\hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right] + \mathcal{HOT} \right\}$$

$\mathcal{HOT}$ : Higher Order Terms / Terme höherer Ordnung

Monopole Moment / Monopolmoment

$$Q_e = \iiint_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \mathrm{d}^3\underline{\mathbf{R}}'$$

Dipole Moment / Dipolmoment

$$\underline{\mathbf{p}}_e = \iiint_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \underline{\mathbf{R}}' \mathrm{d}^3\underline{\mathbf{R}}'$$

Quadrupole Moment / Quadrupolmoment

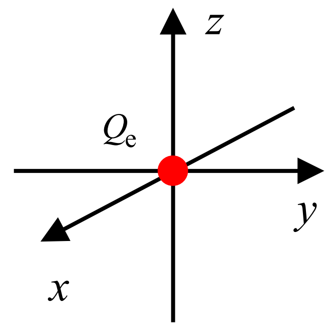
$$\underline{\underline{\mathbf{q}}}_e = \iiint_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \underline{\mathbf{R}}' \underline{\mathbf{R}}' \mathrm{d}^3\underline{\mathbf{R}}'$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono–, Di– und Quadrupol

**Monopole Moment / Monopolmoment**

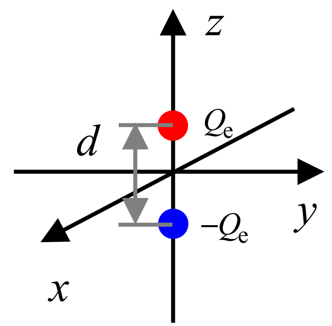
One Point Charge / Eine Punktladung




$$Q_e \neq 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

**Dipole Moment / Dipolmoment**

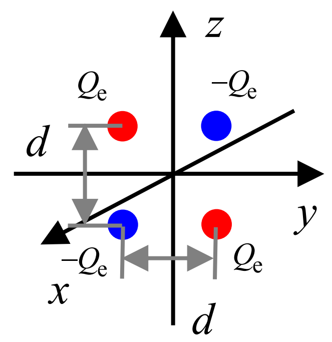
Two Point Charges / Zwei Punktladungen




$$Q_e = 0, \quad \underline{\mathbf{p}}_e \neq \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

**Quadrupole Moment/ Quadrupolmoment**

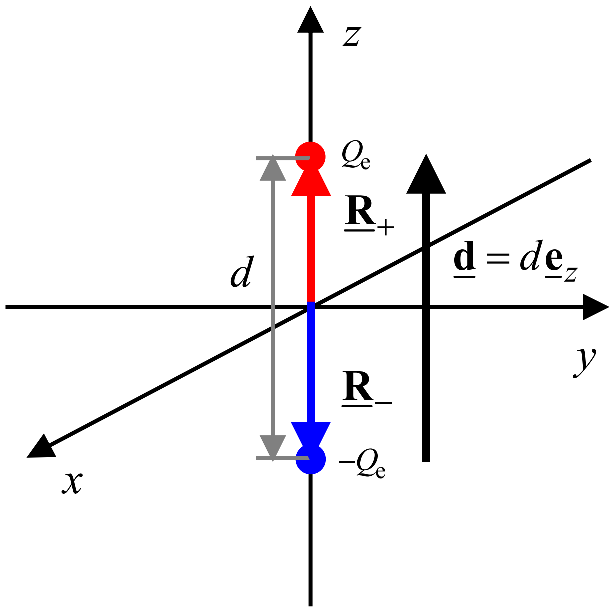
Four Point Charges / Vier Punktladungen




$$Q_e = 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e \neq \underline{\underline{\mathbf{0}}}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Electrostatic Dipole / Elektrostatischer Dipol




Electrostatic Volume Charge Density / Elektrostatische Raumladungsdichte

$$\rho_e(\underline{\mathbf{R}}) = Q_e\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$
$$= Q_e\delta(\underline{\mathbf{R}} - \frac{d}{2}\underline{\mathbf{e}}_z) - Q_e\delta(\underline{\mathbf{R}} + \frac{d}{2}\underline{\mathbf{e}}_z)$$

with $\underline{\mathbf{R}}_+ = \frac{d}{2}\underline{\mathbf{e}}_z$, $\underline{\mathbf{R}}_- = -\frac{d}{2}\underline{\mathbf{e}}_z$

Electrostatic Potential / Elektrostatisches Potential

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$

Electrostatic Field Strength / Elektrostatische Feldstärke

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_+}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_-}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|^3}\right)$$

Electrostatic Dipole Moment / Elektrische Dipolmoment

$$\underline{\mathbf{p}}_e = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\mathrm{d}^3\underline{\mathbf{R}}' = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e\delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\right]\underline{\mathbf{R}}'\mathrm{d}^3\underline{\mathbf{R}}'$$

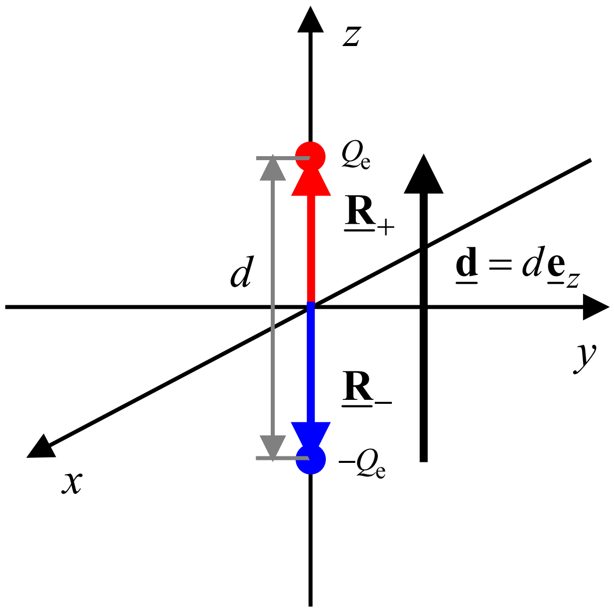
$$= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-} = Q_e\underline{\mathbf{R}}_+ - Q_e\underline{\mathbf{R}}_- = Q_e(\underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_-) = Q_e\underline{\mathbf{d}}$$

Distance Vector / Abstandsvektor

$$\underline{\mathbf{d}} = \underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_- = \frac{d}{2}\underline{\mathbf{e}}_z + \frac{d}{2}\underline{\mathbf{e}}_z = d\underline{\mathbf{e}}_z$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Electrostatic Dipole / Elektrostatischer Dipol




### Electrostatic Dipole Moment / Elektrostatisches Dipolmoment

$$\underline{\mathbf{p}}_e = Q_e \underline{\mathbf{d}} = p_e \hat{\underline{\mathbf{p}}}_e$$

with / mit

$$p_e = Q_e |\underline{\mathbf{d}}| = Q_e |\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|$$

$$\hat{\underline{\mathbf{p}}}_e = \hat{\underline{\mathbf{d}}} = \widehat{|\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|}$$

### Electrostatic Quadrupole Moment / Elektrostatisches Quadrupolmoment

$$\underline{\underline{\mathbf{q}}}_e = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'$$

$$= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\right]\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'$$

$$= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-\underline{\mathbf{R}}_-}$$

$$= Q_e \underline{\mathbf{R}}_+\underline{\mathbf{R}}_+ - Q_e \underline{\mathbf{R}}_-\underline{\mathbf{R}}_-$$

$$= Q_e \left[\left(\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(\frac{d}{2}\underline{\mathbf{e}}_z\right) - \left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\right]$$

$$= Q_e \underbrace{\left[d\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z - d\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z\right]}_{=\underline{\underline{\mathbf{0}}}}$$

$$= \underline{\underline{\mathbf{0}}}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ... (2)

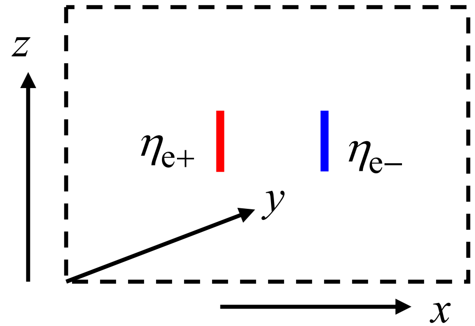
Application: Numerical Solution of Unbounded Static Problems / Anwendung: Numerische Lösung von unbegrenzten statischen Problemen

$$\Delta\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon_0}$$

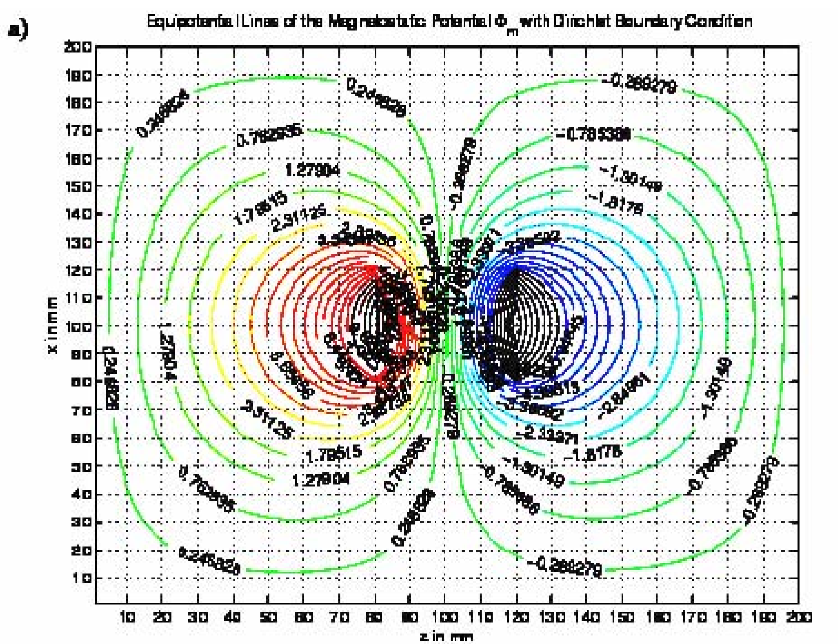
$$\rho_e(\underline{\mathbf{R}}) = \eta_{e+}(y,z)\delta(x-x_+) + \eta_{e-}(y,z)\delta(x-x_-)$$

$$\eta_{e+}(y,z) = \begin{cases} \eta_{e0} & \begin{matrix} y_- \le y \le y_+ \\ z_- \le z \le z_+ \end{matrix} \\ 0 & \text{else / sonst} \end{cases}$$

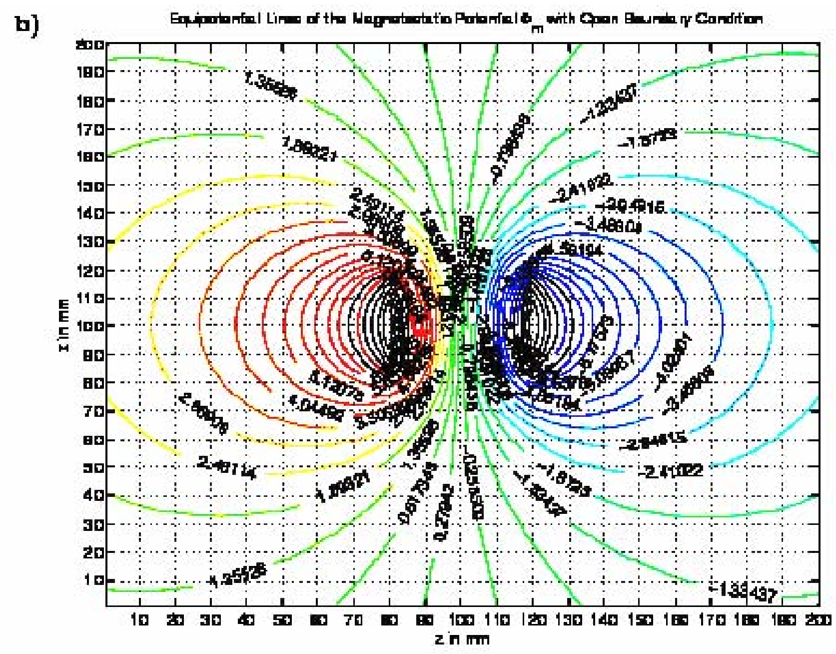
$$= -\eta_{e-}(y,z)$$




With Dirichlet Boundary Condition / Mit Dirichlet Randbedingung




With Open Boundary Condition (OBC) / Mit offener Randbedingung (ORB)

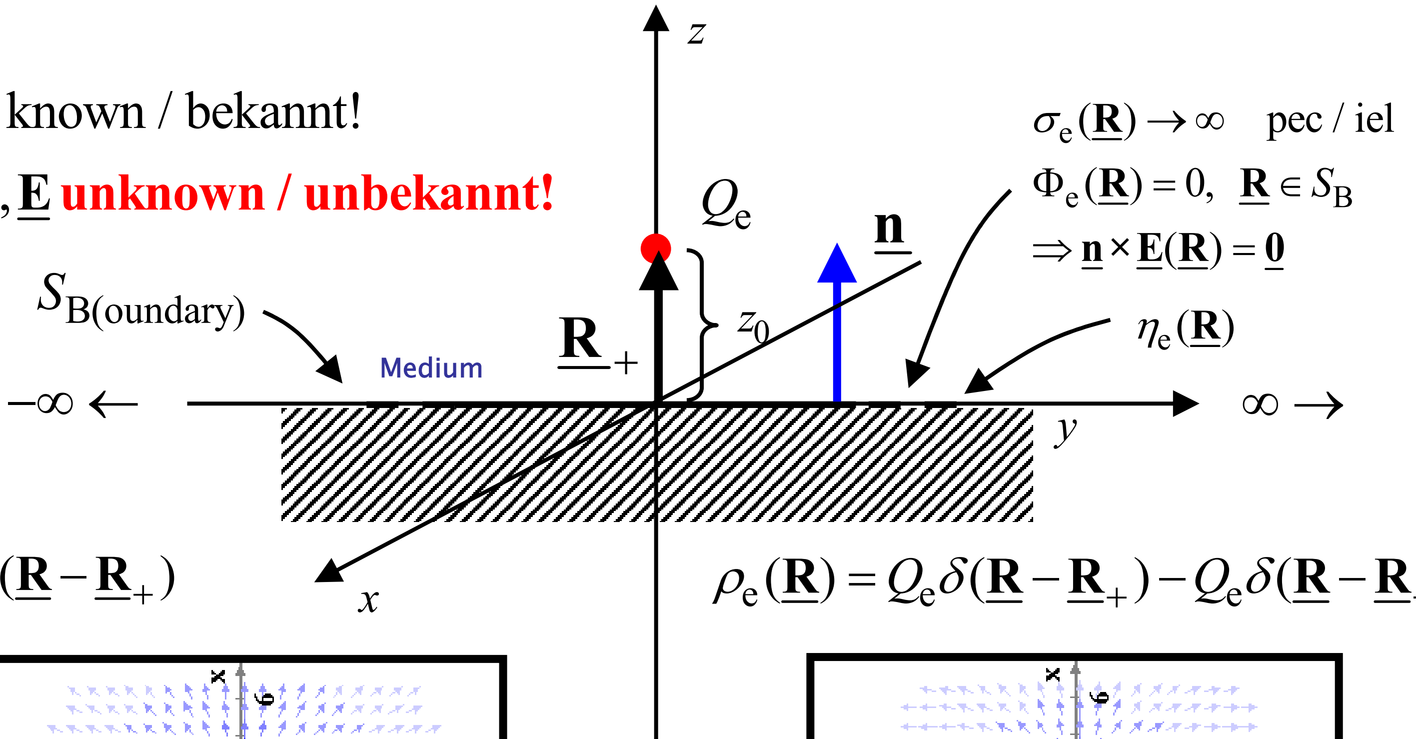


# ES Fields – Method of Images / ES–Felder – Spiegelungsmethode

## Boundary Value Problem (BVP) – Randwertproblem (RWP)

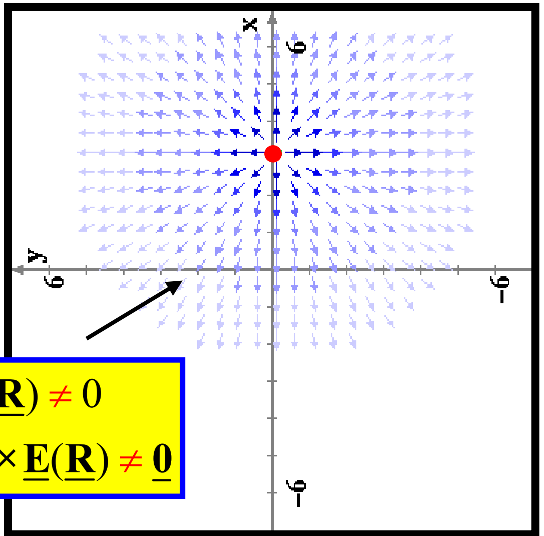
$Q_e$ known / bekannt!

$\Phi_e, \underline{\mathbf{E}}$ **unknown / unbekannt!**

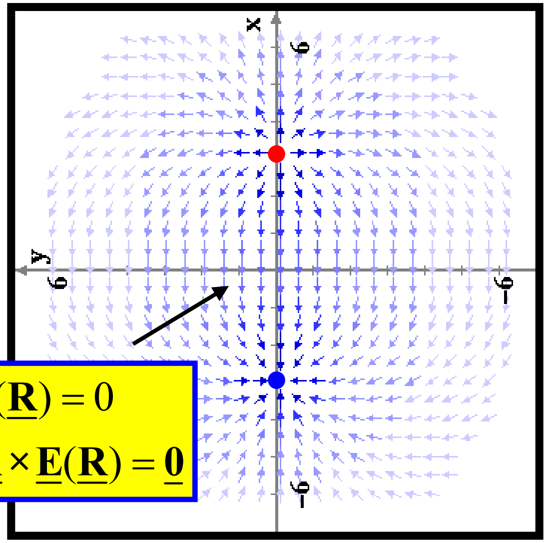



$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$$

$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$



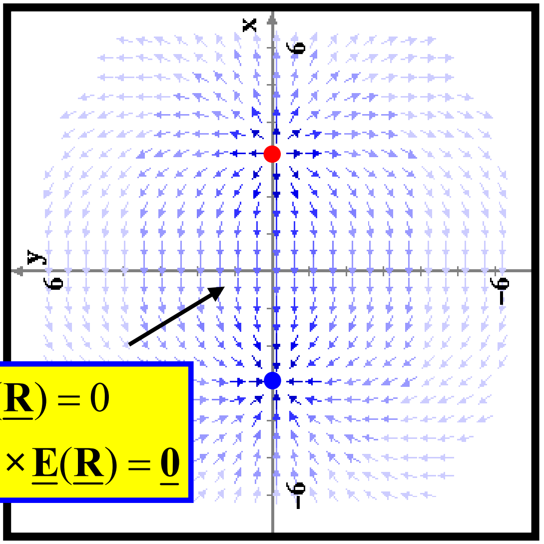
$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) \neq 0$
$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) \neq \underline{\mathbf{0}}$



$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$
$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

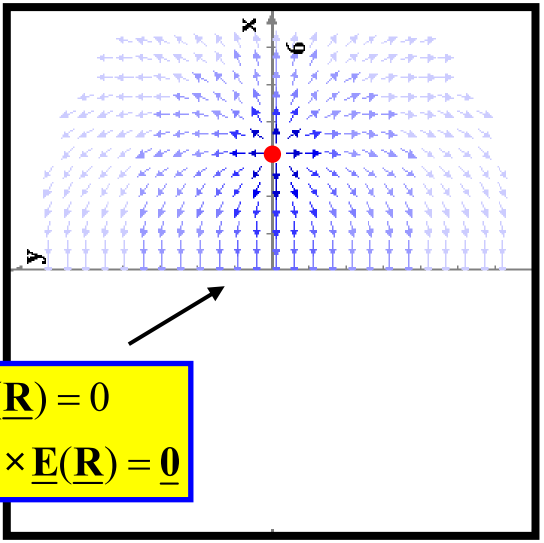
## Method of Images / Spiegelungsmethode




$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$



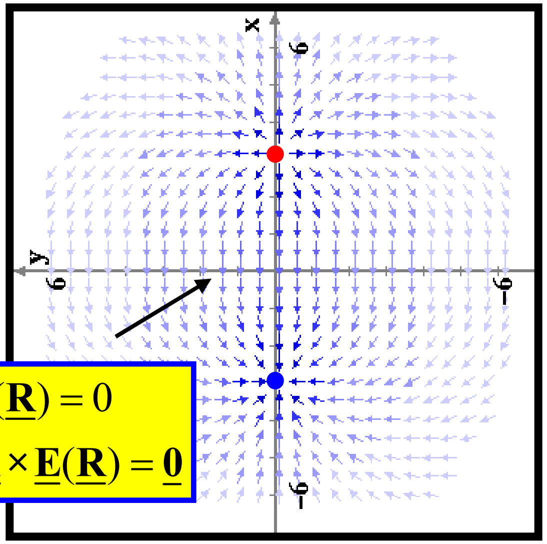

$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

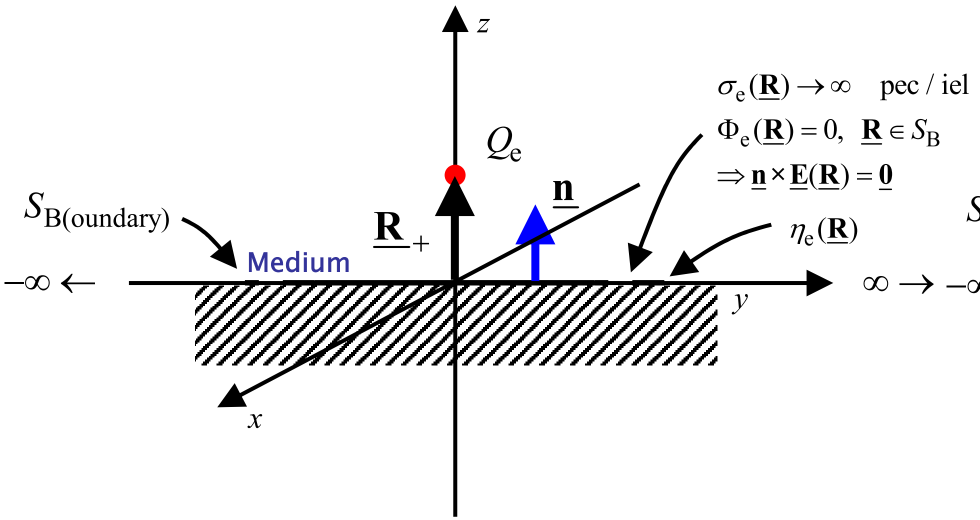
## Method of Images / Spiegelungsmethode




$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

**Problem:**




**Solution / Lösung:**

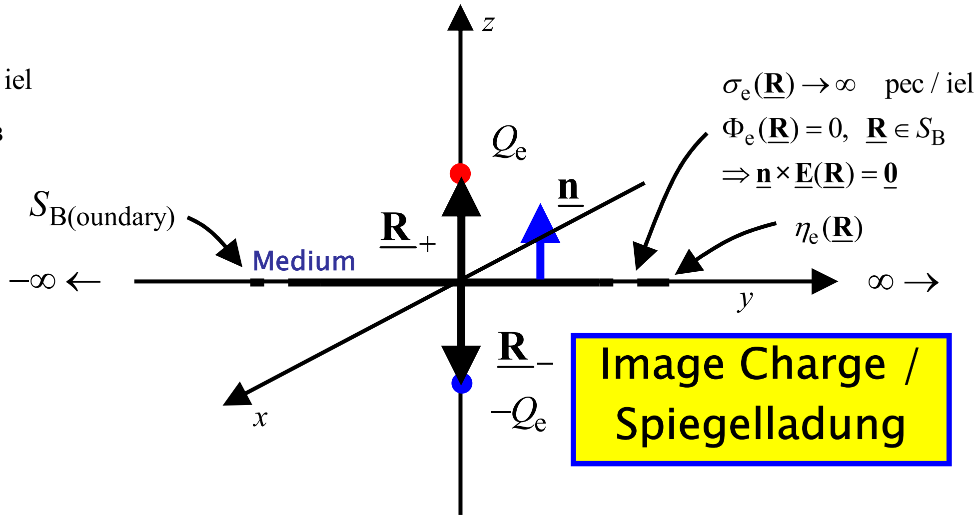
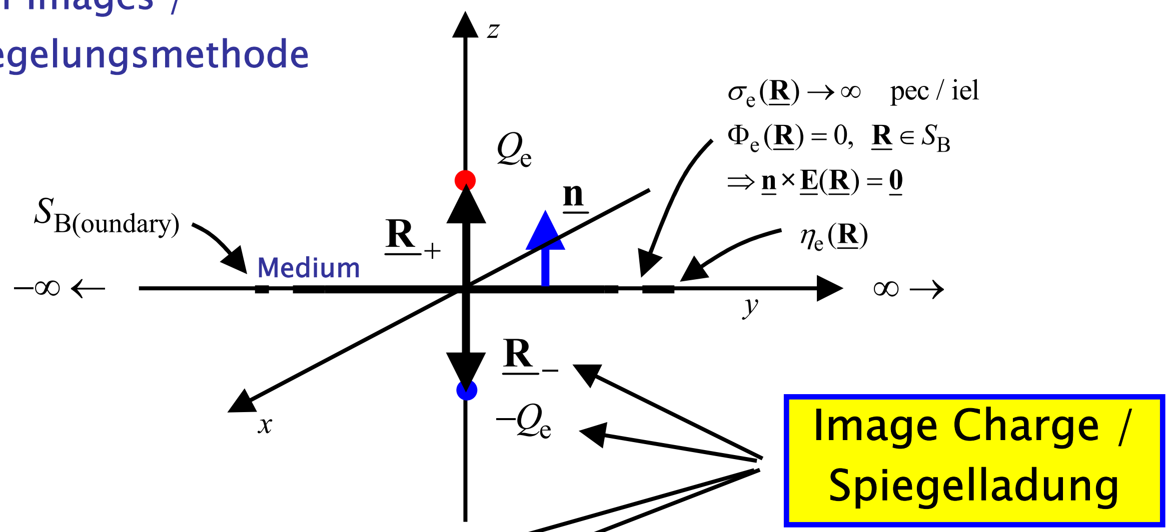



Image Charge / Spiegelladung

# ES Fields – Method of Images / ES Felder – Spiegelungsmethode

Solution by Applying the Method of Images /
Lösung durch Anwendung der Spiegelungsmethode




$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with
mit

$$\underline{\mathbf{R}}_+ = z_0 \underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0 \underline{\mathbf{e}}_z$$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode

$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_-)$$

with / mit $\underline{\mathbf{R}}_+ = z_0\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0\underline{\mathbf{e}}_z$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

# ES Fields – Method of Images / ES–Felder – Spiegelungsmethode

## Boundary Value Problem (BVP) – Randwertproblem (RWP)

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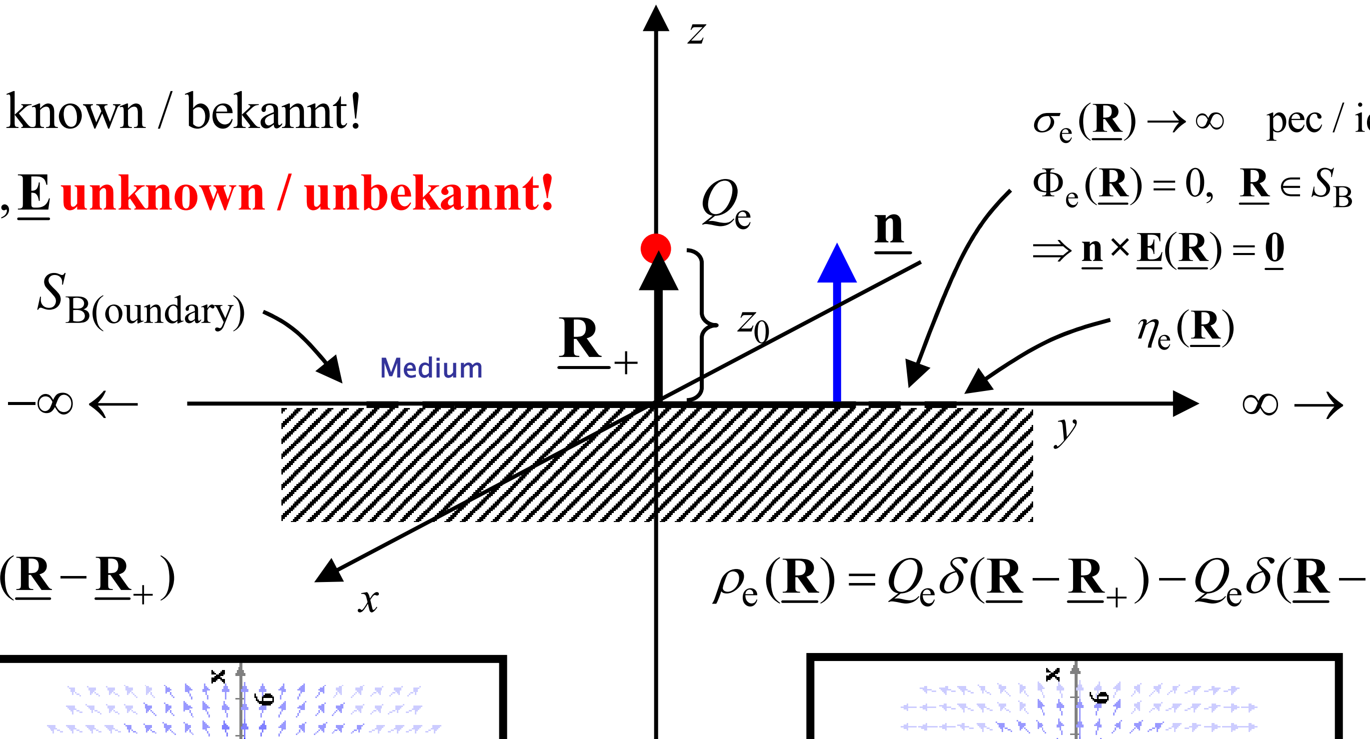
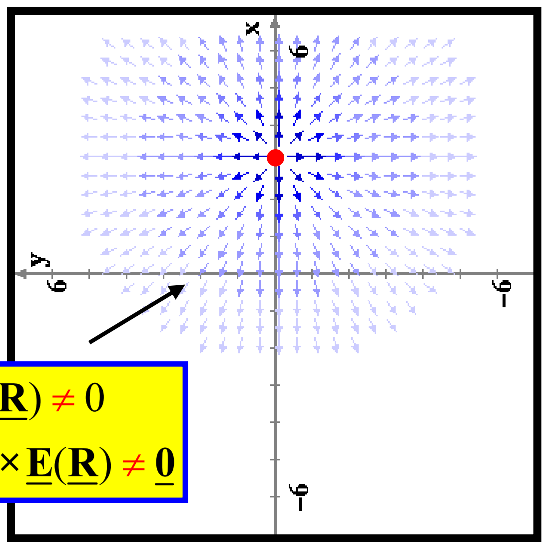
$\Phi_e, \underline{\mathbf{E}}$ **unknown / unbekannt!**




$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$$

$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$




$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) \neq 0$

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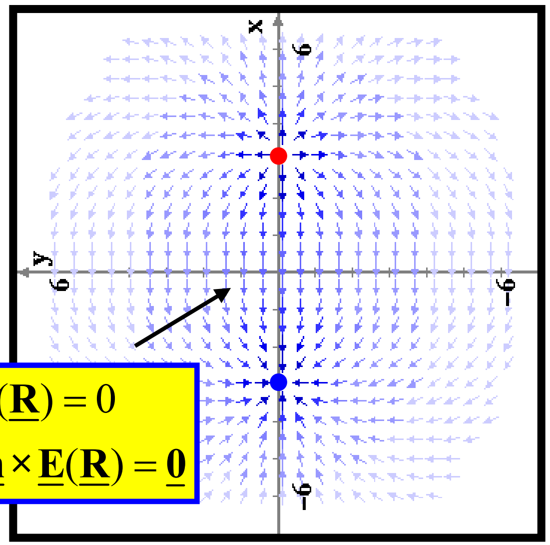


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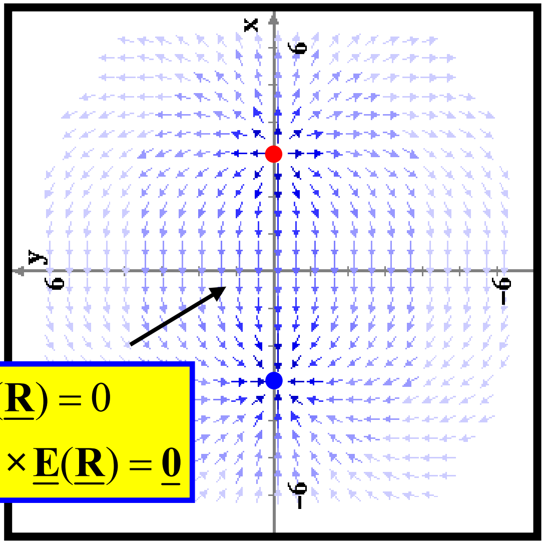
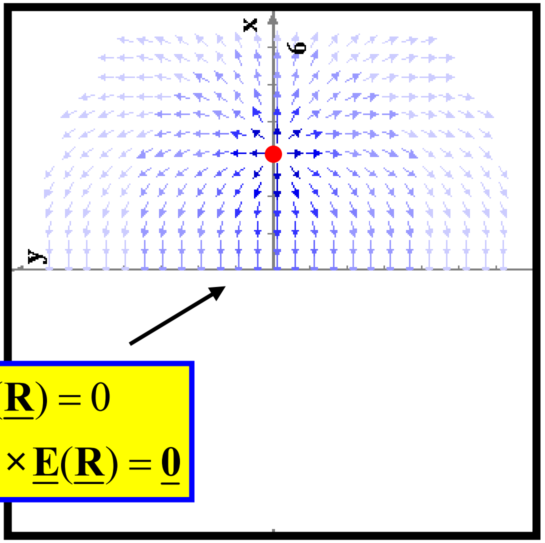
# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode




$$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$$
$$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$$

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$

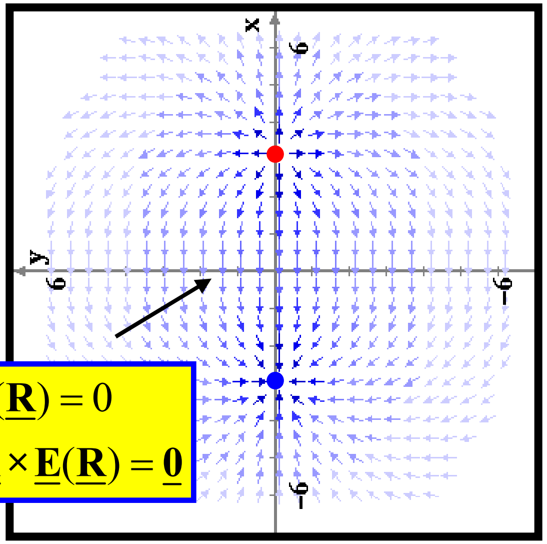



$$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$$
$$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode

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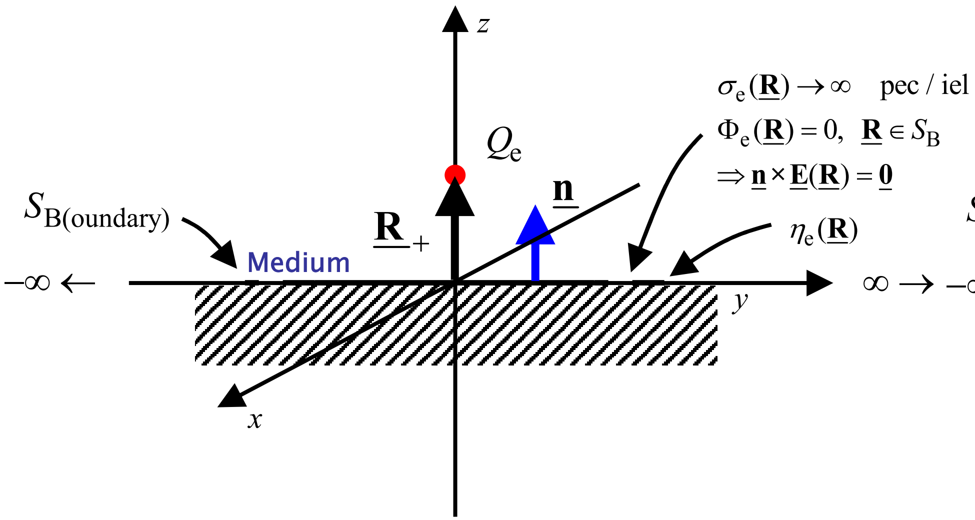
**Problem:**




**Solution / Lösung:**

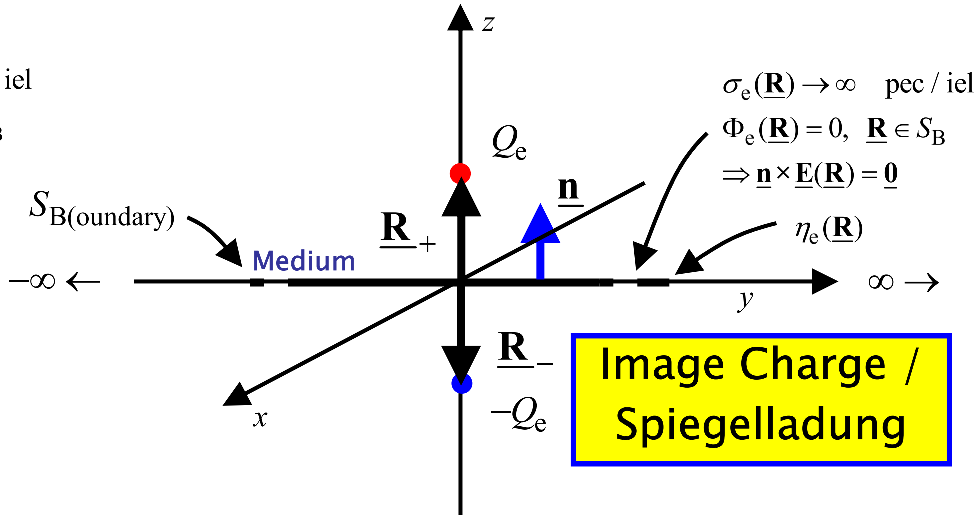
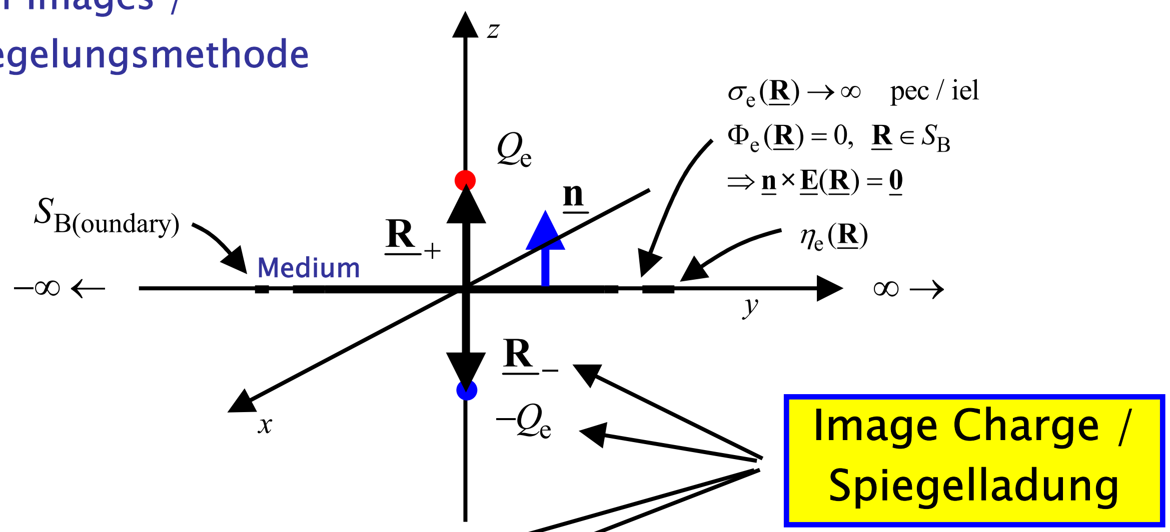



Image Charge / Spiegelladung

# ES Fields – Method of Images / ES Felder – Spiegelungsmethode

Solution by Applying the Method of Images /
Lösung durch Anwendung der Spiegelungsmethode




$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with
mit

$$\underline{\mathbf{R}}_+ = z_0 \underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0 \underline{\mathbf{e}}_z$$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0} \left( \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|} \right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode

$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_-)$$

with / mit $\underline{\mathbf{R}}_+ = z_0\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0\underline{\mathbf{e}}_z$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

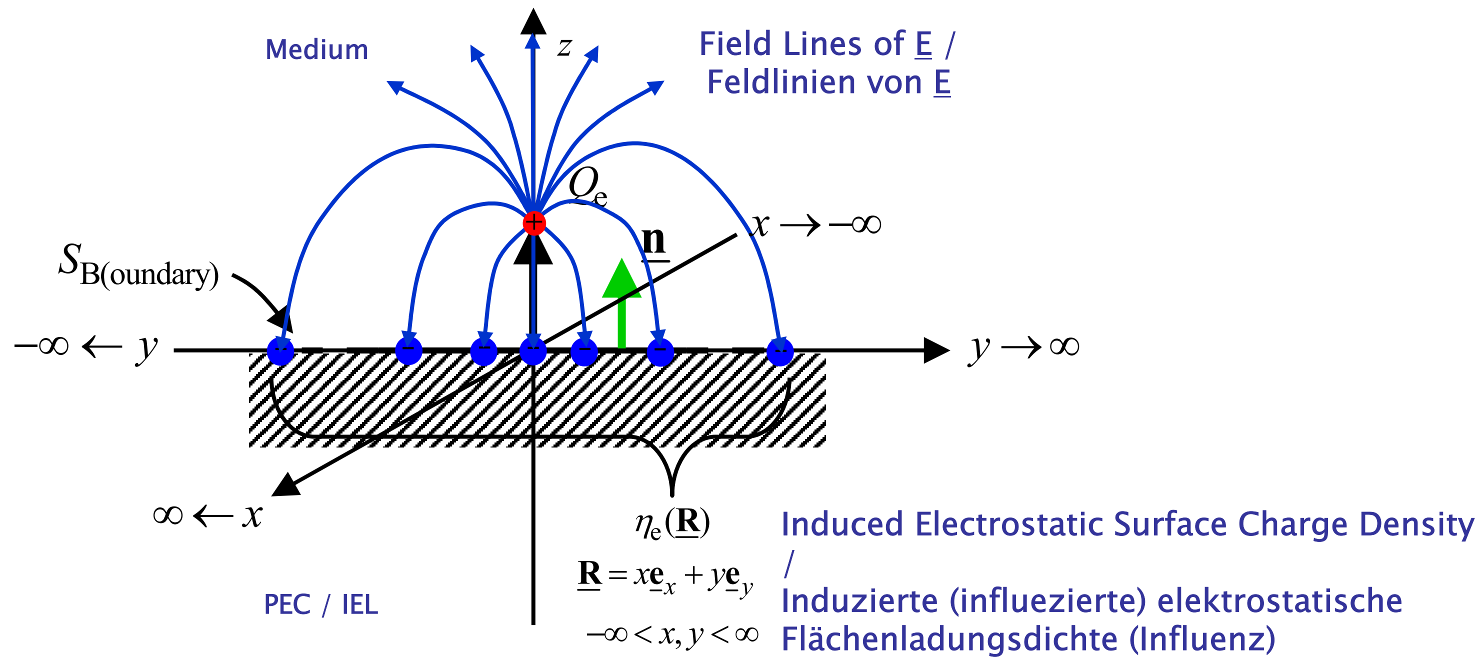
$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode




Without the Method of Images we have to Solve the Following Integral Equation for the Unknown Induced Electrostatic Surface Charge / Ohne die Spiegelungsmethode muss man die folgende Integralgleichung für die induzierte (influezierte) elektrostatische Flächenladungsdichte lösen

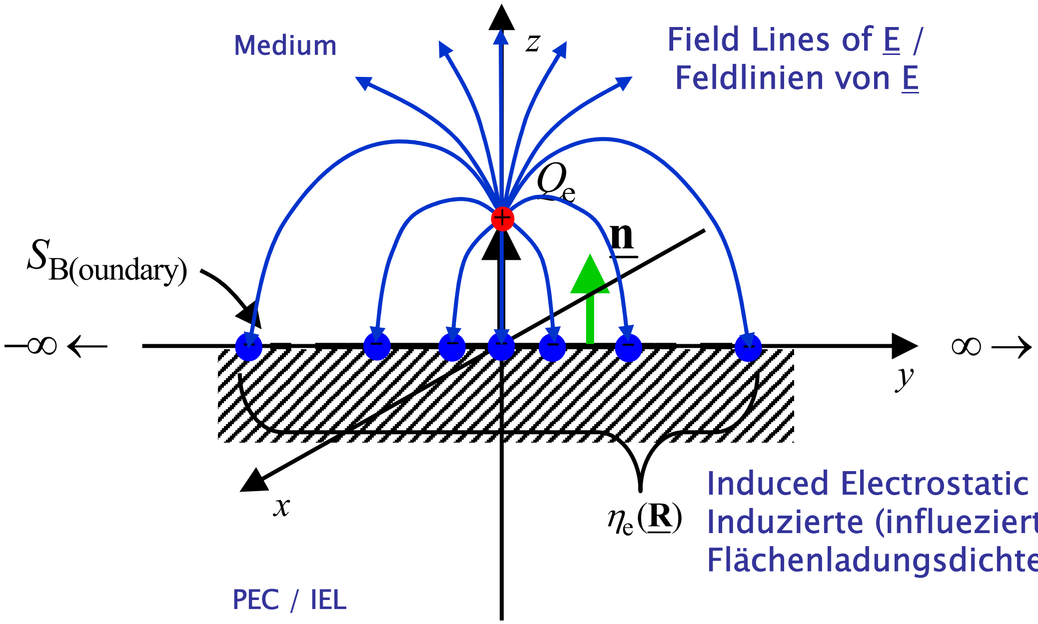
Unknown / Unbekannt

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_e}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} + \iint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{\eta_e(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+|}\, d^2\underline{\mathbf{R}}'\,\Bigg|_{z=0}\right] = 0$$

for / für $\Phi_e(\underline{\mathbf{R}})\big|_{z=0} = 0$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode




If D is known from the Method of Images / Falls D über die Spiegelungsmethode bekannt ist

$$\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B} = \begin{matrix}\text{known}\\ \text{bekannt}\end{matrix} \; !$$

$\eta_e(\underline{\mathbf{R}})$ is Defined by the Normal Component of D / $\eta_e(\underline{\mathbf{R}})$ ist definiert über die Normalkomponente von D

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B}$$

$$= \underline{\mathbf{n}} \bullet \frac{Q_e}{4\pi}\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{\underline{\mathbf{R}}\in S_B} \quad \begin{matrix}\text{for}\\ \text{für}\end{matrix} \quad z = 0$$

$$= \frac{Q_e}{4\pi}\underline{\mathbf{e}}_z \bullet \left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{z=0}$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode

$$
\begin{aligned}
\eta_{\mathrm{e}}(\underline{\mathbf{R}}) &= \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}} \in S_{\mathrm{B}}} \\
&= \frac{Q_{\mathrm{e}}}{4\pi} \left[ \frac{\underline{\mathbf{e}}_z \bullet \left(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+\right)}{\left|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+\right|^3} - \frac{\underline{\mathbf{e}}_z \bullet \left(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-\right)}{\left|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-\right|^3} \right]\Bigg|_{z=0} \\
&= \frac{Q_{\mathrm{e}}}{4\pi} \left[ \frac{z - z_0}{\left[x^2 + y^2 + \left(z - z_0\right)^2\right]^{3/2}} - \frac{z + z_0}{\left[x^2 + y^2 + \left(z - z_0\right)^2\right]^{3/2}} \right]\Bigg|_{z=0} \\
&= \frac{Q_{\mathrm{e}}}{4\pi} \left[ \frac{-z_0}{\left[x^2 + y^2 + z_0^2\right]^{3/2}} - \frac{z_0}{\left[x^2 + y^2 + z_0^2\right]^{3/2}} \right] \\
&= -\frac{Q_{\mathrm{e}}}{2\pi} \frac{z_0}{\left[x^2 + y^2 + z_0^2\right]^{3/2}} \\
&= -\frac{Q_{\mathrm{e}}}{2\pi} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}}
\end{aligned}
$$

# Electrostatic (ES) Fields / Elektrostatische (ES) Felder

## Method of Images / Spiegelungsmethode

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\big|_{\underline{\mathbf{R}} \in S_B} = -\frac{Q_e}{2\pi} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}}$$

⇨ Total Electric Charge at the *xy* Plane at *z*=0 / Gesamtladung auf der *xy* Ebene bei *z*=0

$$Q_e^{tot} = -\frac{Q_e}{2\pi} \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} r \, dr \, d\varphi$$

$$= -\frac{Q_e}{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} r \, dr \underbrace{\int_{\varphi=0}^{2\pi} d\varphi}_{=2\pi}$$

$$= -Q_e z_0 \int_{r=0}^{\infty} \frac{r}{\left[r^2 + z_0^2\right]^{3/2}} dr$$

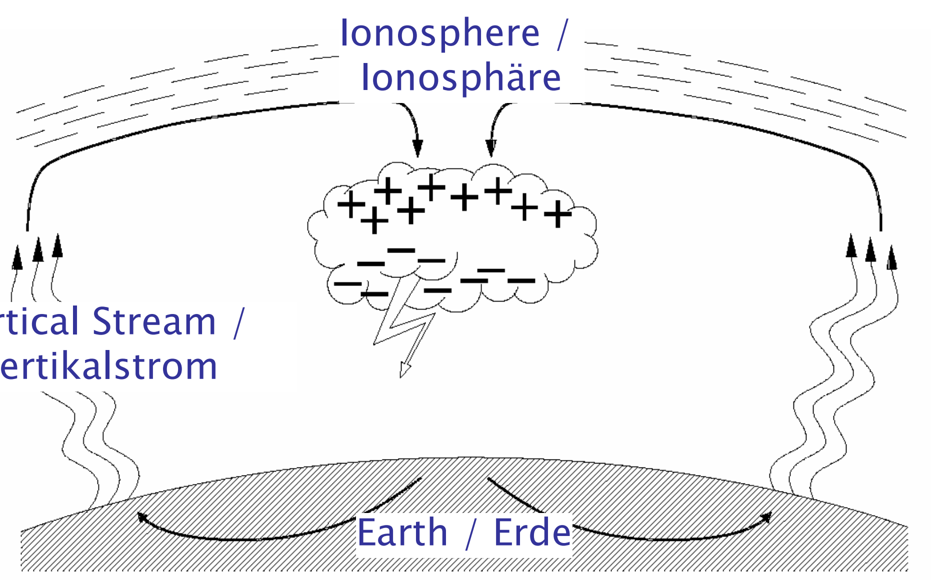
$$\int \frac{x}{\left[x^2 + a^2\right]^{3/2}} dx = -\frac{1}{\sqrt{x^2 + a^2}}$$

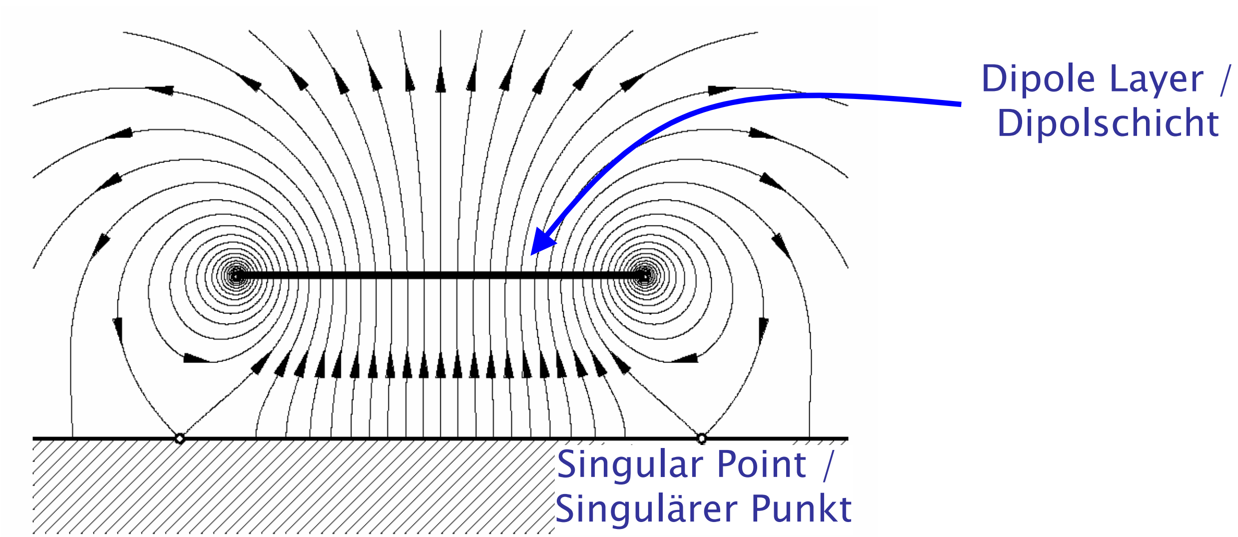
$$= Q_e z_0 \left[ \underbrace{\frac{1}{\sqrt{(r \to \infty)^2 + z_0^2}}}_{\to 0} - \frac{1}{z_0} \right]$$

$$= -Q_e$$

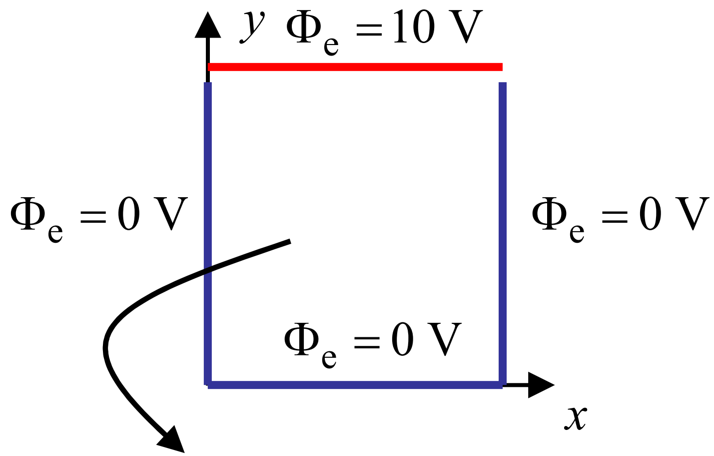
$$Q_e^{tot} = -Q_e$$

# ES Fields – Method of Images – Applications / ES Felder – Spiegelungsmethode – Anwendungen



# Electrostatic (ES) Fields – Separation of Variables – Example / Elektrostatische (ES) Felder – Separation der Variablen – Beispiel




$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y) = 0$$

➡ **Separation of Variables / Separation der Variablen !**

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\Delta\Phi_{\mathrm{e}}(x,y,z)=0$$

Elliptic Partial Differential Equation / Elliptische partielle Differentialgleichung

Laplace Equation in Cartesian Coordinates / Laplace-Gleichung in Kartesischen Koordinaten

3-D / 3D

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi_{\mathrm{e}}(x,y,z)=0$$

Function of Three Variables / Funktion von drei Variablen

2-D / 2D

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_{\mathrm{e}}(x,y)=0$$

Function of Two Variables / Funktion von zwei Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_{\mathrm{e}}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_{\mathrm{e}}(x,y)=0$$

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=0$$

Solution Strategy: Reduce the Partial Differential Equation (PDE) to an Ordinary Differential Equation (ODE) and Find a Solution of the PDE by Solving the ODE

Lösungsstrategie: Reduziere die partielle Differentialgleichung (PDG) auf eine gewöhnliche (ordinäre) Differentialgleichung (GDG) und finde eine Lösung der PDG durch Lösung der GDG

Ansatz of Separation / Separationsansatz

$$\Phi_\mathrm{e}(x,y)=\overbrace{X(x)\;Y(y)}$$

Product of 2 Functions / Produkt aus 2 Funktionen

Function of 2 Variables: $x$ and $y$ / Funktion von 2 Variablen: $x$ und $y$

Function of $x$ only / Nur eine Funktion von $x$

Function of $y$ only / Nur eine Funktion von $y$

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_{\mathrm{e}}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_{\mathrm{e}}(x,y)=0$$

Ansatz of Separation / Separationsansatz

$$\Phi_{\mathrm{e}}(x,y)=X(x)\,Y(y)$$

⇨ Inserted in the Above Laplace Equation Yields / Eingesetzt in die obere Laplace-Gleichung ergibt

$$\frac{\partial^2}{\partial x^2}\Phi_{\mathrm{e}}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_{\mathrm{e}}(x,y)=\frac{\partial^2}{\partial x^2}\left[X(x)\,Y(y)\right]+\frac{\partial^2}{\partial y^2}\left[X(x)\,Y(y)\right]$$

$$=Y(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+X(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)$$

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=\underbrace{Y(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+X(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)}$$

$$\frac{1}{X(x)Y(y)}$$

$$\frac{1}{X(x)Y(y)}\overbrace{\left[Y(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+X(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)\right]}=\frac{1}{X(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+\frac{1}{Y(y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)=0$$

$$\underbrace{\underbrace{\frac{1}{X(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)}_{=\text{Function of }x\text{ / Funktion von }x}}_{=-\alpha^2}+\underbrace{\underbrace{\frac{1}{Y(y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)}_{=\text{Function of }y\text{ / Funktion von }y}}_{=-\beta^2}=0$$

$$\frac{1}{X(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+\frac{1}{Y(y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)=\underbrace{\left(-\alpha^2\right)+\left(-\beta^2\right)}_{=0}$$

Separation Condition / Separationsbedingung $\alpha^2+\beta^2=0$

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -\alpha^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = -\beta^2 Y(y)$$

Separation Condition /
Separationsbedingung

$$\alpha^2 + \beta^2 = 0$$

With / Mit $\alpha^2 = -\beta^2 = k^2$

We Obtain Two ODE /
Wir erhalten zwei GDG

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -k^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = k^2 Y(y)$$

Solutions of these Equations are /
Lösungen dieser Gleichungen sind

$X(x) \sim \cos(kx)$ or / oder $\sim \sin(kx)$

$Y(y) \sim \cosh(ky)$ or / oder $\sim \sinh(ky)$

For $k = 0$ these Solutions Degenerate to /
Für $k = 0$ diese Lösungen degenerieren zu

$X(x) \sim \text{const.}$ or / oder $\sim x$

$Y(y) \sim \text{const.}$ or / oder $\sim y$

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

We Obtain Two ODE /
Wir erhalten zwei GDG

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -k^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = k^2 Y(y)$$

$$X(x) = \cos(kx)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) = -k\frac{\mathrm{d}}{\mathrm{d}x}\sin(kx) = -k^2 \underbrace{\cos(kx)}_{=X(x)}$$

$$Y(y) = \cosh(ky)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = \frac{\mathrm{d}^2}{\mathrm{d}y^2}\cosh(ky) = k\frac{\mathrm{d}}{\mathrm{d}y}\sinh(ky) = k^2 \underbrace{\cosh(y)}_{=Y(y)}$$

---

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(kx) = -k\sin(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) = -k^2\cos(kx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh(kx) = k\sinh(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cosh(kx) = k^2\cosh(kx)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(kx) = k\cos(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(kx) = -k^2\sin(kx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh(kx) = k\cosh(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sinh(kx) = k^2\sinh(kx)$$

# Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

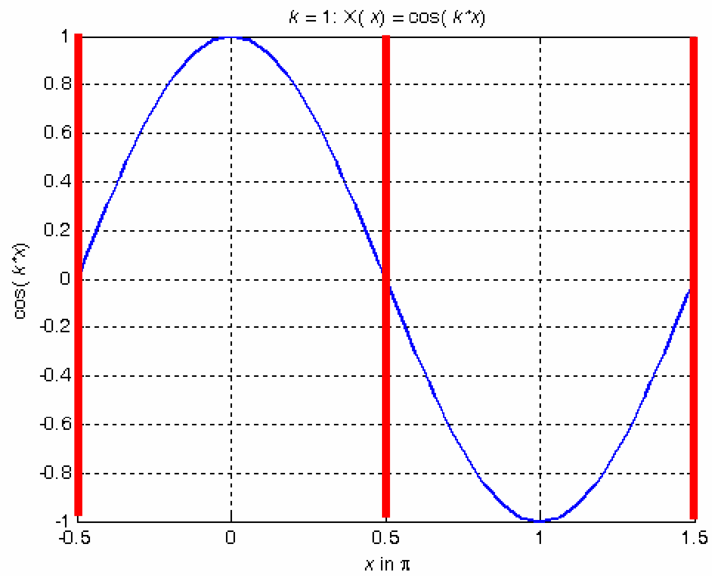
## Solutions of the 2-D Laplace Equation in the Cartesian Coordinate System / Lösungen der 2D-Laplace-Gleichung im Kartesischen Koordinatensystem

$$\Phi_\mathrm{e}(x,y) = X(x)\,Y(y) =$$

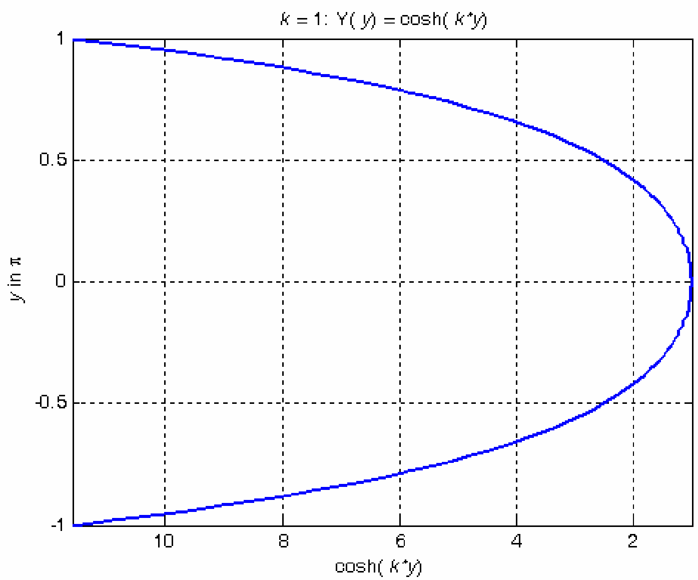
| $k=0$ | $k^2 \geq 0$ | $k^2 \leq 0 \; (k \to jk')$ |
|---|---|---|
| const. | $\cos(kx)\cosh(ky)$ | $\cosh(k'x)\cos(k'y)$ |
| $y$ | $\cos(kx)\sinh(ky)$ | $\cosh(k'x)\sin(k'y)$ |
| $x$ | $\sin(kx)\cosh(ky)$ | $\sinh(k'x)\cos(k'y)$ |
| $xy$ | $\sin(kx)\sinh(ky)$ | $\sinh(k'x)\sin(k'y)$ |
| | $\cos(kx)\,\mathrm{e}^{ky}$ | $\mathrm{e}^{k'x}\cos(k'y)$ |
| | $\cos(kx)\,\mathrm{e}^{-ky}$ | $\mathrm{e}^{-k'x}\cos(k'y)$ |
| | $\sin(kx)\,\mathrm{e}^{ky}$ | $\mathrm{e}^{k'x}\sin(k'y)$ |
| | $\sin(kx)\,\mathrm{e}^{-ky}$ | $\mathrm{e}^{-k'x}\sin(k'y)$ |

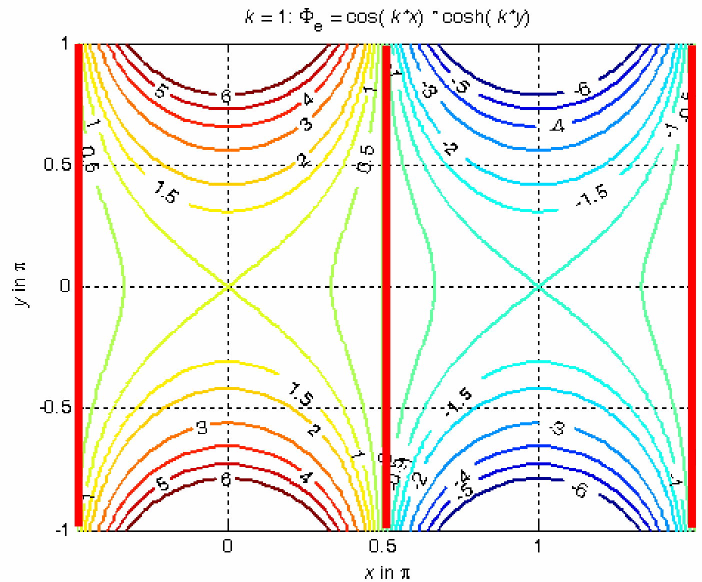
# ES Fields – Separation of Variables / ES Felder – Separation der Variablen (...)

$$\Phi_e(x,y) = X(x)Y(y) = \cos(kx)\cosh(ky)$$



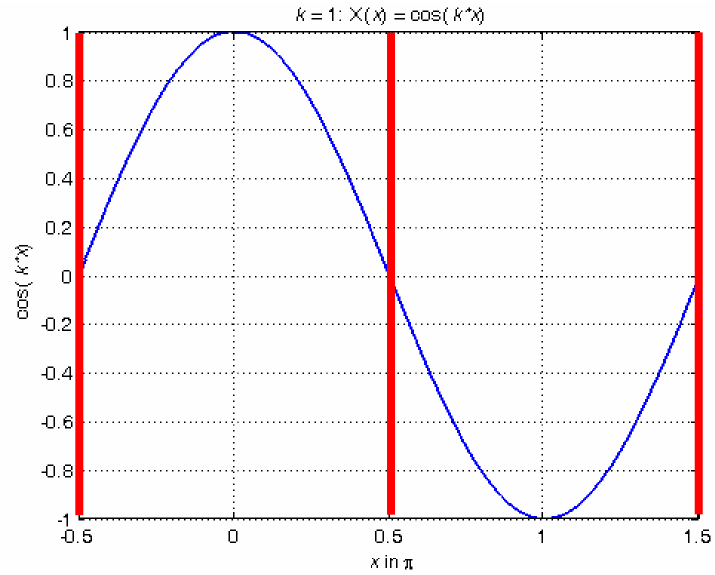

$$\Phi_e(x,y) = 0$$

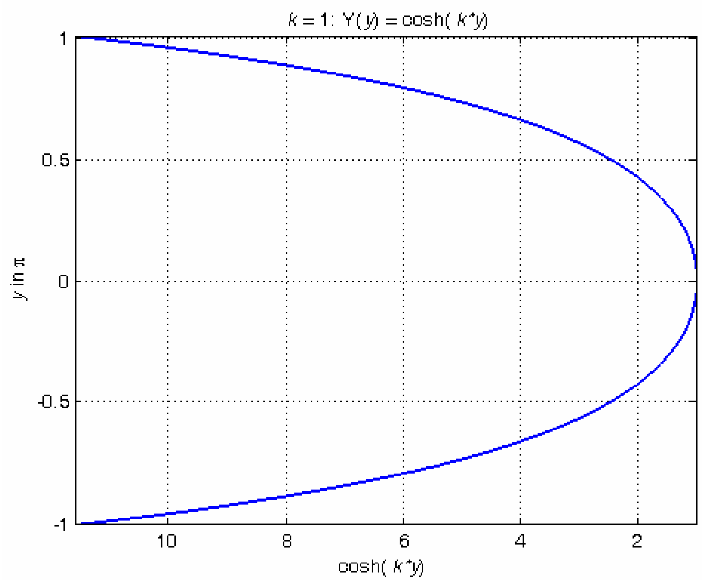


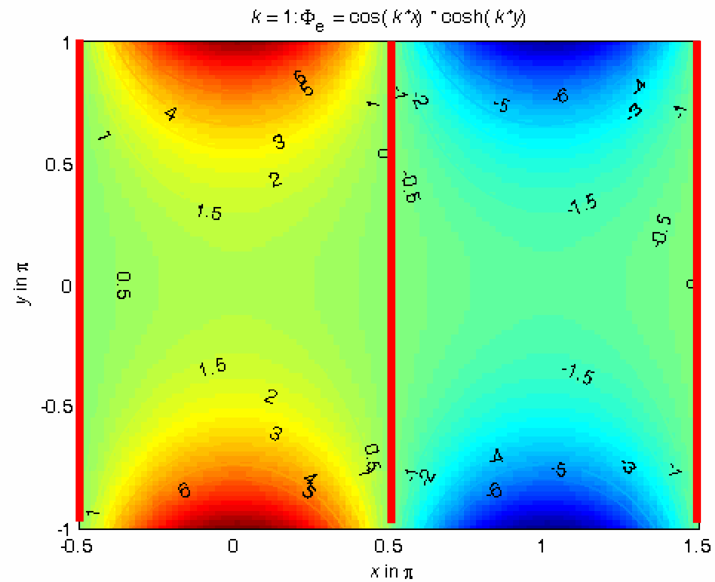



# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_e(x,y) = X(x)Y(y) = \cos(kx)\cosh(ky)$$

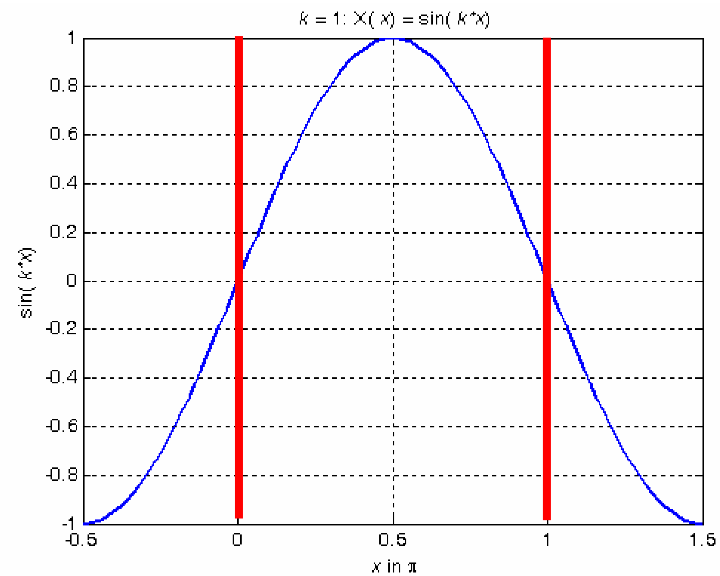


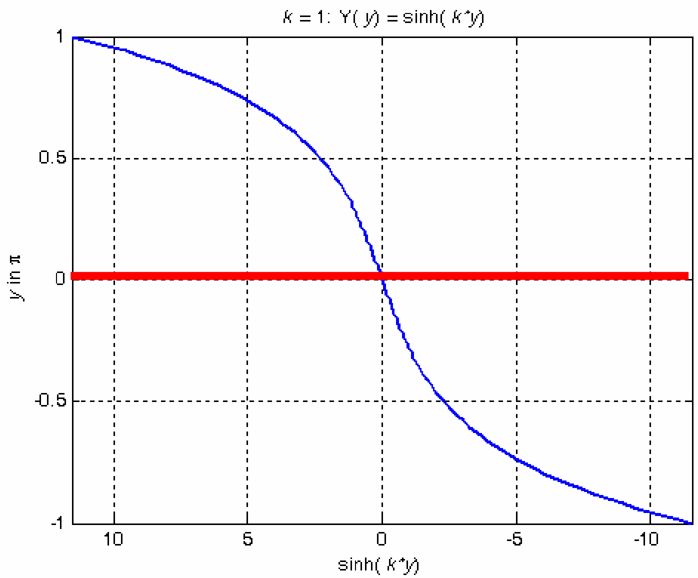



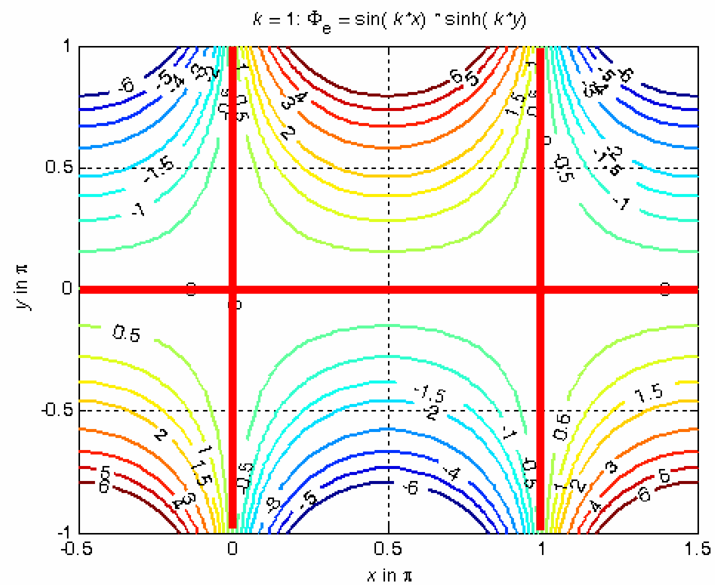
# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_e(x,y) = X(x)Y(y) = \sin(kx)\sinh(ky)$$



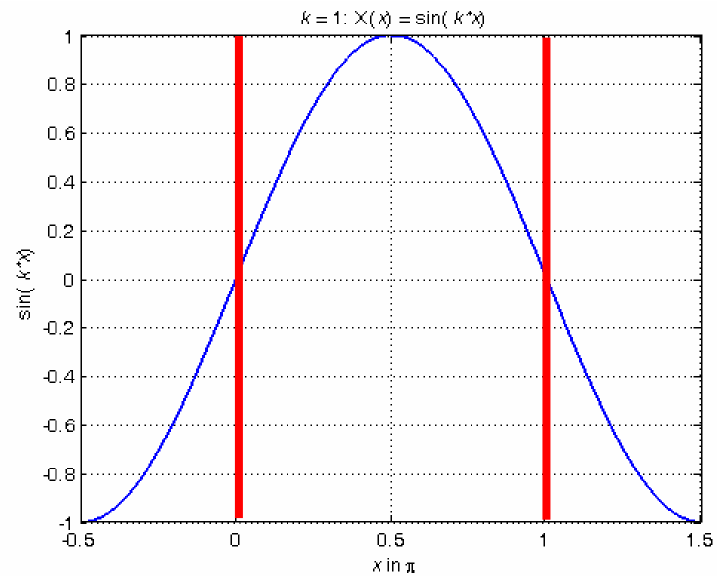

$$\Phi_e(x,y) = 0$$

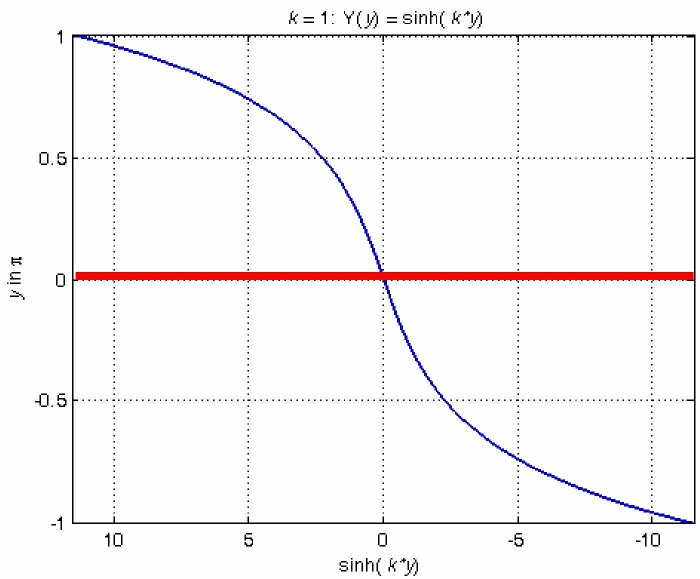


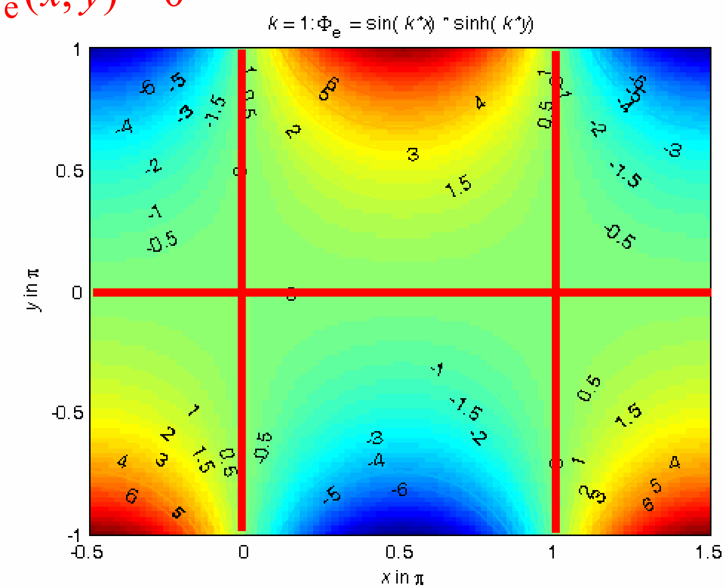
# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_e(x,y) = X(x)Y(y) = \sin(kx)\sinh(ky)$$

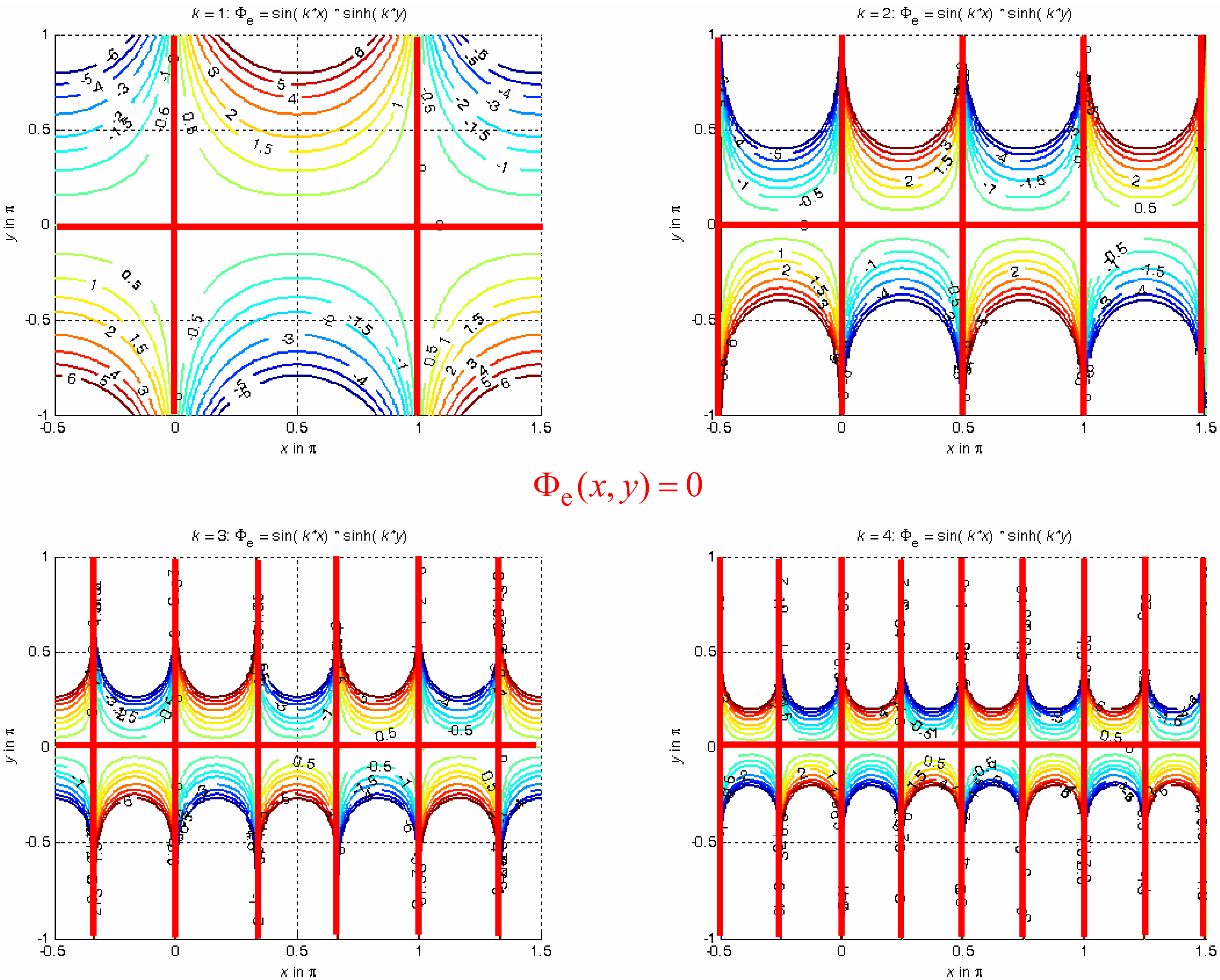



$$\Phi_e(x,y) = 0$$

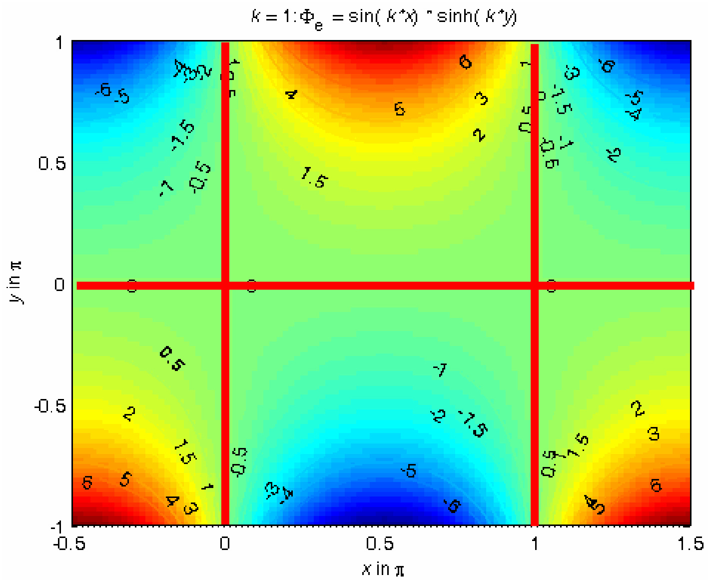


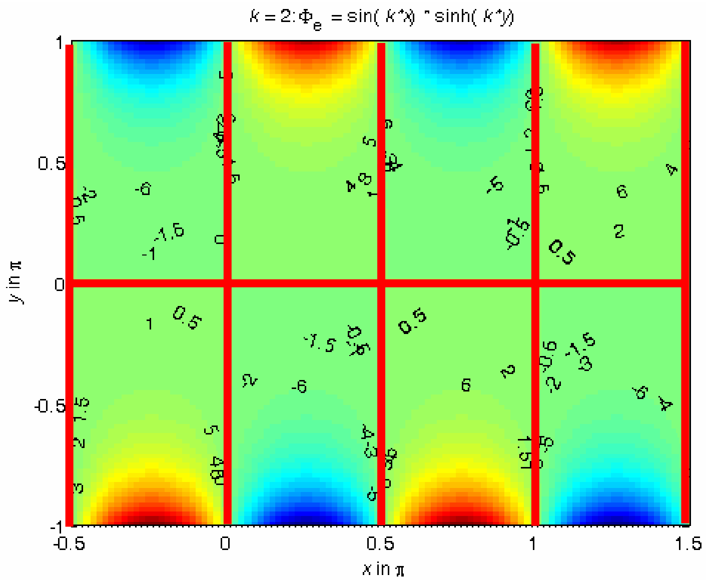
# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



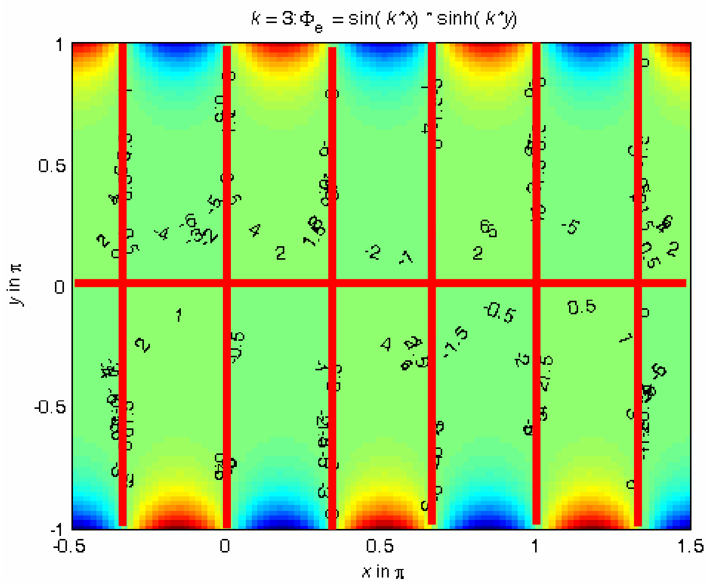

$$\Phi_e(x,y) = 0$$

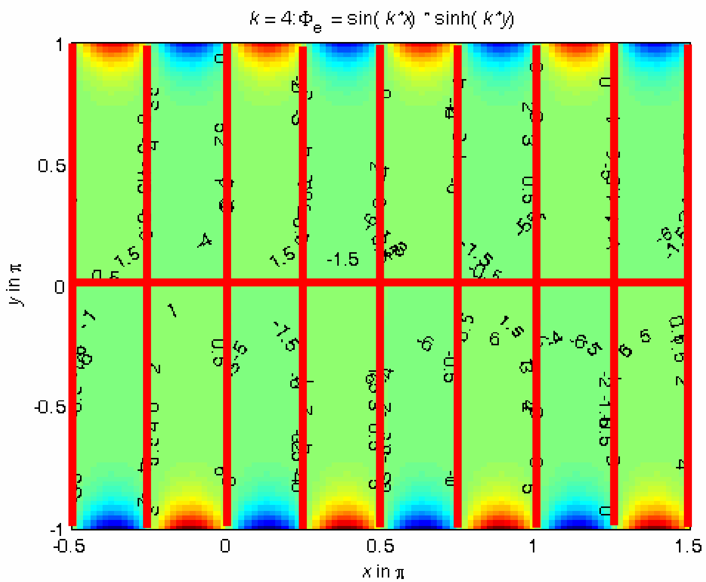
# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)




$$\Phi_{\mathrm{e}}(x, y) = 0$$



# ES Fields – Separation of Variables – Superposition of Modes / ES Felder – Separation der Variablen – Superposition von Moden (...)

Superposition of Modes to Ensure Boundary Conditions /
Superposition von Moden zur Erfüllung von Randbedingungen:

Each solution of the Laplace equation – eigen solution, mode – obtained by the separation of variables displays lines (surfaces) of vanishing potential. At these lines (surfaces) we could place a Dirichlet boundary with $\Phi_e(x,y) = 0$ V /
Jede Lösung der Laplace-Gleichung – Eigenlösung, Mode –, die man über die Methode der Separation bestimmt, weist Linien (Flächen) mit dem Null-Potential auf.
Auf diesen Linien (Flächen) kann man eine Dirichlet-Rand mit $\Phi_e(x,y) = 0$ V platzieren.

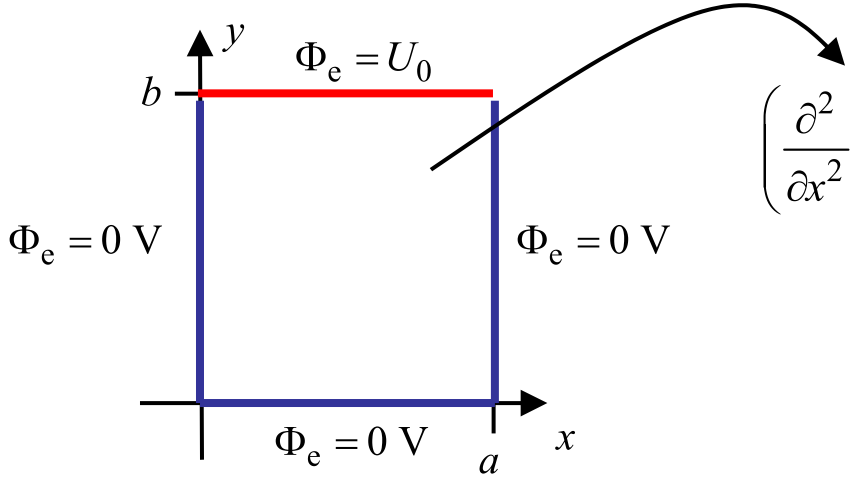
For Example, Consider the Solution / Betrachte beispielsweise die Lösung

$$\Phi_\mathrm{e}(x,y) = \Phi_{\mathrm{e}0} \sin(kx) \sinh(ky)$$

This Functions is Zero for / Diese Funktion ist gleich null für

$$\Phi_\mathrm{e}(x,y) = 0 \quad \begin{cases} y = 0 & & \text{because / weil} & \sinh(ky) = \sinh(0) = 0 \\ x = \dfrac{n\pi}{k} & n = -\infty, \ldots -1, 0, 1, \ldots \infty & \text{because / weil} & \sin(kx) = \sin(n\pi) = 0 \end{cases}$$

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)




$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y)=0 \quad \begin{cases} 0<x<a \\ 0<y<b \end{cases}$$

We Set / Wir setzen:

$$\Phi_e(x,y)=\Phi_{e0}\sin(kx)\sinh(ky) \qquad k=\frac{n\pi}{a} \qquad \Phi_e(x,y)=\Phi_{e0}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

then it follows / dann folgt

$$\Phi_e(x,y)=0 \quad \begin{matrix} x=0 \\ x=a \\ y=0 \end{matrix}$$

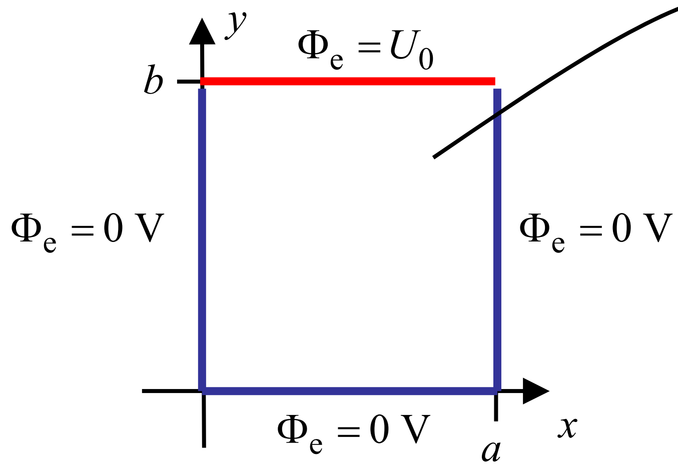
$$y=b:\Phi_e=U_0$$

$$\Phi_e(x,y=b)=\Phi_{e0}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)\neq U_0$$

# ES Fields – Separation of Variables – Superposition of Modes / ES Felder – Separation der Variablen – Superposition von Moden (...)

$$\Rightarrow \quad \Phi_{\mathrm{e}}(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right)$$

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)




$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_\mathrm{e}(x,y)=0 \quad \begin{cases} 0<x<a \\ 0<y<b \end{cases}$$

$$\Rightarrow \quad \Phi_\mathrm{e}(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$A_n$ ?!

Adjust die Coefficients $A_n$, $n = 1,2,\ldots,\infty$ in Order to Ensure the Inhomogeneous Dirichlet Boundary Condition $\Phi_\mathrm{e}(x,b) = U_0$ at the Top Boundary. / Die Koeffizienten $A_n$, $n = 1,2,\ldots,\infty$ sind so zu bestimmen, dass die inhomogene Dirichlet-Randbedingung $\Phi_\mathrm{e}(x,b) = U_0$ am oberen Rand erfüllt wird.

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Adjust die Coefficients $A_n$, $n = 1,2,\ldots,\infty$ in Order to Ensure the Inhomogeneous Dirichlet Boundary Condition $\Phi_\mathrm{e}(x,b) = U_0$ at the Top Boundary. / Die Koeffizienten $A_n$, $n = 1,2,\ldots,\infty$ sind so zu bestimmen, dass die inhomogene Dirichlet-Randbedingung $\Phi_\mathrm{e}(x,b) = U_0$ am oberen Rand erfüllt wird.

$$\Phi_\mathrm{e}(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

1. Determine / Bestimme $\left.\Phi_\mathrm{e}(x,y)\right|_{y=b}$

$$\Phi_\mathrm{e}(x,y=b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

2. Multiply Both Sides with / Multipliziere beide Seiten mit $\sin\left(\frac{m\pi}{a}x\right)$

$$\int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \int_{x=0}^{a}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right)\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x$$

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Orthogonal "Eigen"functions /
Orthogonale „Eigen"funktionen

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \begin{cases} \frac{a}{2} & n=m \\ 0 & n\neq m \end{cases} = \frac{a}{2}\delta_{nm}$$

Kronecker Delta /
Kronecker-Delta

$$\delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n\neq m \end{cases}$$

$$\int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \underbrace{\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x}_{=\frac{a}{2}\delta_{nm}}$$

3. It Follows for $m=n$ /
Es folgt für $m=n$

$$\int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \frac{a}{2}\sum_{n=m} A_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{a}{2}A_n \sinh\left(\frac{n\pi}{a}b\right)$$

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)} \int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x$$

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_{x=0}^{a} \Phi_\mathrm{e}(x,b) \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x$$

$$\Phi_\mathrm{e}(x,b) = U_0$$

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_{x=0}^{a} U_0 \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x$$

$$= \frac{2U_0}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_{x=0}^{a} \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x$$

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x = -\frac{a}{n\pi} \cos\left(\frac{n\pi}{a} x\right)\Bigg|_{x=0}^{a}$$

$$= -\frac{a}{n\pi}\left[\cos\left(\frac{n\pi}{a} a\right) - \underbrace{\cos(0)}_{=1}\right]$$

$$= \frac{a}{n\pi}\left[1 - \cos(n\pi)\right]$$

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$A_n = \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \underbrace{\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x}$$

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \frac{a}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \frac{a}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{2U_0}{n\pi\sinh\left(\frac{n\pi}{a}b\right)} \underbrace{\left[1-\cos(n\pi)\right]}$$

$$\cos(n\pi) = \begin{cases} -1 & n = 1,3,5,\ldots \\ 1 & n = 2,4,6 \end{cases}$$

$$1-\cos(n\pi) = \begin{cases} 2 & n = 1,3,5,\ldots \\ 0 & n = 2,4,6 \end{cases}$$

$$\Rightarrow \quad A_n = \begin{cases} \dfrac{4U_0}{n\pi} \dfrac{1}{\sinh\left(\frac{n\pi}{a}b\right)} & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots \end{cases}$$

# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

## Solution / Lösung

**Infinite Series / Unendliche Reihe**

$$\Phi_{\mathrm{e}}(x,y)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

## Coefficients / Koeffizienten

with / mit

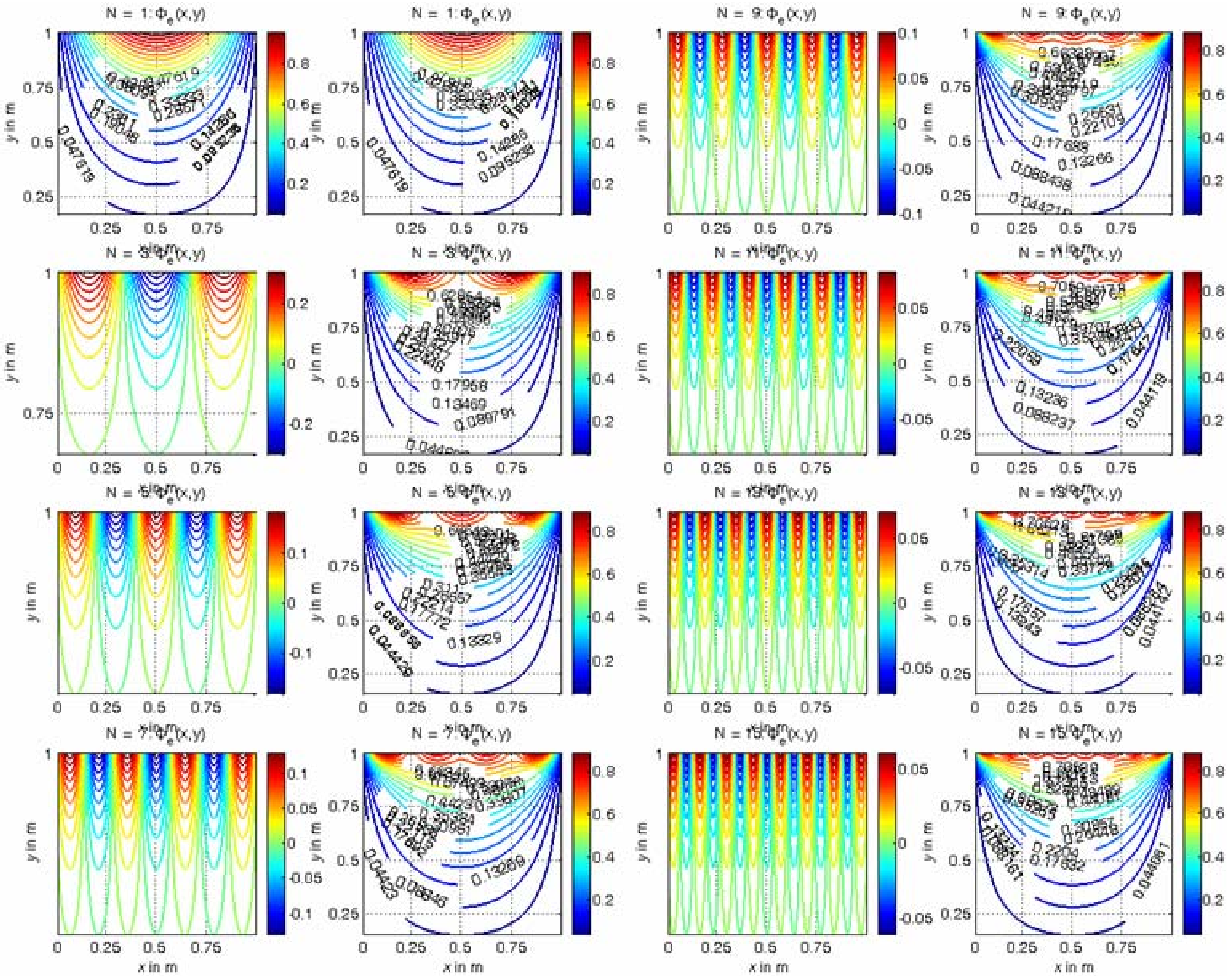
$$A_n=\begin{cases}\dfrac{4U_0}{n\pi}\dfrac{1}{\sinh\left(\dfrac{n\pi}{a}b\right)} & n=1,3,5,\ldots\\[2ex] 0 & n=2,4,6,\ldots\end{cases}$$

## Complete Solution / Komplette Lösung

$$\Rightarrow\quad \Phi_{\mathrm{e}}(x,y)=\frac{4U_0}{\pi}\sum_{\substack{n=1\\ \text{odd / ungerade}}}^{\infty}\frac{1}{n}\frac{\sinh\left(\dfrac{n\pi}{a}y\right)}{\sinh\left(\dfrac{n\pi}{a}b\right)}\sin\left(\frac{n\pi}{a}x\right)$$

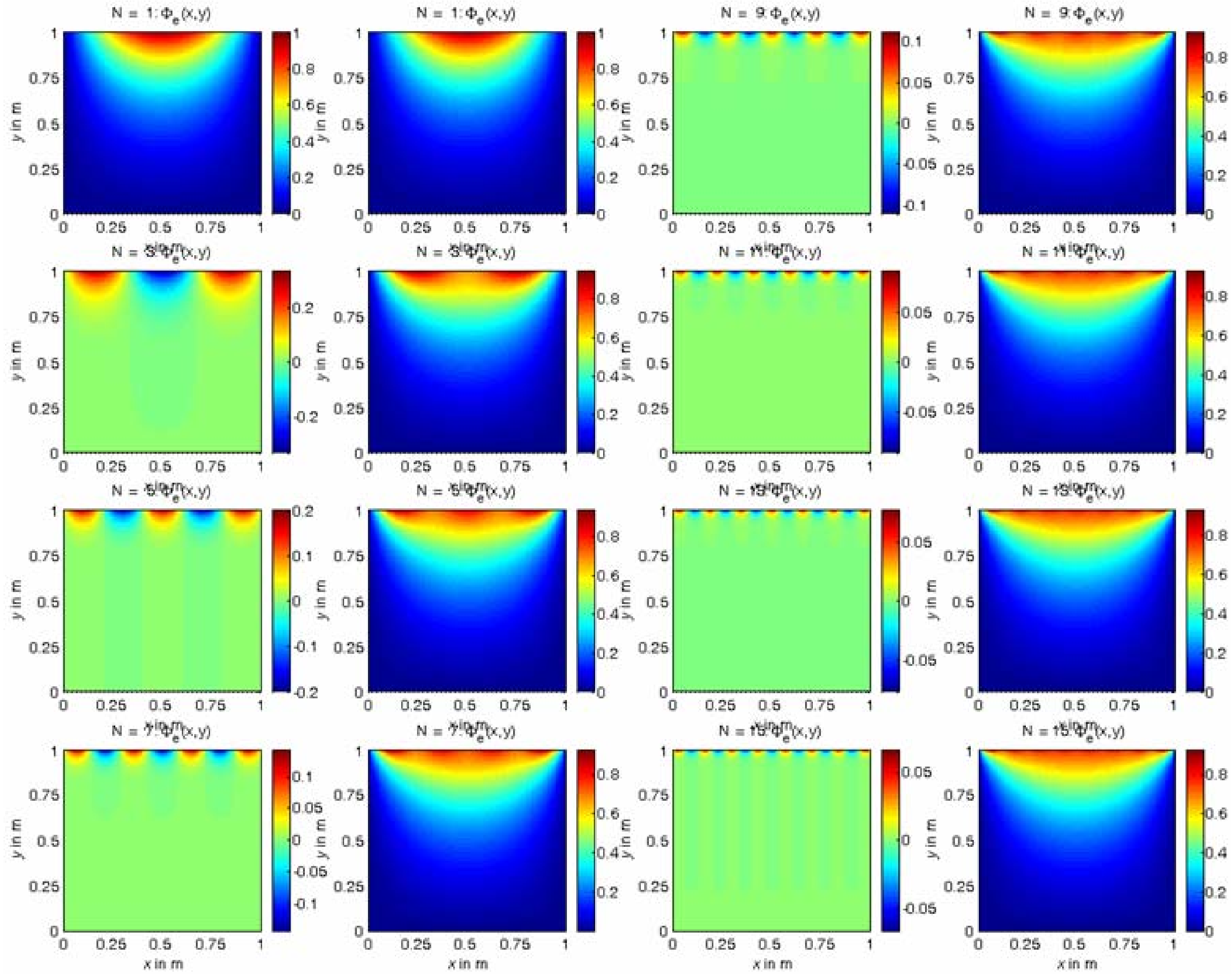
# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$n$th Mode / $n$-ter Mode | $\sum$ Modes / Moden | $n$th Mode / $n$-ter Mode | $\sum$ Modes / Moden



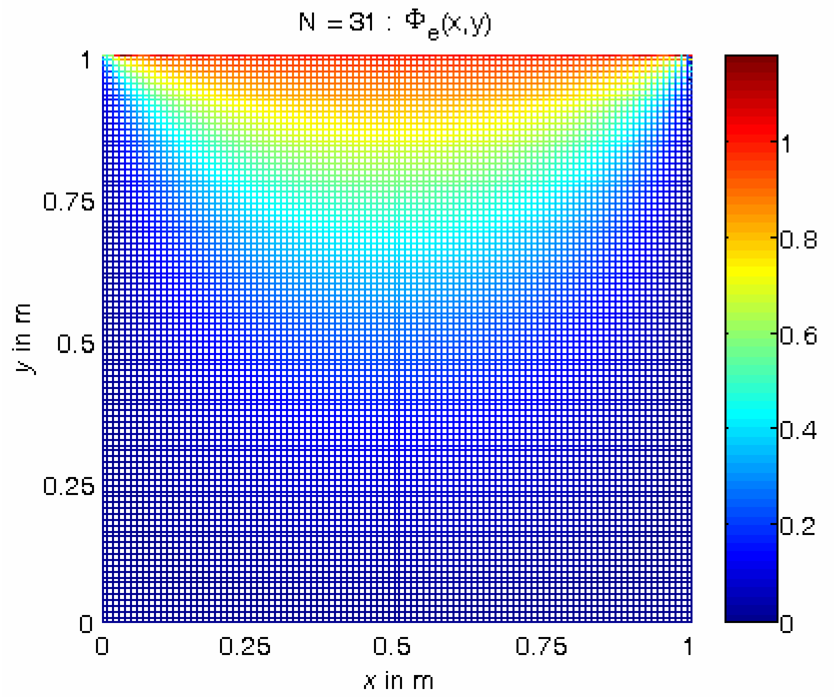
# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$n$th Mode / $n$-ter Mode | $\sum$ Modes / Moden | $n$th Mode / $n$-ter Mode | $\sum$ Modes / Moden

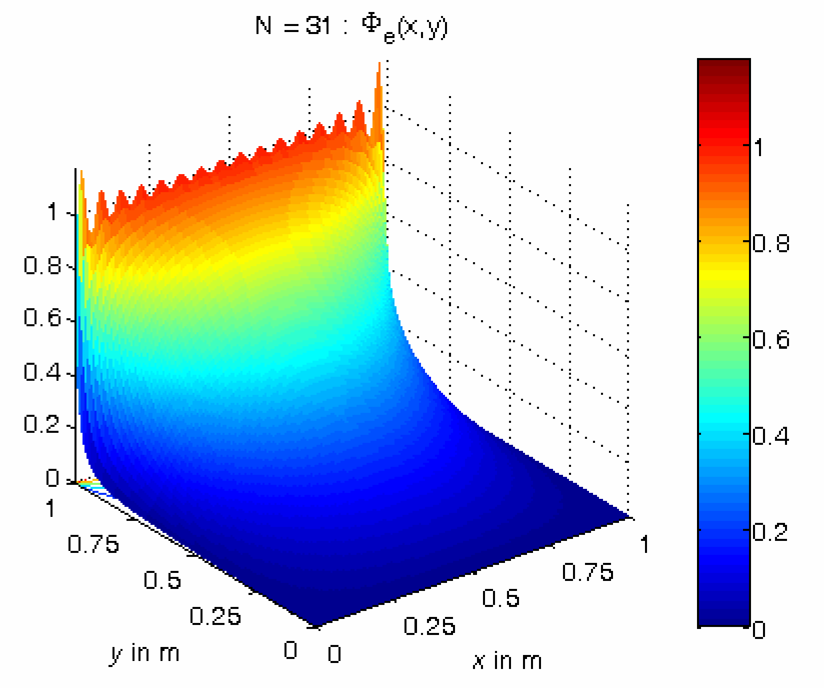


# ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$\sum_{n=1}^{31}$ Modes / Moden




$\sum_{n=1}^{31}$ Modes / Moden



# End of Lecture 7 /
# Ende der 7. Vorlesung