Elektromagnetische Feldtheorie I (EFT I) / Electromagnetic Field Theory I (EFT I)

8th Lecture / 8. Vorlesung

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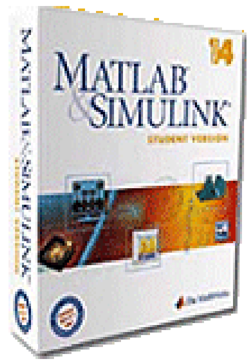
Anmerkung zu MATLAB, SciLab, MuPAD Pro 3



Weblink: http://www.mathworks.de

MathWorks Studentenversionen – MATLAB Studenten Version
Ab sofort verfügbar! Die MATLAB Studenten-Version Release 14
Die MATLAB Studenten-Version ist eine voll funktionsfähige, professionelle Version von MATLAB (Simulink ist in der Studentenversion auf 1000 Blöcke beschränkt), die für Studenten an Hochschulen und Bildungsinstitutionen gedacht ist, an denen ein akademischer Grad erworben werden kann.

MATLAB – Studentenversion 14 — 87,00 €
87,00 EUR (inkl. MwSt.) Studierende
inkl. ServicePack 1, Englisch, WinNT / Win2000 / WinXP / Mac OS / Linux ServicePack 2 nicht verfügbar!
Nur für Studenten an Universitäten und Fachhochschulen
Artikel-Nr.: 185754-005

Weblink: https://www.academic-center.de/cgi-bin/product/P13511

Scilab

A Free Scientific Software Package — 0,00 €

Weblink: http://scilabsoft.inria.fr/

Weblink: http://schule.mupad.de/

MuPAD Light 3.1.1 for Linux/nicht für Windows! — 0,00 €

MuPAD Light 2.5.3 for Linux/Windows — 0,00 €

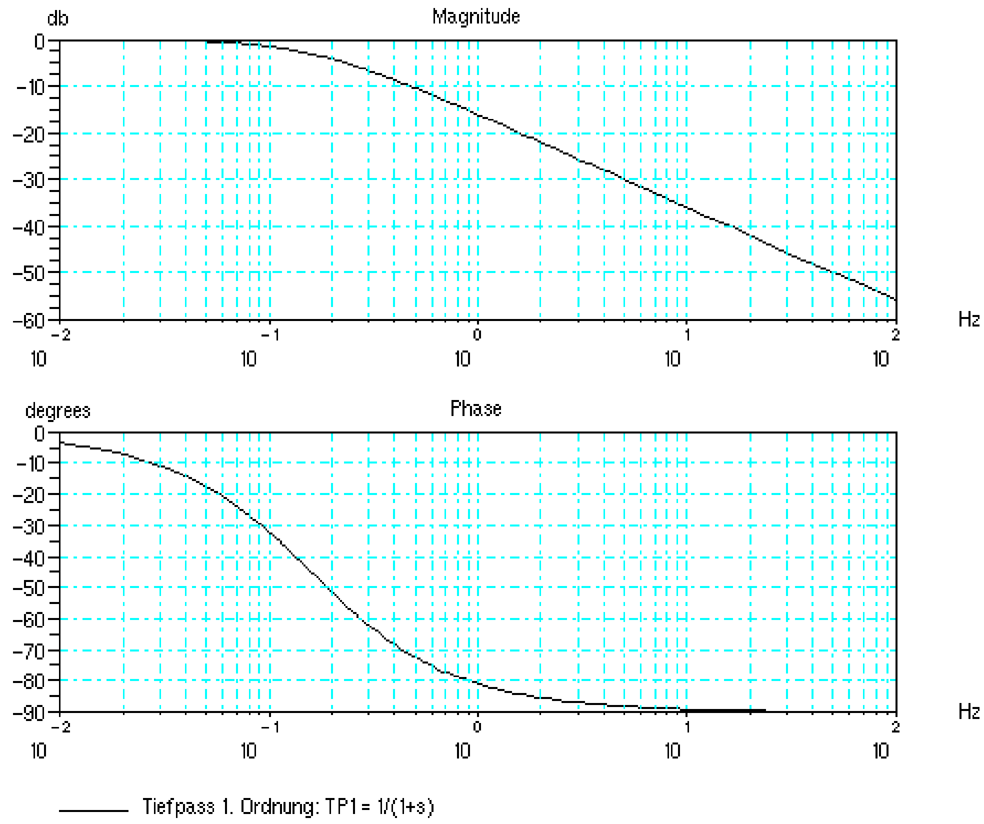
Lizenz: MuPAD Pro 3.1.1, Privat — 90,00 €
Zeitlich unbefristete MuPAD Lizenz für Lehrer bzw. Privatpersonen. Betriebssystem: Windows und Apple MacOS X (Sonderaktion!). Nur Lizenzschlüssel.

Anmerkung zu SciLab

A Free Scientific Software Package

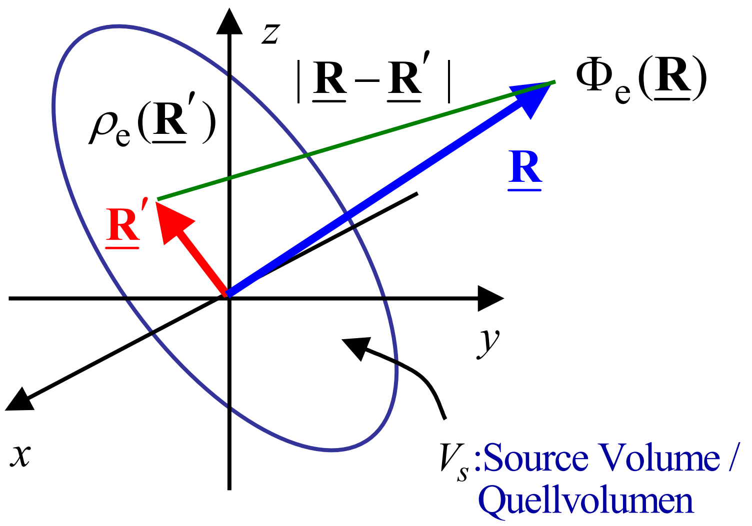
Weblink: http://scilabsoft.inria.fr/

```
// SciLab-Programm:
// Bode-Diagramm für Tiefpass 1. Ordnung
clf;
s=poly(0,'s')
h=syslin('c',1/(s+1))
title='Tiefpass 1. Ordnung: TP1 = 1/(1+s)';
bode(h,0.01,100,title);
```



SciLab-Skript-Sprache ist der MATLAB-Skript-Sprache und der C-, C++-Programmiersprache sehr ähnlich!

ES Fields – Green's Function / ES Felder – Greensche Funktion



$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V_s}\frac{\rho_\mathrm{e}(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}d^3\underline{\mathbf{R}}'$$

$$= \frac{1}{\varepsilon_0}\iiint_{V_s}\underbrace{\frac{1}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}}_{=G_\mathrm{e}^\mathrm{ES}(\underline{\mathbf{R}}-\underline{\mathbf{R}}')}\rho_\mathrm{e}(\underline{\mathbf{R}}')d^3\underline{\mathbf{R}}'$$

Electrostatic Green's Function / Elektrostatische Greensche Funktion

$$G_\mathrm{e}^\mathrm{ES}(\underline{\mathbf{R}}-\underline{\mathbf{R}}') = \frac{1}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} \qquad \text{for / für } \underline{\mathbf{R}} \neq \underline{\mathbf{R}}'$$

with $\Delta G_\mathrm{e}^\mathrm{ES}(\underline{\mathbf{R}}-\underline{\mathbf{R}}') = -\delta\left(\underline{\mathbf{R}}-\underline{\mathbf{R}}'\right)$

Normalized Potential of a Point Charge / Normiertes Potential einer Punktladung

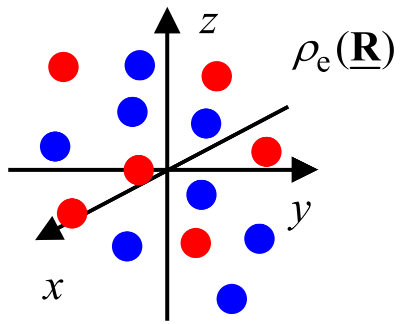
Electrostatic Potential of an Electrostatic Point Charge / Elektrostatisches Potential einer elektrostatischen Punktladung

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} \qquad \text{for / für } \rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+)$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

Arbitrary Point Charge /
Beliebige Punktsladungsverteilungen



$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} \rho_\mathrm{e}(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'$$

Expansion of $\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}$ in a Taylor Series for $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ yields:

Entwicklung von $\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}$ in eine Taylor-Reihe für $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ ergibt:

$$\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|} = \frac{1}{R} + \frac{1}{R^3}\underline{\mathbf{R}}\bullet\underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^5}\underline{\mathbf{R}}\bullet\left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}'\bullet\underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right]\bullet\underline{\mathbf{R}} + \mathcal{HOT}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \underline{\hat{\mathbf{R}}} \bullet \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^3} \underline{\hat{\mathbf{R}}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\underline{\mathbf{I}}} \right] \bullet \underline{\hat{\mathbf{R}}} + \mathcal{HOT} \right\} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3 \underline{\mathbf{R}}'$$

$$\begin{aligned}
\underline{\hat{\mathbf{R}}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\underline{\mathbf{I}}} \right] \bullet \underline{\hat{\mathbf{R}}} &= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\hat{\mathbf{R}}} \bullet \underbrace{\underline{\underline{\mathbf{I}}} \bullet \underline{\hat{\mathbf{R}}}}_{=\underline{\hat{\mathbf{R}}}} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underbrace{\underline{\hat{\mathbf{R}}} \bullet \underline{\hat{\mathbf{R}}}}_{=1} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underbrace{\underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\underline{\mathbf{I}}}} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\underline{\mathbf{I}}} \\
&= \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left(3\underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right)
\end{aligned}$$

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \underline{\mathbf{R}}' \bullet \underline{\hat{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3} \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left(3\underline{\hat{\mathbf{R}}}\underline{\hat{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right) + \mathcal{HOT} \right\} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3 \underline{\mathbf{R}}'$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\Bigg\{\frac{1}{R}\underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty}\rho_\mathrm{e}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=Q_\mathrm{e}}$$

$$+\frac{1}{R^2}\underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty}\rho_\mathrm{e}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{p}}_\mathrm{e}}\bullet\underline{\hat{\mathbf{R}}}$$

$$+\frac{1}{2}\frac{1}{R^3}\left[\underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty}\rho_\mathrm{e}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\underline{\mathbf{q}}}_\mathrm{e}}:\left(3\underline{\hat{\mathbf{R}}}\,\underline{\hat{\mathbf{R}}}-\underline{\underline{\mathbf{I}}}\right)\right]\Bigg\}+\mathcal{HOT}$$

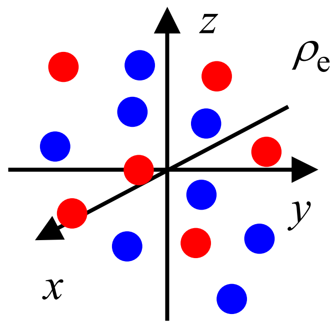
$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{R}Q_\mathrm{e}+\frac{1}{R^2}\underline{\mathbf{p}}_\mathrm{e}\bullet\underline{\hat{\mathbf{R}}}+\frac{1}{2}\frac{1}{R^3}\underline{\underline{\mathbf{q}}}_\mathrm{e}:\left[3\underline{\hat{\mathbf{R}}}\,\underline{\hat{\mathbf{R}}}-\underline{\underline{\mathbf{I}}}\right]+\mathcal{HOT}\right\}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono–, Di– und Quadrupol ...

Arbitrary Point Charge / Beliebige Punktsladungsverteilungen



$\rho_e(\underline{\mathbf{R}})$

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_e(\underline{\mathbf{R}}') \mathrm{d}^3\underline{\mathbf{R}}'$$

Expansion of $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in a Taylor Series for $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ yields: / Entwicklung von $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in eine Taylor-Reihe für $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ ergibt:

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{1}{R} Q_e + \frac{1}{R^2} \underline{\mathbf{p}}_e \bullet \hat{\underline{\mathbf{R}}} + \frac{1}{2} \frac{1}{R^3} \underline{\underline{\mathbf{q}}}_e : \left[3\hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right] + \mathcal{HOT} \right\}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

Monopole Moment / Monopolmoment

$$Q_e = \iiint\limits_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \mathrm{d}^3\underline{\mathbf{R}}'$$

Dipole Moment / Dipolmoment

$$\underline{\mathbf{p}}_e = \iiint\limits_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \underline{\mathbf{R}}' \mathrm{d}^3\underline{\mathbf{R}}'$$

Quadrupole Moment / Quadrupolmoment

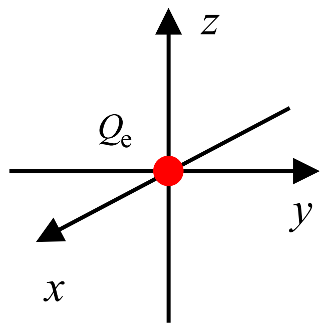
$$\underline{\underline{\mathbf{q}}}_e = \iiint\limits_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \underline{\mathbf{R}}' \underline{\mathbf{R}}' \mathrm{d}^3\underline{\mathbf{R}}'$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono–, Di– und Quadrupol

Monopole Moment / Monopolmoment

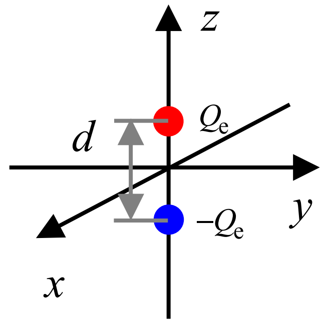
One Point Charge / Eine Punktladung



$$Q_e \neq 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

Dipole Moment / Dipolmoment

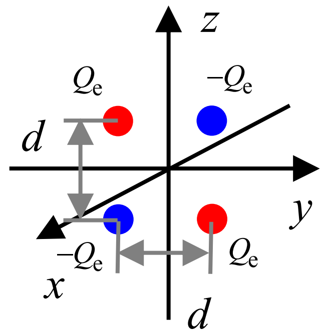
Two Point Charges / Zwei Punktladungen



$$Q_e = 0, \quad \underline{\mathbf{p}}_e \neq \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

Quadrupole Moment/ Quadrupolmoment

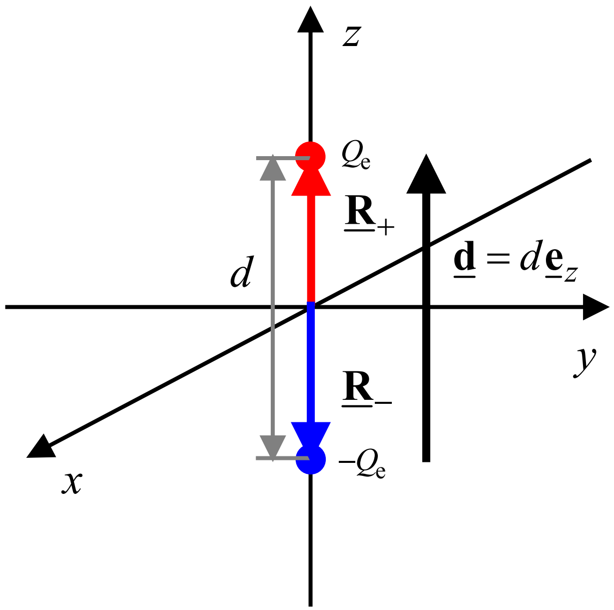
Four Point Charges / Vier Punktladungen



$$Q_e = 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e \neq \underline{\underline{\mathbf{0}}}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Electrostatic Dipole / Elektrostatischer Dipol



Electrostatic Volume Charge Density / Elektrostatische Raumladungsdichte

$$\rho_e(\underline{\mathbf{R}}) = Q_e\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$
$$= Q_e\delta(\underline{\mathbf{R}} - \frac{d}{2}\underline{\mathbf{e}}_z) - Q_e\delta(\underline{\mathbf{R}} + \frac{d}{2}\underline{\mathbf{e}}_z)$$

with $\underline{\mathbf{R}}_+ = \frac{d}{2}\underline{\mathbf{e}}_z$, $\underline{\mathbf{R}}_- = -\frac{d}{2}\underline{\mathbf{e}}_z$

Electrostatic Potential / Elektrostatisches Potential

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$

Electrostatic Field Strength / Elektrostatische Feldstärke

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_+}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_-}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|^3}\right)$$

Electrostatic Dipole Moment / Elektrische Dipolmoment

$$\underline{\mathbf{p}}_e = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}' = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e\delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\right]\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'$$

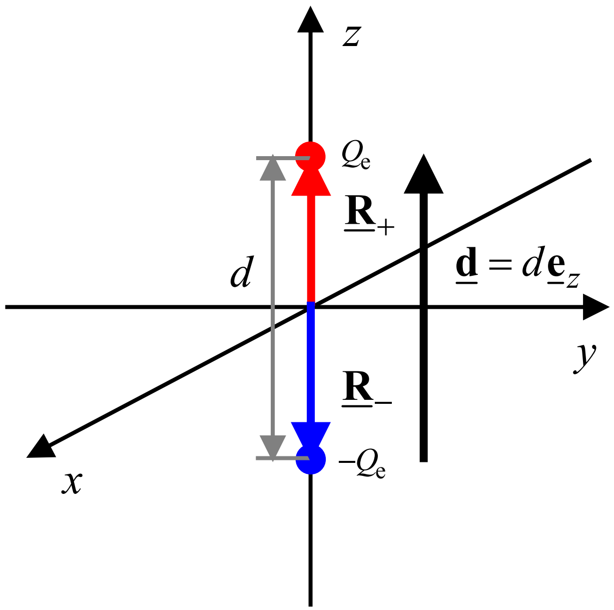
$$= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-} = Q_e\underline{\mathbf{R}}_+ - Q_e\underline{\mathbf{R}}_- = Q_e(\underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_-) = Q_e\underline{\mathbf{d}}$$

Distance Vector / Abstandsvektor

$$\underline{\mathbf{d}} = \underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_- = \frac{d}{2}\underline{\mathbf{e}}_z + \frac{d}{2}\underline{\mathbf{e}}_z = d\underline{\mathbf{e}}_z$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Electrostatic Dipole / Elektrostatischer Dipol



Electrostatic Dipole Moment / Elektrostatisches Dipolmoment

$$\underline{\mathbf{p}}_e = Q_e \underline{\mathbf{d}} = p_e \hat{\underline{\mathbf{p}}}_e$$

with / mit

$$p_e = Q_e |\underline{\mathbf{d}}| = Q_e |\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|$$

$$\hat{\underline{\mathbf{p}}}_e = \hat{\underline{\mathbf{d}}} = \widehat{|\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|}$$

Electrostatic Quadrupole Moment / Elektrostatisches Quadrupolmoment

$$\begin{aligned}
\underline{\underline{\mathbf{q}}}_e &= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}' \\
&= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-) \right] \underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}' \\
&= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-\underline{\mathbf{R}}_-} \\
&= Q_e \underline{\mathbf{R}}_+\underline{\mathbf{R}}_+ - Q_e \underline{\mathbf{R}}_-\underline{\mathbf{R}}_- \\
&= Q_e \left[\left(\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(\frac{d}{2}\underline{\mathbf{e}}_z\right) - \left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(-\frac{d}{2}\underline{\mathbf{e}}_z\right) \right] \\
&= Q_e \underbrace{\left[d\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z - d\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z \right]}_{=\underline{\underline{\mathbf{0}}}} \\
&= \underline{\underline{\mathbf{0}}}
\end{aligned}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ... (2)

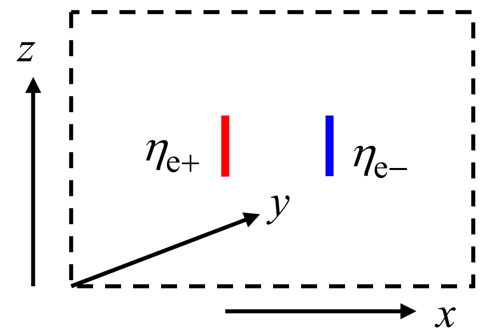
Application: Numerical Solution of Unbounded Static Problems / Anwendung: Numerische Lösung von unbegrenzten statischen Problemen

$$\Delta\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon_0}$$

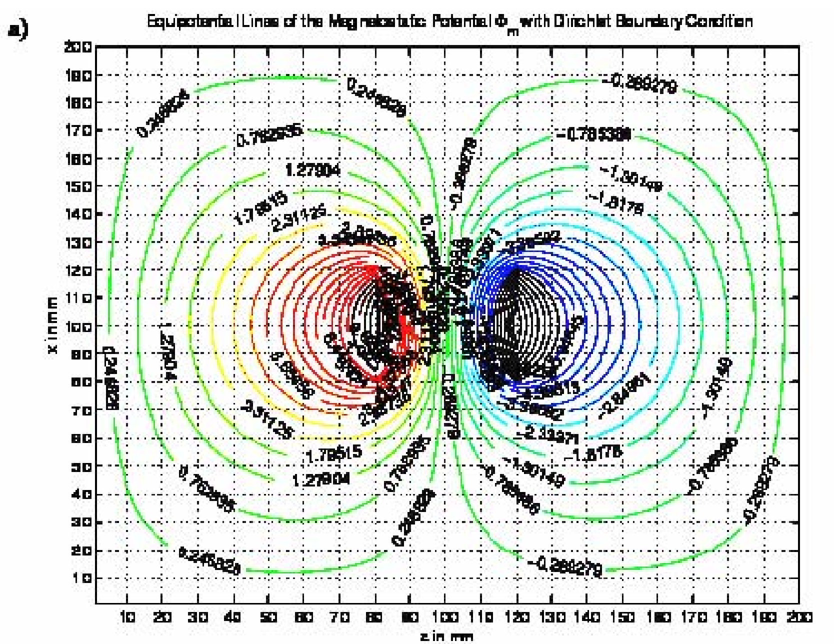
$$\rho_e(\underline{\mathbf{R}}) = \eta_{e+}(y,z)\delta(x-x_+) + \eta_{e-}(y,z)\delta(x-x_-)$$

$$\eta_{e+}(y,z) = \begin{cases} \eta_{e0} & y_- \le y \le y_+ \\ & z_- \le z \le z_+ \\ 0 & \text{else / sonst} \end{cases}$$

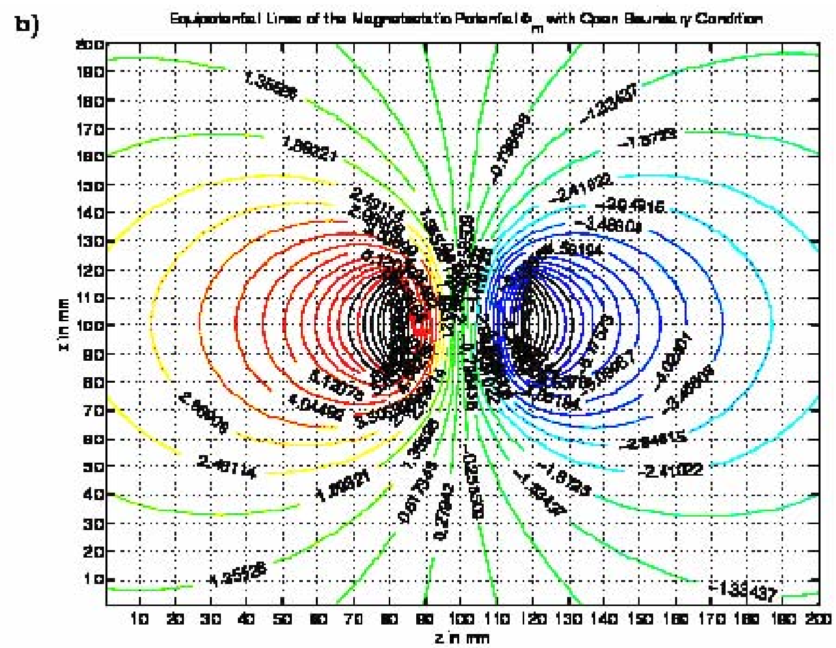
$$= -\eta_{e-}(y,z)$$



With Dirichlet Boundary Condition / Mit Dirichlet Randbedingung



With Open Boundary Condition (OBC) / Mit offener Randbedingung (ORB)



ES Fields – Method of Images / ES–Felder – Spiegelungsmethode

Boundary Value Problem (BVP) – Randwertproblem (RWP)

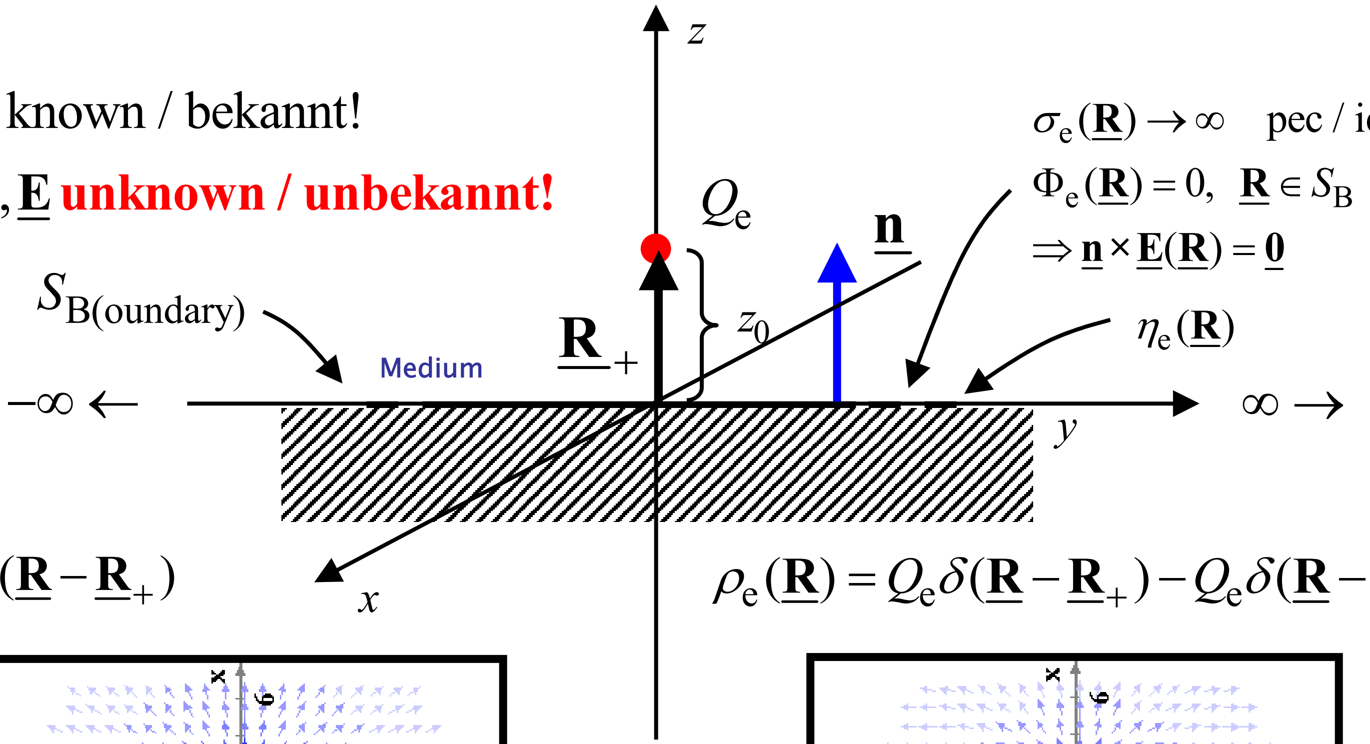
Q_e known / bekannt!

$\Phi_\mathrm{e}, \underline{\mathbf{E}}$ **unknown / unbekannt!**

$\sigma_\mathrm{e}(\underline{\mathbf{R}}) \to \infty$ pec / iel

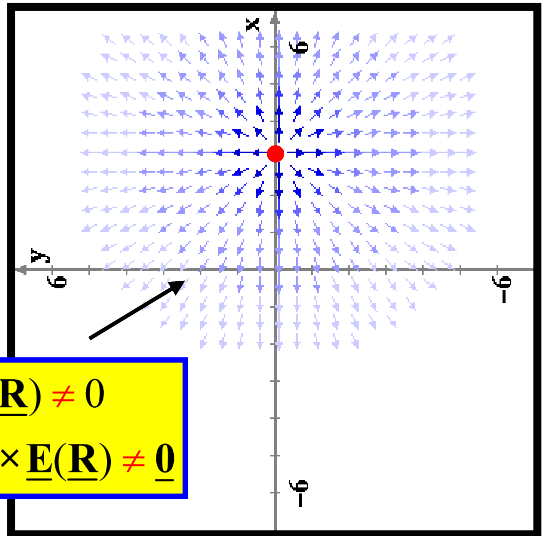
$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0, \quad \underline{\mathbf{R}} \in S_\mathrm{B}$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$



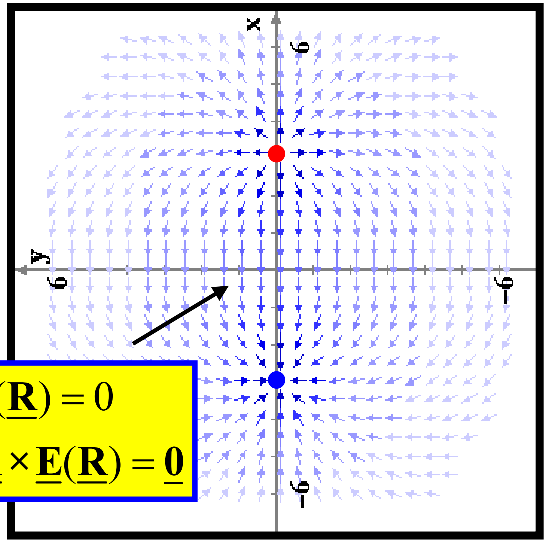
$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$$

$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$



$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) \neq 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) \neq \underline{\mathbf{0}}$

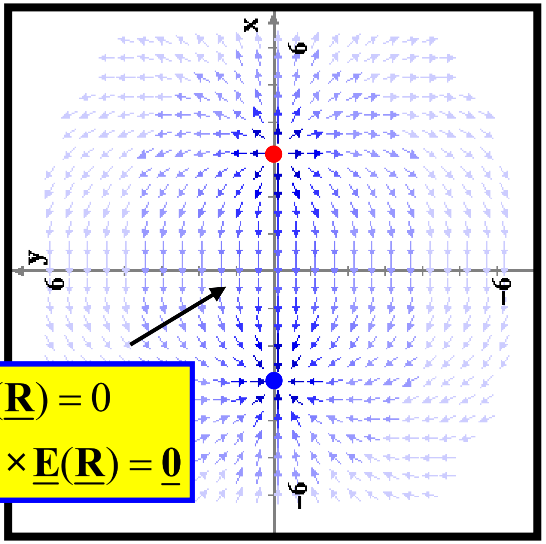


$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

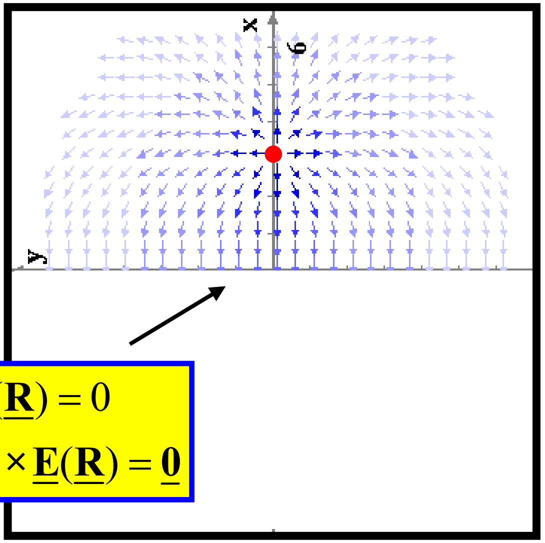
Method of Images / Spiegelungsmethode



$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$



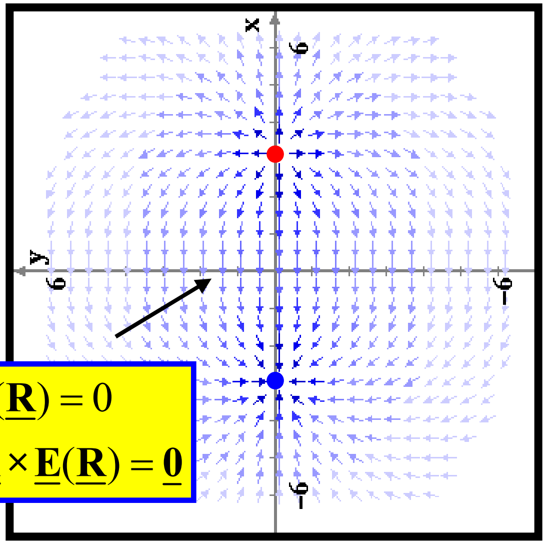
$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

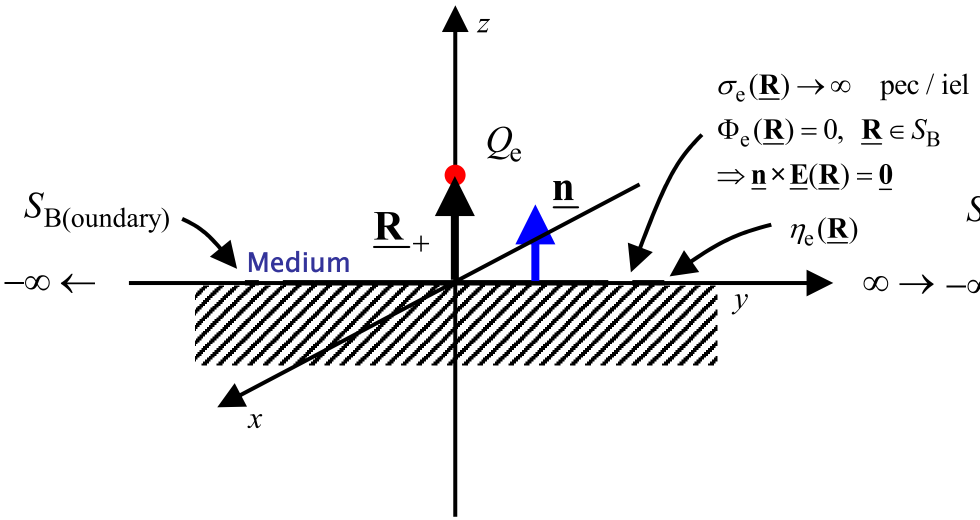
Method of Images / Spiegelungsmethode



$\underline{\mathbf{R}} \in S_B : \Phi_e(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

Problem:



Solution / Lösung:

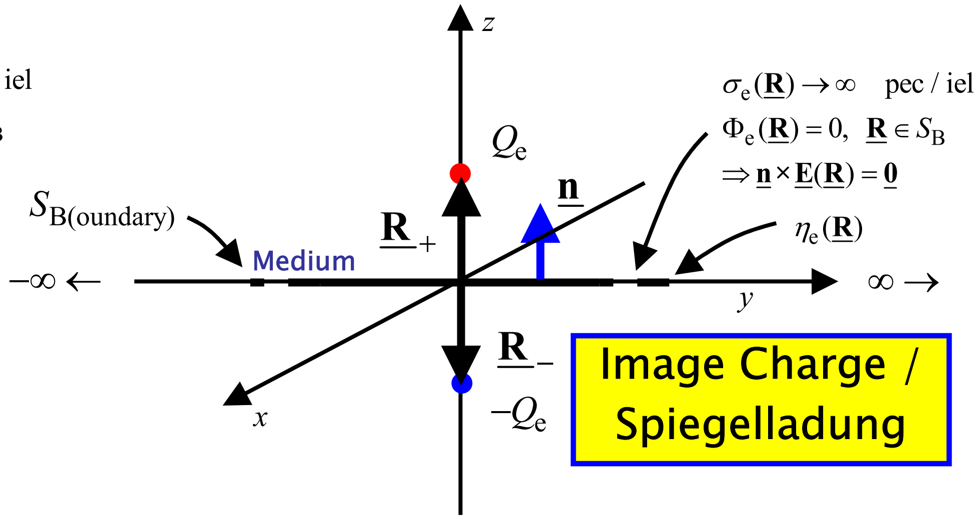
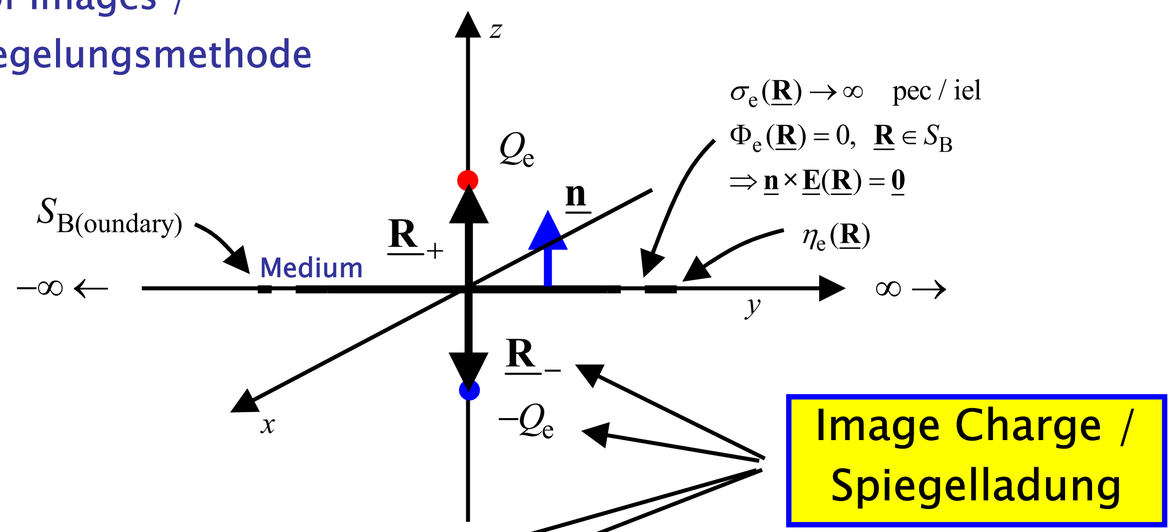


Image Charge / Spiegelladung

ES Fields – Method of Images / ES Felder – Spiegelungsmethode

Solution by Applying the Method of Images /
Lösung durch Anwendung der Spiegelungsmethode



$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with
mit

$$\underline{\mathbf{R}}_+ = z_0 \underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0 \underline{\mathbf{e}}_z$$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0} \left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|} \right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_-)$$

with / mit $\underline{\mathbf{R}}_+ = z_0\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0\underline{\mathbf{e}}_z$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

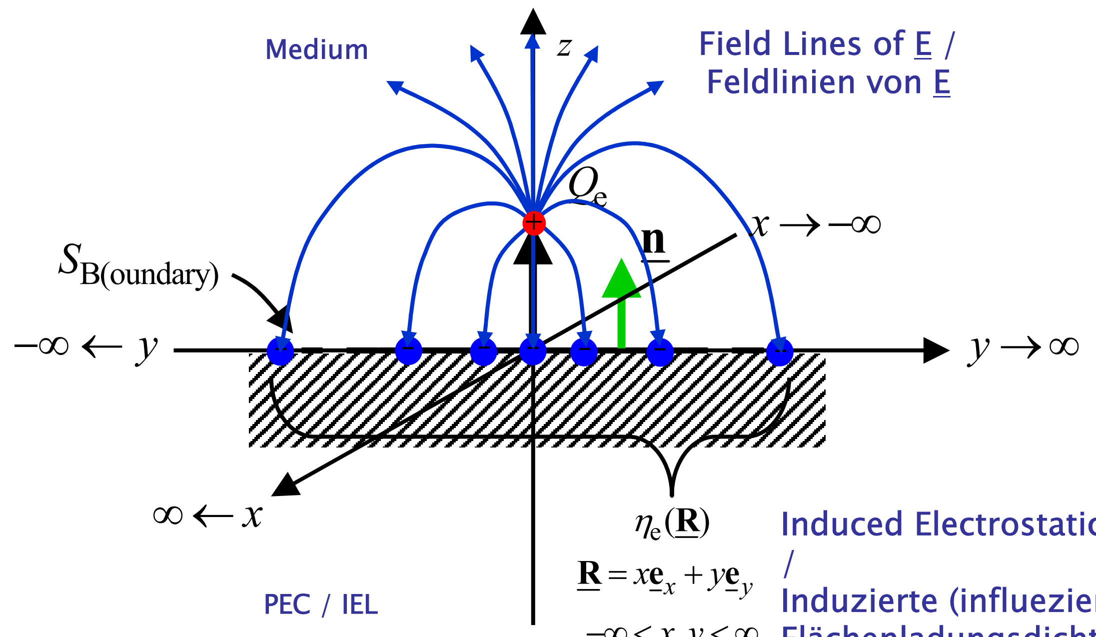
$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi}\left(\dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



Induced Electrostatic Surface Charge Density / Induzierte (influezierte) elektrostatische Flächenladungsdichte (Influenz)

Without the Method of Images we have to Solve the Following Integral Equation for the Unknown Induced Electrostatic Surface Charge / Ohne die Spiegelungsmethode muss man die folgende Integralgleichung für die induzierte (influezierte) elektrostatische Flächenladungsdichte lösen

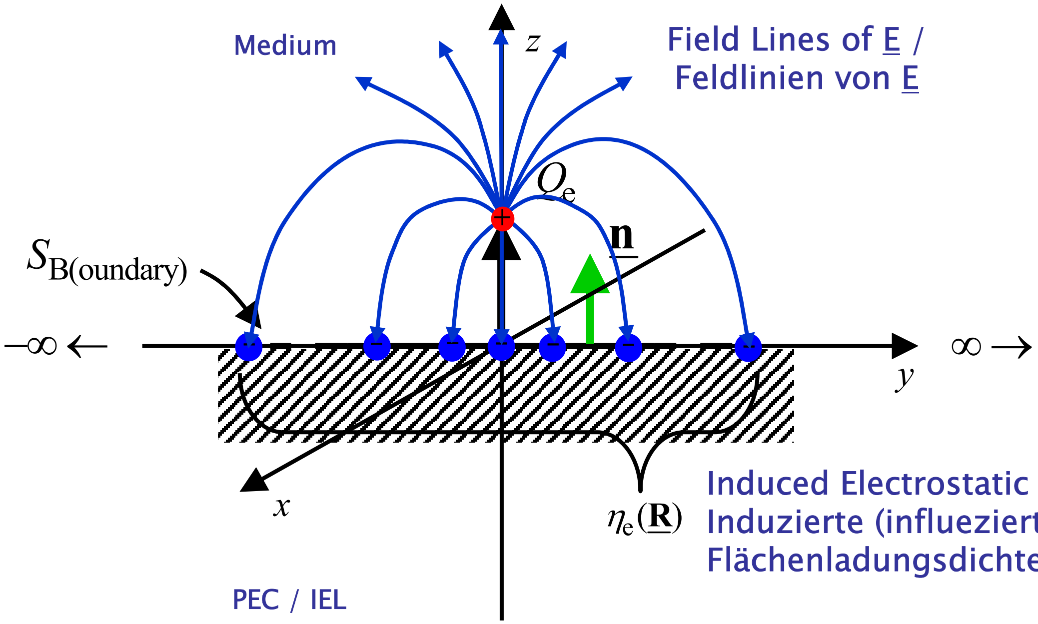
Unknown / Unbekannt

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_e}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} + \iint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{\eta_e(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} d^2\underline{\mathbf{R}}' \Bigg|_{z=0}\right] = 0$$

for / für $\Phi_e(\underline{\mathbf{R}})\big|_{z=0} = 0$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



Induced Electrostatic Surface Charge Density / Induzierte (influezierte) elektrostatische Flächenladungsdichte (Influenz)

If $\underline{D}$ is known from the Method of Images / Falls $\underline{D}$ über die Spiegelungsmethode bekannt ist

$$\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B} = \begin{matrix}\text{known}\\ \text{bekannt}\end{matrix}\ !$$

$\eta_e(\underline{\mathbf{R}})$ is Defined by the Normal Component of $\underline{\mathbf{D}}$ / $\eta_e(\underline{\mathbf{R}})$ ist definiert über die Normalkomponente von $\underline{\mathbf{D}}$

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}}\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B}$$

$$= \underline{\mathbf{n}}\bullet\frac{Q_e}{4\pi}\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{\underline{\mathbf{R}}\in S_B} \quad \begin{matrix}\text{for}\\ \text{für}\end{matrix} \quad z=0$$

$$= \frac{Q_e}{4\pi}\underline{\mathbf{e}}_z\bullet\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{z=0}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$
\begin{aligned}
\eta_{\mathrm{e}}(\underline{\mathbf{R}}) &= \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_{\mathrm{B}}} \\
&= \frac{Q_{\mathrm{e}}}{4\pi}\left[\frac{\underline{\mathbf{e}}_z \bullet \left(\underline{\mathbf{R}}-\underline{\mathbf{R}}_{+}\right)}{\left|\underline{\mathbf{R}}-\underline{\mathbf{R}}_{+}\right|^3} - \frac{\underline{\mathbf{e}}_z \bullet \left(\underline{\mathbf{R}}-\underline{\mathbf{R}}_{-}\right)}{\left|\underline{\mathbf{R}}-\underline{\mathbf{R}}_{-}\right|^3}\right]\Bigg|_{z=0} \\
&= \frac{Q_{\mathrm{e}}}{4\pi}\left[\frac{z-z_0}{\left[x^2+y^2+\left(z-z_0\right)^2\right]^{3/2}} - \frac{z+z_0}{\left[x^2+y^2+\left(z-z_0\right)^2\right]^{3/2}}\right]\Bigg|_{z=0} \\
&= \frac{Q_{\mathrm{e}}}{4\pi}\left[\frac{-z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}} - \frac{z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}}\right] \\
&= -\frac{Q_{\mathrm{e}}}{2\pi}\frac{z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}} \\
&= -\frac{Q_{\mathrm{e}}}{2\pi}\frac{z_0}{\left[r^2+z_0^2\right]^{3/2}}
\end{aligned}
$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\eta_\mathrm{e}(\underline{\mathbf{R}}) = \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}} \in S_\mathrm{B}} = -\frac{Q_\mathrm{e}}{2\pi} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}}$$

⇨ Total Electric Charge at the *xy* Plane at *z*=0 / Gesamtladung auf der *xy* Ebene bei *z*=0

$$Q_\mathrm{e}^\mathrm{tot} = -\frac{Q_\mathrm{e}}{2\pi} \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} r\,\mathrm{d}r\,\mathrm{d}\varphi$$

$$= -\frac{Q_\mathrm{e}}{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} r\,\mathrm{d}r \underbrace{\int_{\varphi=0}^{2\pi} \mathrm{d}\varphi}_{=2\pi}$$

$$= -Q_\mathrm{e} z_0 \int_{r=0}^{\infty} \frac{r}{\left[r^2 + z_0^2\right]^{3/2}} \mathrm{d}r$$

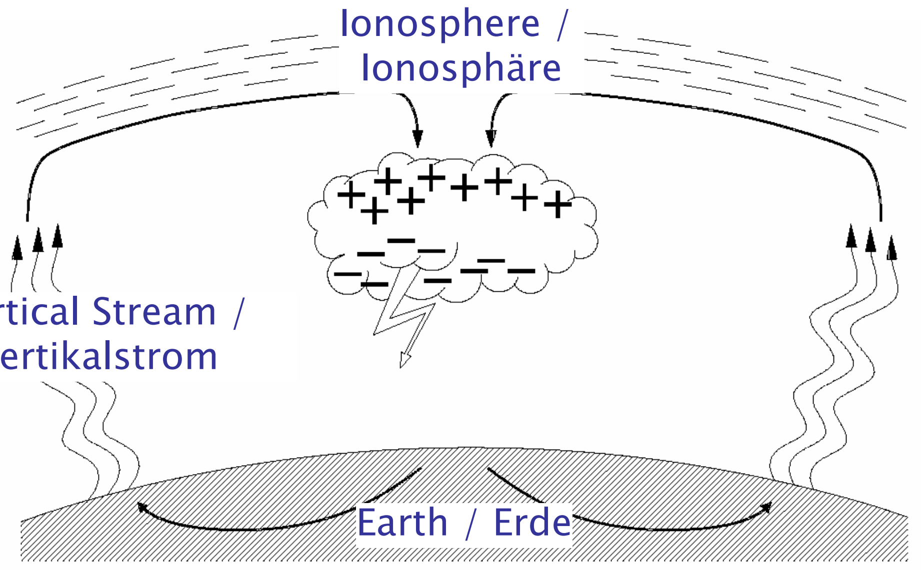
$$\int \frac{x}{\left[x^2 + a^2\right]^{3/2}} dx = -\frac{1}{\sqrt{x^2 + a^2}}$$

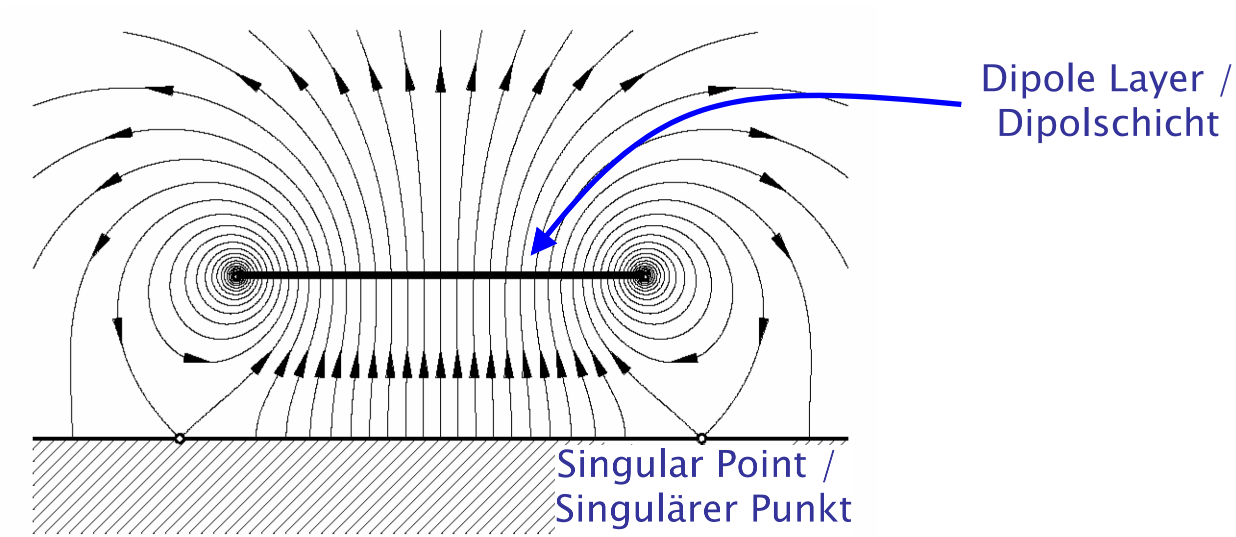
$$= Q_\mathrm{e} z_0 \left[\underbrace{\frac{1}{\sqrt{(r \to \infty)^2 + z_0^2}}}_{\to 0} - \frac{1}{z_0} \right]$$

$$= -Q_\mathrm{e}$$

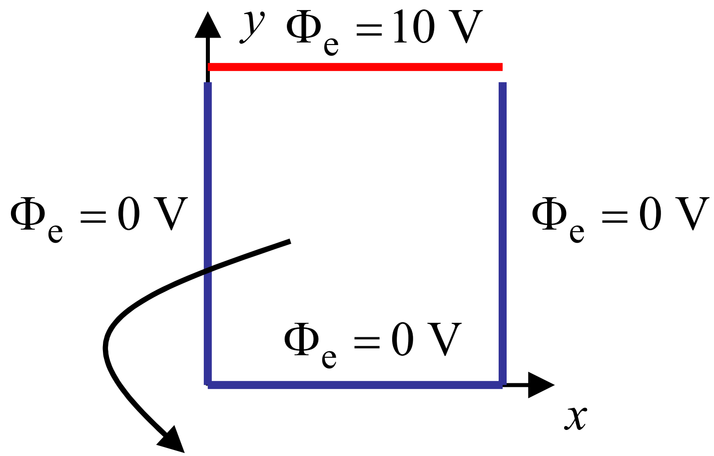
$$\boxed{Q_\mathrm{e}^\mathrm{tot} = -Q_\mathrm{e}}$$

ES Fields – Method of Images – Applications / ES Felder – Spiegelungsmethode – Anwendungen





Electrostatic (ES) Fields – Separation of Variables – Example / Elektrostatische (ES) Felder – Separation der Variablen – Beispiel



$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_\mathrm{e}(x,y)=0$$

➡ **Separation of Variables / Separation der Variablen !**

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\Delta\Phi_\mathrm{e}(x,y,z)=0$$

Elliptic Partial Differential Equation / Elliptische partielle Differentialgleichung

Laplace Equation in Cartesian Coordinates / Laplace-Gleichung in Kartesischen Koordinaten

3-D / 3D

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi_\mathrm{e}(x,y,z)=0$$

Function of Three Variables / Funktion von drei Variablen

2-D / 2D

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_\mathrm{e}(x,y)=0$$

Function of Two Variables / Funktion von zwei Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=0$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=0$$

Solution Strategy: Reduce the Partial Differential Equation (PDE) to an Ordinary Differential Equation (ODE) and Find a Solution of the PDE by Solving the ODE

Lösungsstrategie: Reduziere die partielle Differentialgleichung (PDG) auf eine gewöhnliche (ordinäre) Differentialgleichung (GDG) und finde eine Lösung der PDG durch Lösung der GDG

Ansatz of Separation / Separationsansatz

$$\Phi_e(x,y)=\overbrace{X(x)\;Y(y)}$$

Product of 2 Functions / Produkt aus 2 Funktionen

Function of 2 Variables: x and y / Funktion von 2 Variablen: x und y

Function of x only / Nur eine Funktion von x

Function of y only / Nur eine Funktion von y

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=0$$

Ansatz of Separation / Separationsansatz

$$\Phi_\mathrm{e}(x,y)=X(x)\,Y(y)$$

⇨ Inserted in the Above Laplace Equation Yields / Eingesetzt in die obere Laplace-Gleichung ergibt

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)&=\frac{\partial^2}{\partial x^2}\left[X(x)\,Y(y)\right]+\frac{\partial^2}{\partial y^2}\left[X(x)\,Y(y)\right]\\&=Y(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+X(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)\end{aligned}$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=\underbrace{Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)}$$

$$\frac{1}{X(x)Y(y)}$$

$$\frac{1}{X(x)Y(y)}\overbrace{\left[Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)\right]}=\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=0$$

$$\underbrace{\underbrace{\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)}_{=\text{Function of } x\,/\,\text{Funktion von } x}}_{=-\alpha^2}+\underbrace{\underbrace{\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)}_{=\text{Function of } y\,/\,\text{Funktion von } y}}_{=-\beta^2}=0$$

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=\underbrace{\left(-\alpha^2\right)+\left(-\beta^2\right)}_{=0}$$

Separation Condition / Separationsbedingung $\quad \alpha^2+\beta^2=0$ ⇐

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -\alpha^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = -\beta^2 Y(y)$$

Separation Condition /
Separationsbedingung

$$\alpha^2 + \beta^2 = 0$$

With / Mit $\alpha^2 = -\beta^2 = k^2$

We Obtain Two ODE /
Wir erhalten zwei GDG

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -k^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = k^2 Y(y)$$

Solutions of these Equations are /
Lösungen dieser Gleichungen sind

$X(x) \sim \cos(kx)$ or / oder $\sim \sin(kx)$

$Y(y) \sim \cosh(ky)$ or / oder $\sim \sinh(ky)$

For $k = 0$ these Solutions Degenerate to /
Für $k = 0$ diese Lösungen degenerieren zu

$X(x) \sim \text{const.}$ or / oder $\sim x$

$Y(y) \sim \text{const.}$ or / oder $\sim y$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

We Obtain Two ODE / Wir erhalten zwei GDG

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -k^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = \ \ k^2 Y(y)$$

$$X(x) = \cos(kx)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) = -k\frac{\mathrm{d}}{\mathrm{d}x}\sin(kx) = -k^2 \underbrace{\cos(kx)}_{=X(x)}$$

$$Y(y) = \ \cosh(ky)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = \ \frac{\mathrm{d}^2}{\mathrm{d}y^2}\cosh(ky) = \ k\frac{\mathrm{d}}{\mathrm{d}y}\sinh(ky) = \ k^2 \underbrace{\cosh(y)}_{=Y(y)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(kx) = -k\sin(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) = -k^2\cos(kx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh(kx) = k\sinh(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cosh(kx) = k^2\cosh(kx)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(kx) = k\cos(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(kx) = -k^2\sin(kx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh(kx) = k\cosh(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sinh(kx) = k^2\sinh(kx)$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

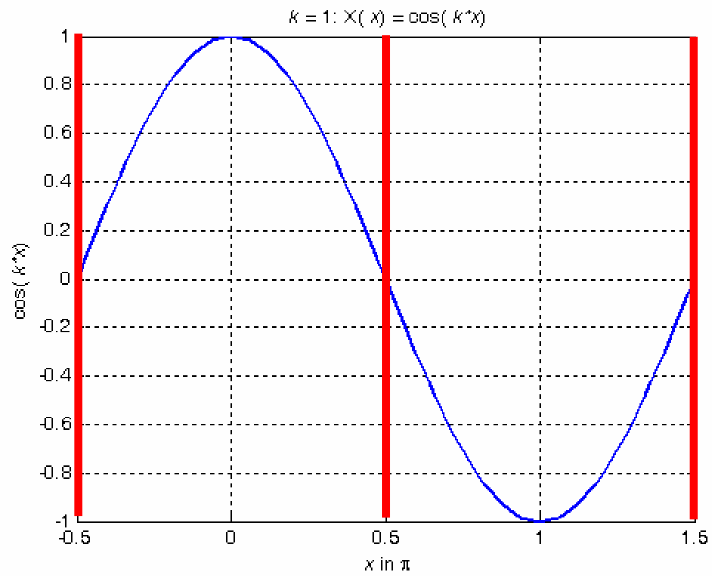
Solutions of the 2-D Laplace Equation in the Cartesian Coordinate System / Lösungen der 2D-Laplace-Gleichung im Kartesischen Koordinatensystem

$$\Phi_e(x,y) = X(x)\,Y(y) =$$

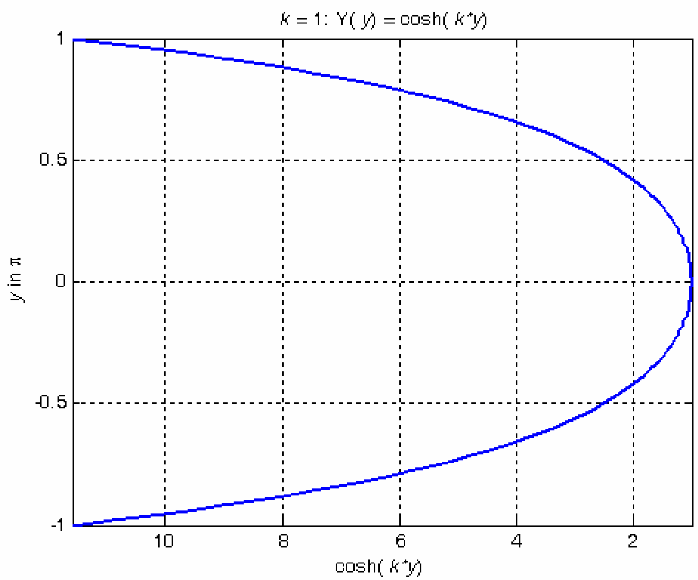
$k=0$	$k^2 \geq 0$	$k^2 \leq 0 \; (k \to jk')$
const.	$\cos(kx)\cosh(ky)$	$\cosh(k'x)\cos(k'y)$
y	$\cos(kx)\sinh(ky)$	$\cosh(k'x)\sin(k'y)$
x	$\sin(kx)\cosh(ky)$	$\sinh(k'x)\cos(k'y)$
xy	$\sin(kx)\sinh(ky)$	$\sinh(k'x)\sin(k'y)$
	$\cos(kx)\,e^{ky}$	$e^{k'x}\cos(k'y)$
	$\cos(kx)\,e^{-ky}$	$e^{-k'x}\cos(k'y)$
	$\sin(kx)\,e^{ky}$	$e^{k'x}\sin(k'y)$
	$\sin(kx)\,e^{-ky}$	$e^{-k'x}\sin(k'y)$

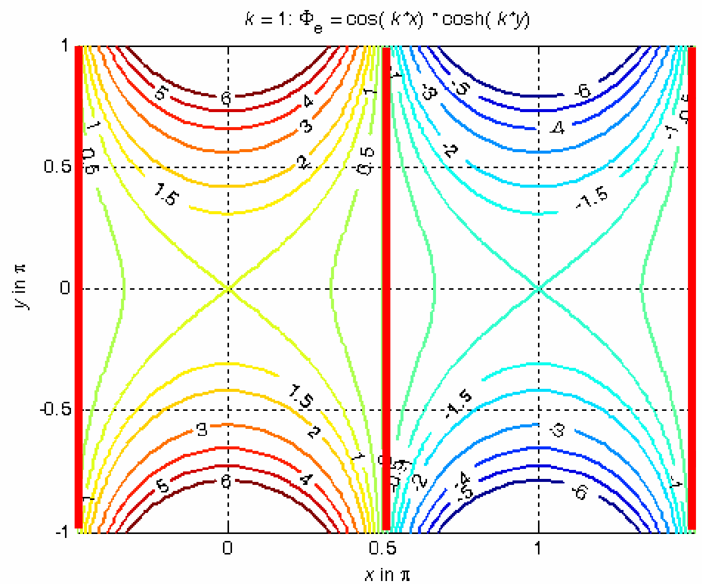
ES Fields – Separation of Variables / ES Felder – Separation der Variablen (...)

$$\Phi_{\mathrm{e}}(x,y) = X(x)Y(y) = \cos(kx)\cosh(ky)$$



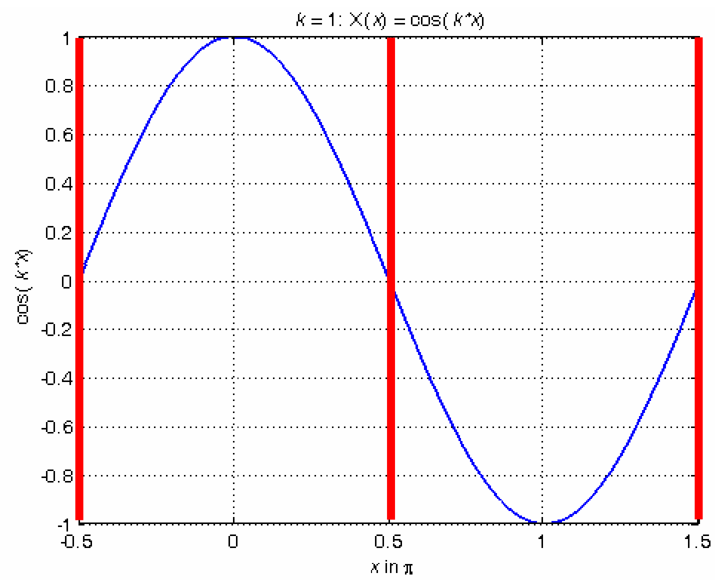
$$\Phi_{\mathrm{e}}(x,y) = 0$$

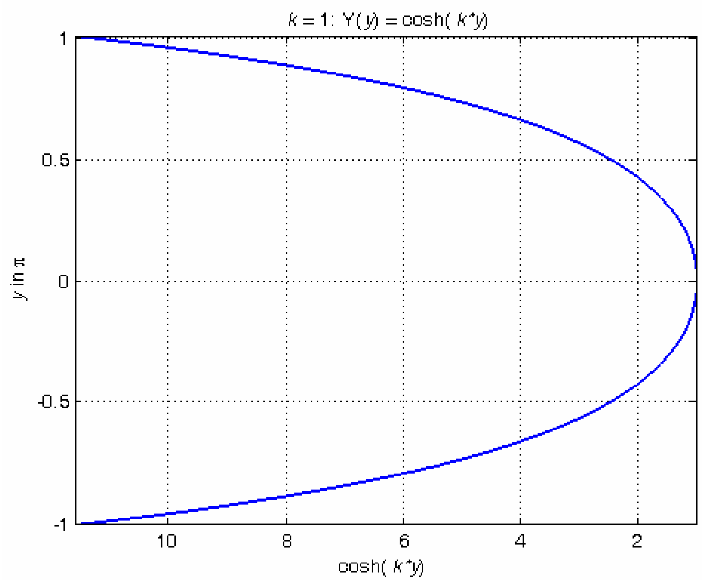


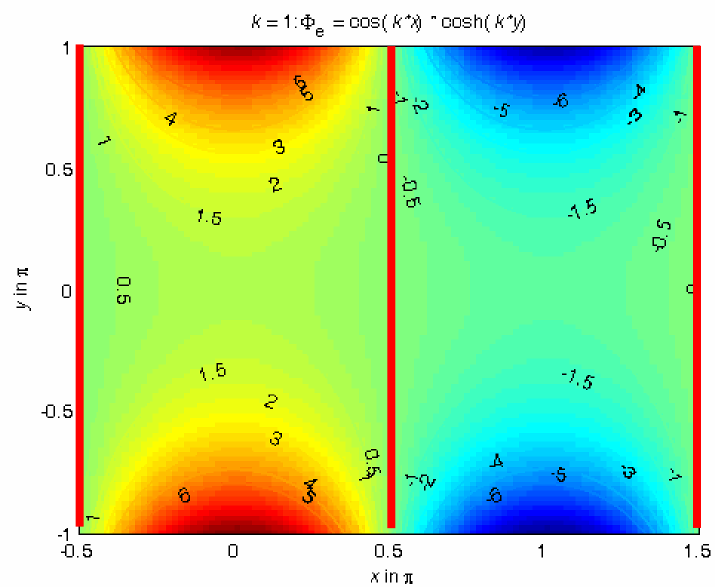


ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_{\mathrm{e}}(x,y) = X(x)Y(y) = \cos(kx)\cosh(ky)$$

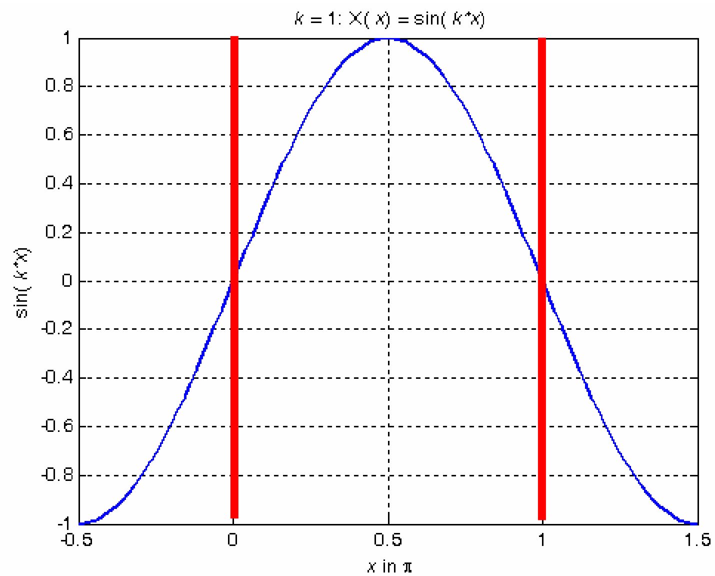




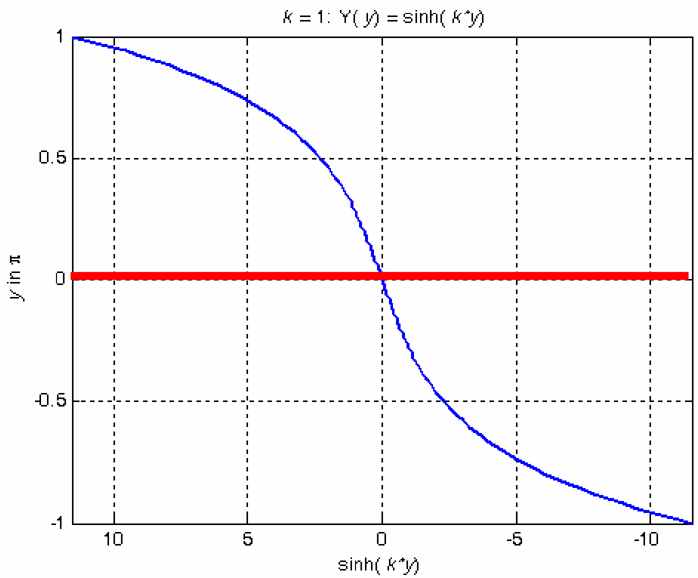


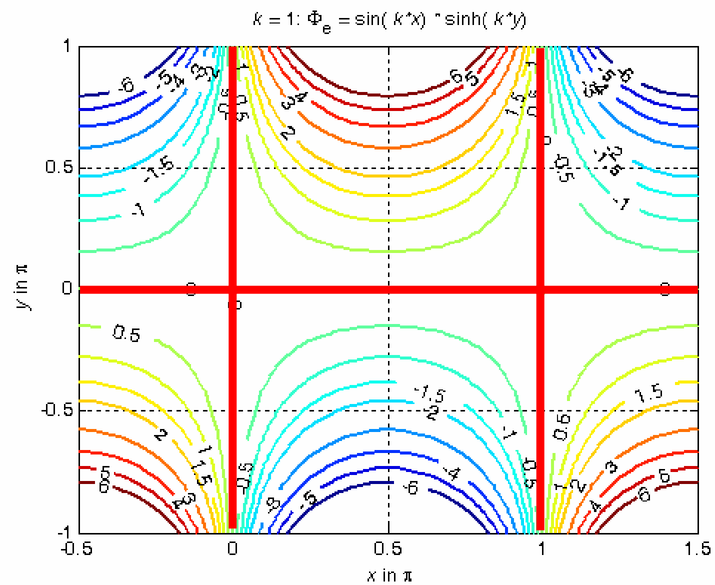
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_e(x,y) = X(x)Y(y) = \sin(kx)\sinh(ky)$$



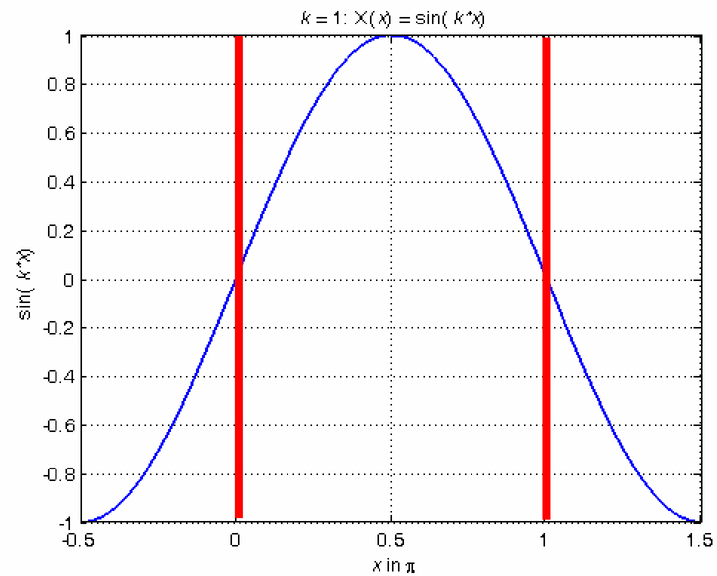
$$\Phi_e(x,y) = 0$$

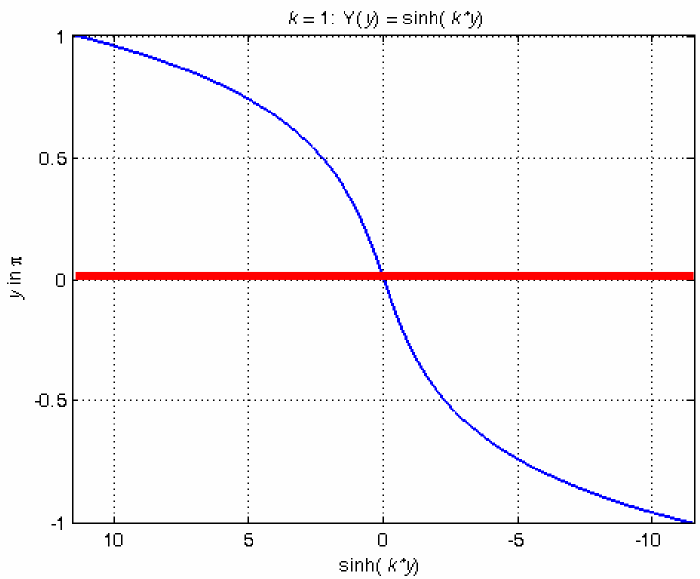




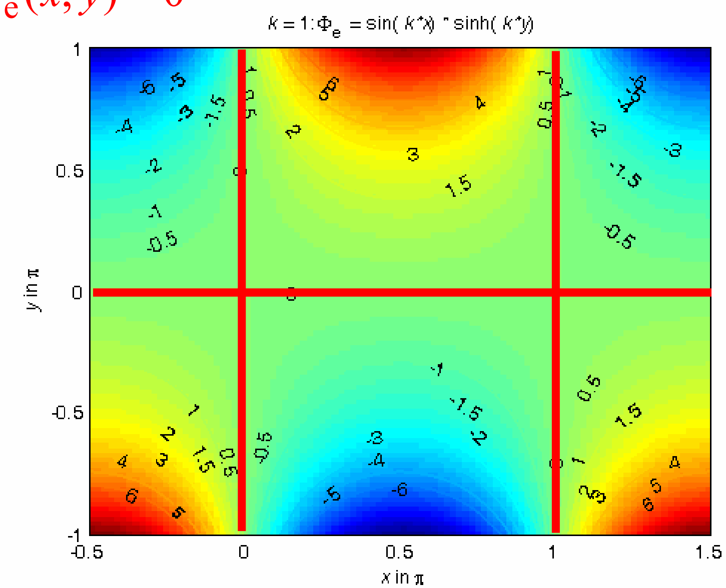
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_e(x,y) = X(x)Y(y) = \sin(kx)\sinh(ky)$$

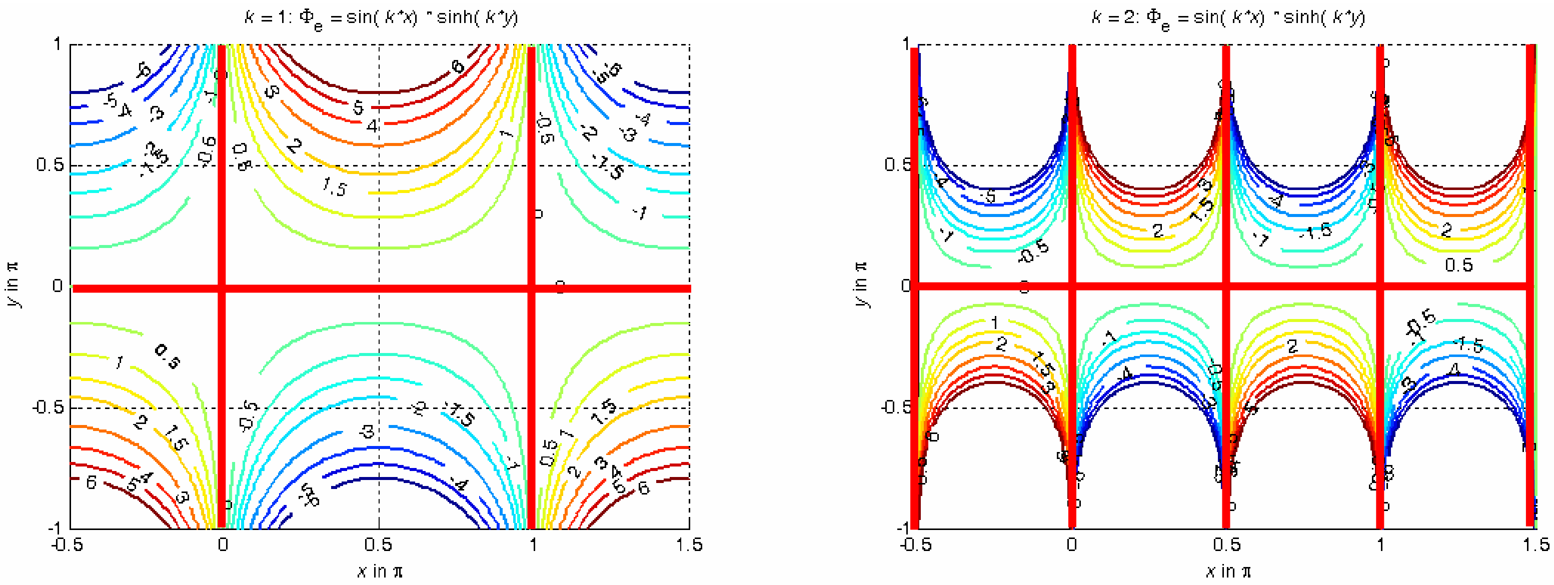




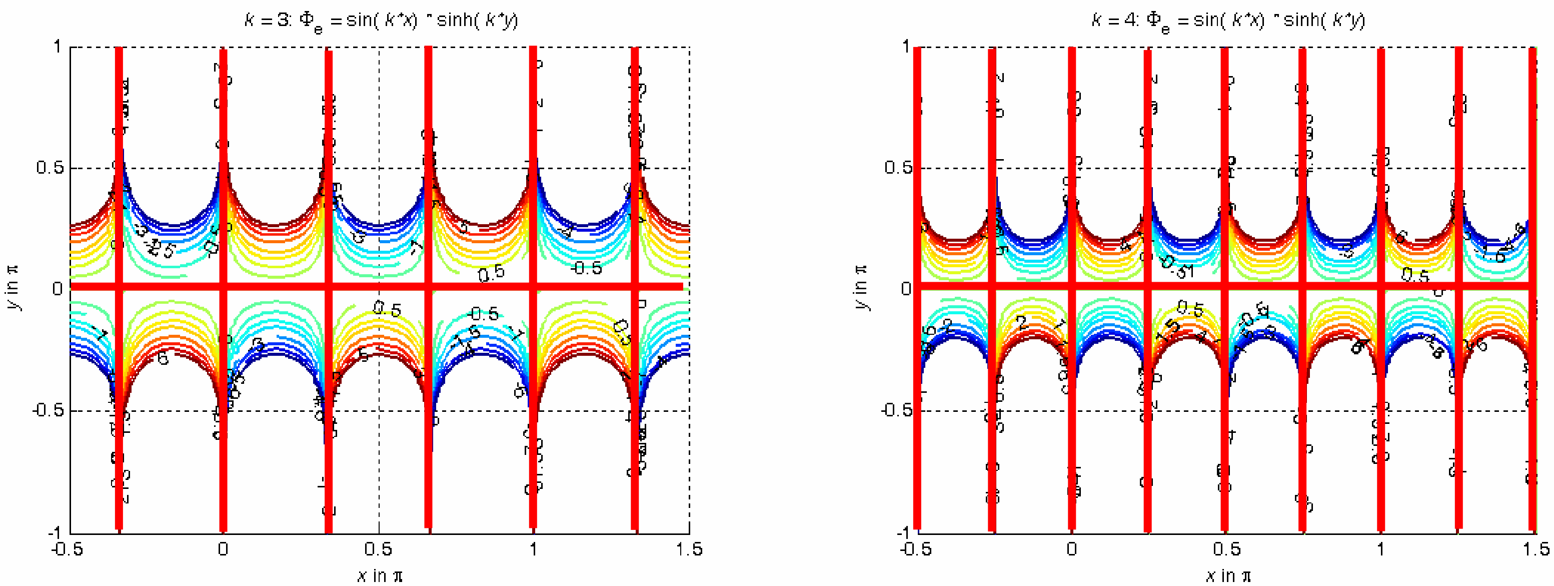
$\Phi_e(x,y) = 0$



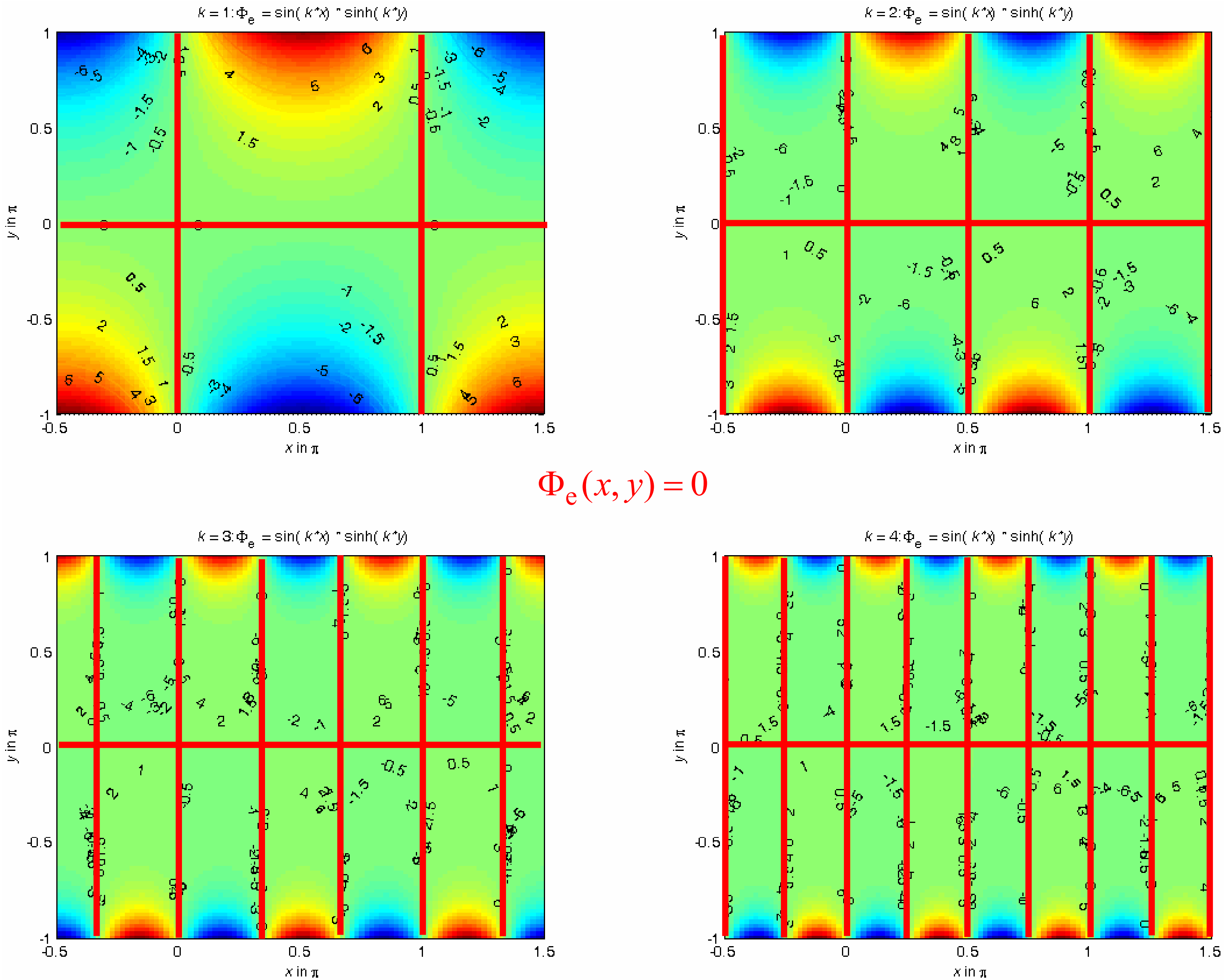
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



$$\Phi_\mathrm{e}(x,y)=0$$



ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



ES Fields – Separation of Variables – Superposition of Modes / ES Felder – Separation der Variablen – Superposition von Moden (...)

Superposition of Modes to Ensure Boundary Conditions /
Superposition von Moden zur Erfüllung von Randbedingungen:

Each solution of the Laplace equation – eigen solution, mode – obtained by the separation of variables displays lines (surfaces) of vanishing potential. At these lines (surfaces) we could place a Dirichlet boundary with $\Phi_e(x,y) = 0$ V /
Jede Lösung der Laplace-Gleichung – Eigenlösung, Mode –, die man über die Methode der Separation bestimmt, weist Linien (Flächen) mit dem Null-Potential auf.
Auf diesen Linien (Flächen) kann man eine Dirichlet-Rand mit $\Phi_e(x,y) = 0$ V platzieren.

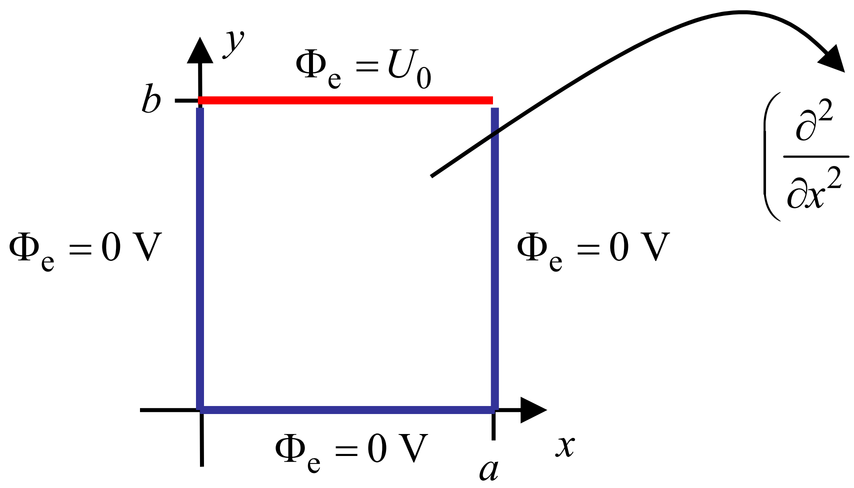
For Example, Consider the Solution / Betrachte beispielsweise die Lösung

$$\Phi_e(x,y) = \Phi_{e0} \sin(kx) \sinh(ky)$$

This Functions is Zero for / Diese Funktion ist gleich null für

$$\Phi_e(x,y) = 0 \begin{cases} y = 0 & & \text{because / weil} & \sinh(ky) = \sinh(0) = 0 \\ x = \dfrac{n\pi}{k} & n = -\infty, \ldots -1, 0, 1, \ldots \infty & \text{because / weil} & \sin(kx) = \sin(n\pi) = 0 \end{cases}$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y) = 0 \quad \begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$$

We Set / Wir setzen:

$$\Phi_e(x,y) = \Phi_{e0}\sin(kx)\sinh(ky) \qquad k = \frac{n\pi}{a} \qquad \Phi_e(x,y) = \Phi_{e0}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

then it follows / dann folgt

$$\Phi_e(x,y) = 0 \quad \begin{matrix} x = 0 \\ x = a \\ y = 0 \end{matrix}$$

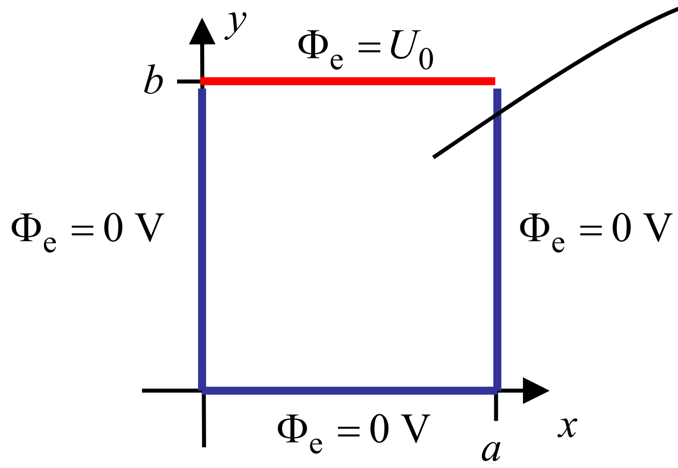
$$y = b : \Phi_e = U_0$$

$$\Phi_e(x, y = b) = \Phi_{e0}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right) \neq U_0$$

ES Fields – Separation of Variables – Superposition of Modes / ES Felder – Separation der Variablen – Superposition von Moden (...)

$$\Rightarrow \quad \Phi_{\mathrm{e}}(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right)$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y) = 0 \quad \begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$$

$$\Rightarrow \quad \Phi_e(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

A_n **?!**

Adjust die Coefficients A_n, $n = 1,2,\ldots,\infty$ in Order to Ensure the Inhomogeneous Dirichlet Boundary Condition $\Phi_e(x,b) = U_0$ at the Top Boundary. / Die Koeffizienten A_n, $n = 1,2,\ldots,\infty$ sind so zu bestimmen, dass die inhomogene Dirichlet–Randbedingung $\Phi_e(x,b) = U_0$ am oberen Rand erfüllt wird.

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Adjust die Coefficients A_n, $n = 1,2,\ldots,\infty$ in Order to Ensure the Inhomogeneous Dirichlet Boundary Condition $\Phi_e(x,b) = U_0$ at the Top Boundary. / Die Koeffizienten A_n, $n = 1,2,\ldots,\infty$ sind so zu bestimmen, dass die inhomogene Dirichlet-Randbedingung $\Phi_e(x,b) = U_0$ am oberen Rand erfüllt wird.

$$\Phi_e(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

1. Determine / Bestimme $\Phi_e(x,y)\big|_{y=b}$

$$\Phi_e(x,y=b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

2. Multiply Both Sides with / Multipliziere beide Seiten mit $\sin\left(\frac{m\pi}{a}x\right)$

$$\int_{x=0}^{a} \Phi_e(x,b)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \int_{x=0}^{a}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right)\int_{x=0}^{a}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Orthogonal "Eigen"functions / Orthogonale „Eigen"funktionen

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \begin{cases} \frac{a}{2} & n=m \\ 0 & n \neq m \end{cases} = \frac{a}{2}\delta_{nm}$$

Kronecker Delta / Kronecker-Delta

$$\delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

$$\int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \underbrace{\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x}_{=\frac{a}{2}\delta_{nm}}$$

3. It Follows for $m = n$ / Es folgt für $m = n$

$$\int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \frac{a}{2}\sum_{n=m} A_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{a}{2} A_n \sinh\left(\frac{n\pi}{a}b\right)$$

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)} \int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_{x=0}^{a} \Phi_\mathrm{e}(x,b) \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x$$

$$\Phi_\mathrm{e}(x,b) = U_0$$

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_{x=0}^{a} U_0 \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x$$

$$= \frac{2U_0}{a \sinh\left(\frac{n\pi}{a} b\right)} \int_{x=0}^{a} \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x$$

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x = -\frac{a}{n\pi} \cos\left(\frac{n\pi}{a} x\right)\Bigg|_{x=0}^{a}$$

$$= -\frac{a}{n\pi}\left[\cos\left(\frac{n\pi}{a} a\right) - \underbrace{\cos(0)}_{=1}\right]$$

$$= \frac{a}{n\pi}\left[1 - \cos(n\pi)\right]$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$A_n = \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \underbrace{\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x}$$

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \frac{a}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \frac{a}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{2U_0}{n\pi\sinh\left(\frac{n\pi}{a}b\right)} \underbrace{\left[1-\cos(n\pi)\right]}$$

$$\cos(n\pi) = \begin{cases} -1 & n = 1,3,5,\ldots \\ 1 & n = 2,4,6 \end{cases}$$

$$1-\cos(n\pi) = \begin{cases} 2 & n = 1,3,5,\ldots \\ 0 & n = 2,4,6 \end{cases}$$

$$\Rightarrow \quad A_n = \begin{cases} \dfrac{4U_0}{n\pi} \dfrac{1}{\sinh\left(\frac{n\pi}{a}b\right)} & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots \end{cases}$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Solution / Lösung

Infinite Series / Unendliche Reihe

$$\Phi_\mathrm{e}(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

Coefficients / Koeffizienten

with / mit

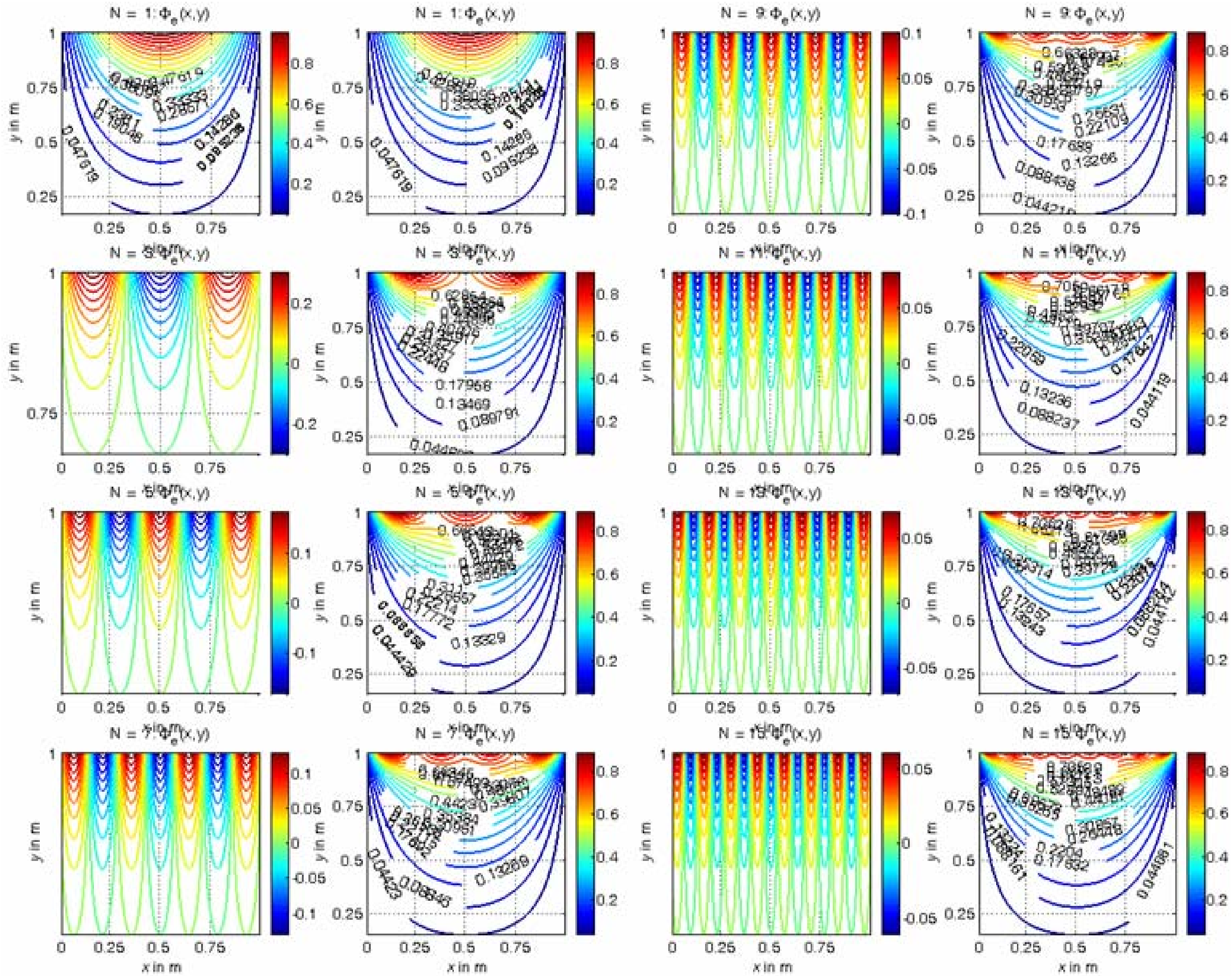
$$A_n=\begin{cases}\dfrac{4U_0}{n\pi}\dfrac{1}{\sinh\left(\dfrac{n\pi}{a}b\right)} & n=1,3,5,\ldots\\[2ex] 0 & n=2,4,6,\ldots\end{cases}$$

Complete Solution / Komplette Lösung

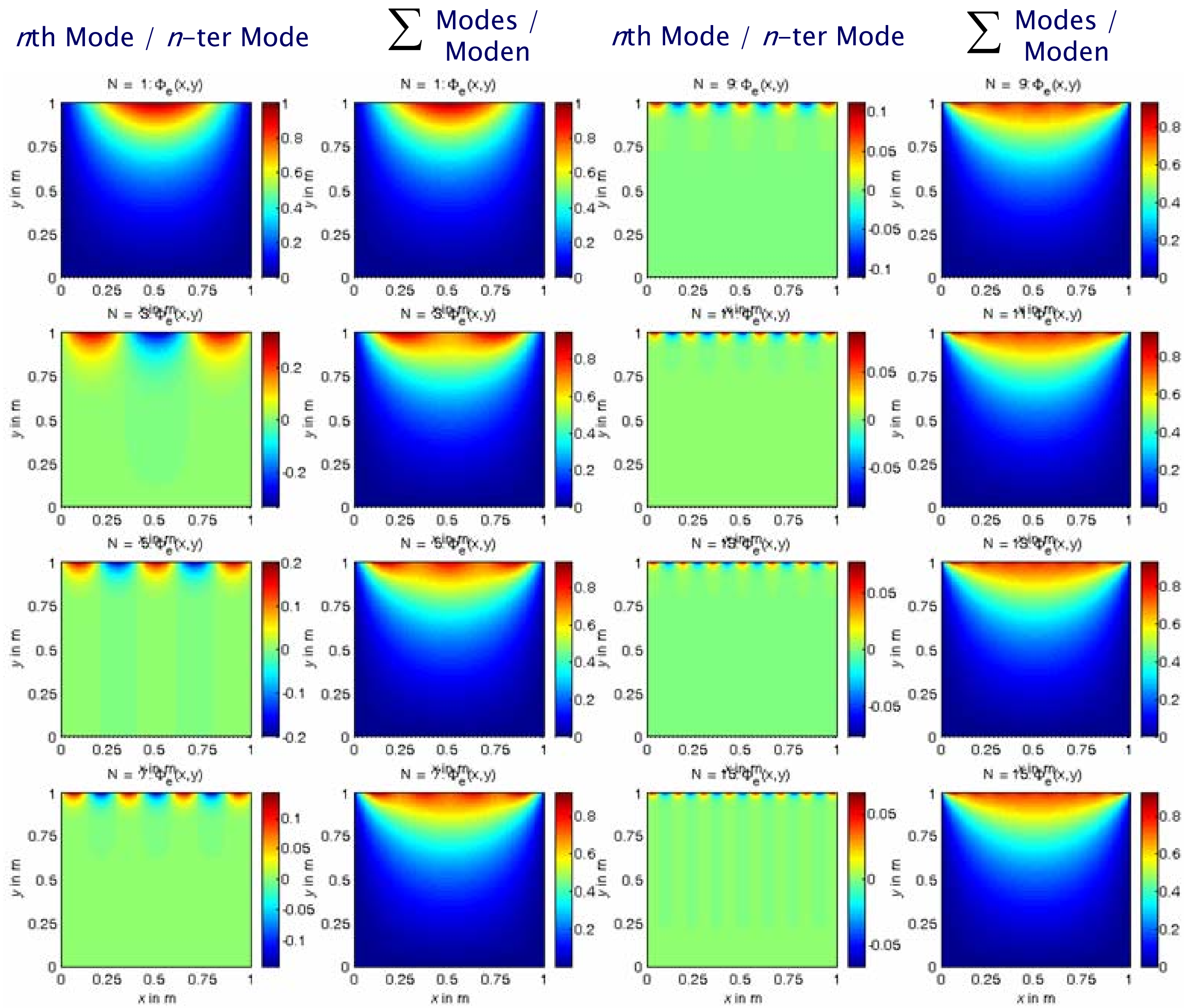
$$\Rightarrow\quad \Phi_\mathrm{e}(x,y)=\frac{4U_0}{\pi}\sum_{\substack{n=1\\ \text{odd /}\\ \text{ungerade}}}^{\infty}\frac{1}{n}\frac{\sinh\left(\dfrac{n\pi}{a}y\right)}{\sinh\left(\dfrac{n\pi}{a}b\right)}\sin\left(\frac{n\pi}{a}x\right)$$

ES Fields – Separation of Variables – Example /
ES Felder – Separation der Variablen – Beispiel (...)

*n*th Mode / *n*–ter Mode | $\sum$ Modes / Moden | *n*th Mode / *n*–ter Mode | $\sum$ Modes / Moden

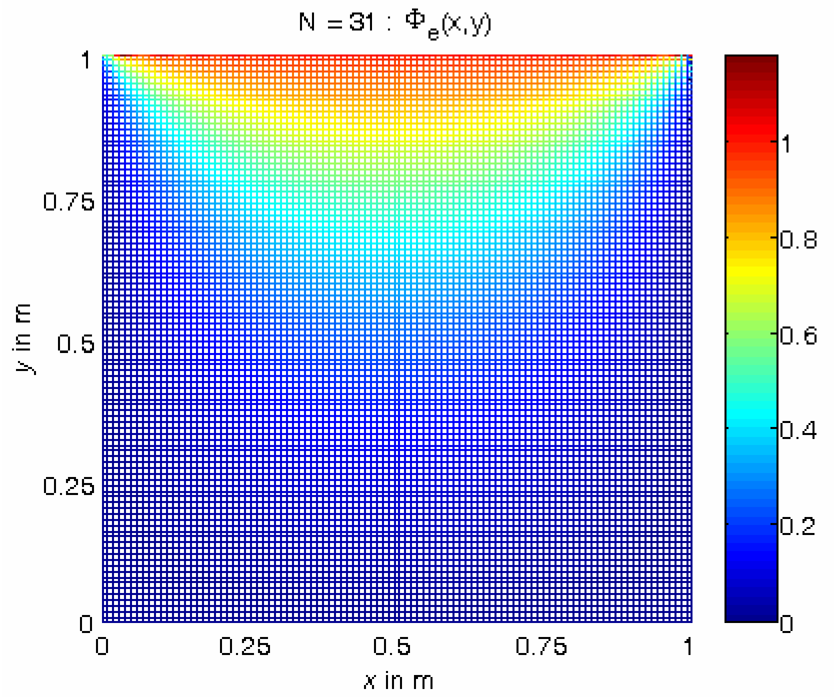


ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

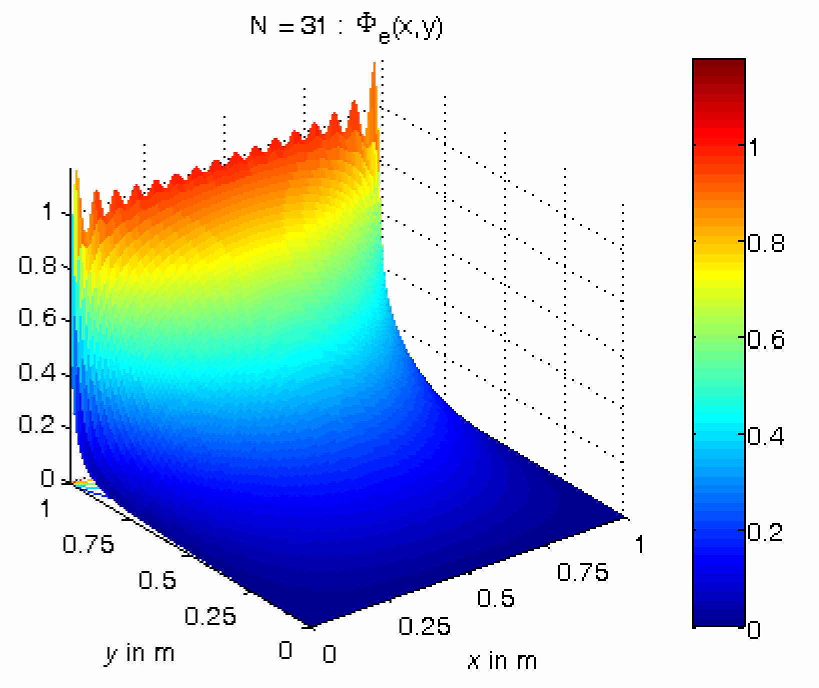


ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$\sum_{n=1}^{31}$ Modes / Moden



$\sum_{n=1}^{31}$ Modes / Moden



End of Lecture 8 /
Ende der 8. Vorlesung