Numerical Methods of Electromagnetic Field Theory I (NFT I)
Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

2nd Lecture / 2. Vorlesung

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1-D Electromagnetic Wave Propagation / 1D elektromagnetische Wellenausbreitung

Constitutive Equations for Vacuum / Konstituierende Gleichungen (Materialgleichungen) für Vakuum

$$\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \mu_0 \underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

Ansatz / Ansatz

$$\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = E_x(z,t)\ \underline{\mathbf{e}}_x$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = H_y(z,t)\underline{\mathbf{e}}_y$$

⇒

The first two Maxwell's Equations are: / Die ersten beiden Maxwellschen Gleichungen lauten:

$$\frac{\partial}{\partial t}\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

⇓

$$\frac{\partial}{\partial t}H_y(z,t) = -\frac{1}{\mu_0}\frac{\partial}{\partial z}E_x(z,t) - \frac{1}{\mu_0}J_{\mathrm{my}}(z,t)$$

$$\frac{\partial}{\partial t}E_x(z,t) = -\frac{1}{\varepsilon_0}\frac{\partial}{\partial z}H_y(z,t) - \frac{1}{\varepsilon_0}J_{\mathrm{ex}}(z,t)$$

$$\frac{\partial^2}{\partial z^2}H_y(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_y(z,t) = \begin{cases} -\dfrac{\partial}{\partial z}J_{\mathrm{ex}}(z,t) + \varepsilon_0\dfrac{\partial}{\partial t}J_{\mathrm{my}}(z,t) & \text{Inhomogeneous 1-D Wave Equation / Inhomogene 1-D Wellengleichung} \\ 0 & \text{Homogeneous 1-D Wave Equation / Homogene 1-D Wellengleichung} \end{cases}$$

$$\frac{\partial^2}{\partial z^2}E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_x(z,t) = \begin{cases} -\dfrac{\partial}{\partial z}J_{\mathrm{my}}(z,t) + \mu_0\dfrac{\partial}{\partial t}J_{\mathrm{ex}}(z,t) & \text{Inhomogeneous 1-D Wave Equation / Inhomogene 1-D Wellengleichung} \\ 0 & \text{Homogeneous 1-D Wave Equation / Homogene 1-D Wellengleichung} \end{cases}$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Backward FD Operator / Rückwärts-FD-Operator

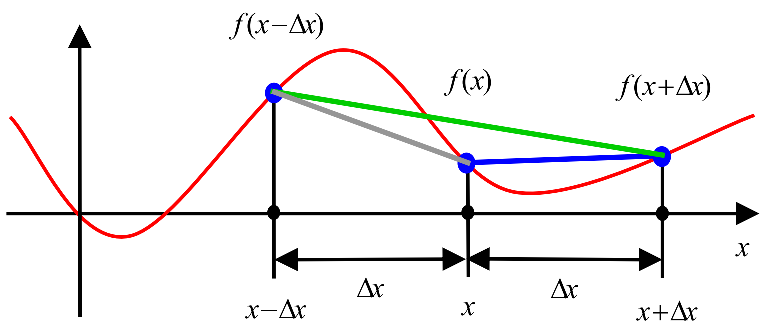
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Forward FD Operator / Vorwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Central FD Operator / Zentraler FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + \mathrm{O}[(\Delta x)^2] \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$



$$\mathrm{O}(\Delta x),\ \mathrm{O}[(\Delta x)^2]$$

Asymptotic approximation error / Asymptotischer Approximationsfehler

Landau symbol “O”, big “oh” / Landau-Symbol „O“, großes „oh“

FD Method - 1-D FD Operator of Second Order / FD-Methode - 1D-FD-Operator zweiter Ordnung

Derivative of the second order / Ableitung der zweiten Ordnung

$$\frac{d^2}{dx^2} f(x) \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} \quad (1)$$

Taylor series expansions / Taylor-Reihenentwicklungen

$$f(x+\Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} + \frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \frac{(\Delta x)^4}{4!}\frac{df(x)}{dx^4} + \bigcirc\left[(\Delta x)^5\right] \quad (2)$$

$$f(x) = f(x) \quad (3)$$

$$f(x-\Delta x) = f(x) - \Delta x \frac{d(x)}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} - \frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} + \bigcirc\left[(\Delta x)^5\right] \quad (4)$$

Multiply (2) with α, (3) with β, and (4) with γ / Multipliziere (2) mit α, (3) mit β und (4) mit γ

$$\alpha f(x+\Delta x) = \alpha f(x) + \alpha\Delta x \frac{df(x)}{dx} + \alpha\frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} + \alpha\frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \alpha\frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} + \bigcirc\left[(\Delta x)^5\right] \quad (5)$$

$$\beta f(x) = \beta f(x) \quad (6)$$

$$\gamma f(x-\Delta x) = \gamma f(x) - \gamma\Delta x \frac{df(x)}{dx} + \gamma\frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} - \gamma\frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \gamma\frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} + \bigcirc\left[(\Delta x)^5\right] \quad (7)$$

FD Method - 1-D FD Operator of Second Order / FD-Methode - 1D-FD-Operator zweiter Ordnung

$$\alpha f(x+\Delta x)+\beta f(x)+\gamma f(x-\Delta x)$$
$$=(\alpha+\beta+\gamma)f(x)+(\alpha-\gamma)\Delta x\frac{\mathrm{d}f(x)}{\mathrm{d}x}+(\alpha+\gamma)\frac{(\Delta x)^2}{2!}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$$
$$+(\alpha-\gamma)\frac{(\Delta x)^3}{3!}\frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3}+(\alpha+\gamma)\frac{(\Delta x)^4}{4!}\frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4}+O\left[(\Delta x)^5\right]$$

$$\alpha-\gamma=0 \quad \rightarrow \quad \gamma=\alpha$$
$$\alpha+\beta+\gamma=0 \rightarrow \quad \beta=-2\alpha$$

With the parameters / Mit den Parametern

$$\alpha=\gamma=1$$
$$\beta=-2$$

$$f(x+\Delta x)-2f(x)+f(x-\Delta x)=2\frac{(\Delta x)^2}{2}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}+2\frac{(\Delta x)^4}{\underbrace{4!}_{1\cdot 2\cdot 3\cdot 4=24}}\frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4}+O\left[(\Delta x)^5\right]$$

$$\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}=\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}+\frac{(\Delta x)^2}{12}\frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4}+O\left[(\Delta x)^3\right]$$

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}\underbrace{-\frac{(\Delta x)^2}{12}\frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4}+O\left[(\Delta x)^3\right]}_{O[(\Delta x)^2]}$$

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}+O[(\Delta x)^2]$$

FD Method - 1-D FD Operators of Second Order / FD-Methode - 1D-FD-Operatoren zweiter Ordnung

Function of one variable / Funktion einer Variablen

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} + O[(\Delta x)^2]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) = \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{(\Delta t)^2} + O[(\Delta t)^2]$$

Function of two variables / Funktion von zwei Variablen

$$\frac{\partial^2}{\partial x^2} f(x,t) = \frac{f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)}{(\Delta x)^2} + O[(\Delta x)^2]$$

$$\frac{\partial^2}{\partial t^2} f(x,t) = \frac{f(x,t+\Delta t) - 2f(x,t) + f(x,t-\Delta t)}{(\Delta t)^2} + O[(\Delta t)^2]$$

Combined Case / Kombinierter Fall

$$\frac{\partial^2}{\partial x^2} f(x,t) + \frac{\partial^2}{\partial t^2} f(x,t)$$

$$= \frac{f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)}{(\Delta x)^2} + \frac{f(x,t+\Delta t) - 2f(x,t) + f(x,t-\Delta t)}{(\Delta t)^2} + O[(\Delta x)^2] + O[(\Delta t)^2]$$

FD Method – 1-D Wave Equation / FD-Methode – 1D Wellengleichung

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = -\frac{\partial}{\partial z}\underbrace{J_{my}(z,t)}_{=0} + \mu_0 \frac{\partial}{\partial t} J_{ex}(z,t)$$
$$= \mu_0 \frac{\partial}{\partial t} J_{ex}(z,t)$$

Central FD Operators / Zentrale FD-Operatoren

$$\frac{\partial^2}{\partial z^2} E_x(z,t) = \frac{E_x(z+\Delta z,t) - 2E_x(z,t) + E_x(z-\Delta z,t)}{(\Delta z)^2} + \bigcirc[(\Delta z)^2]$$

$$\frac{\partial^2}{\partial t^2} E_x(z,t) = \frac{E_x(z,t+\Delta t) - 2E_x(z,t) + E_x(z,t-\Delta t)}{(\Delta t)^2} + \bigcirc[(\Delta t)^2]$$

Backward FD Operator / Rückwärts-FD-Operator

$$\frac{\partial}{\partial t} J_{ex}(z,t) = \frac{J_{ex}(z,t) - J_{ex}(z,t-\Delta t)}{\Delta t} + \bigcirc(\Delta t)$$

$$\frac{E_x(z+\Delta z,t) - 2E_x(z,t) + E_x(z-\Delta z,t)}{(\Delta z)^2} - \frac{1}{c_0^2}\frac{E_x(z,t+\Delta t) - 2E_x(z,t) + E_x(z,t-\Delta t)}{(\Delta t)^2} = \mu_0 \frac{J_{ex}(z,t) - J_{ex}(z,t-\Delta t)}{\Delta t}$$
$$+ \bigcirc[(\Delta z)^2] + \bigcirc[(\Delta t)^2]$$

FD Method – 1D Wave Equation / FD-Methode – 1D Wellengleichung

Explicit FD algorithm in the time domain of 2nd order in space and time /
Expliziter FD-Algorithmus im Zeitbereich 2ter Ordnung in Raum und Zeit

$$E_x(z,t+\Delta t) = 2E_x(z,t) - E_x(z,t-\Delta t) + c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}\left[E_x(z+\Delta z,t) - 2E_x(z,t) + E_x(z-\Delta z,t)\right]$$
$$+ c_0^2 \mu_0 \Delta t\left[J_{\text{ex}}(z,t) - J_{\text{ex}}(z,t-\Delta t)\right] + \bigcirc[(\Delta z)^2] + \bigcirc[(\Delta t)^2]$$

Marching-on-in-time algorithm /
„Marschieren in der Zeit“-Algorithmus

$$z \to n_z \Delta z, \quad n_z = 1,\ldots,N_z$$
$$t \to n_t \Delta t, \quad n_t = 1,\ldots,N_t$$

$$E_x(z,t) \to E_x^{(n_z,n_t)}$$
$$J_{\text{ex}}(z,t) \to J_{\text{ex}}^{(n_z,n_t)}$$

$$E_x^{(n_z,n_t+1)} = 2E_x^{(n_z,n_t)} - E_x^{(n_z,n_t-1)} + c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}\left[E_x^{(n_z+1,n_t)} - 2E_x^{(n_z,n_t)} + E_x^{(n_z-1,n_t)}\right] + c_0^2 \mu_0 \Delta t\left[J_{\text{ex}}^{(n_z,n_t)} - J_{\text{ex}}^{(n_z,n_t-1)}\right]$$

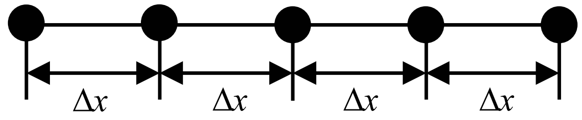
$$\Delta z = ?$$
$$\Delta t = ?$$

FD Method – Properties / FD-Methode - Eigenschaften

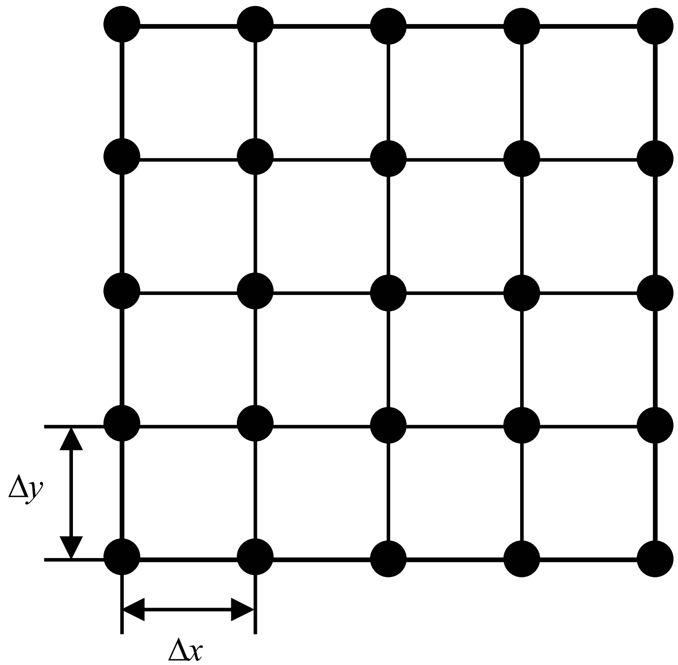
- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\boxed{\begin{matrix}\Delta z = ? \\ \Delta t = ?\end{matrix}}$
- **Consistency / Konsistenz**
- **Dissipation / Dissipation**
- **Stability Condition / Stabilitätsbedingung** $\boxed{\Delta t = f(\Delta z)}$
- **Convergence / Konvergenz**

FD Method – 1-D, 2-D, 3-D Grid System / FD-Methode – 1D-, 2D- und 3D-Gittersystem

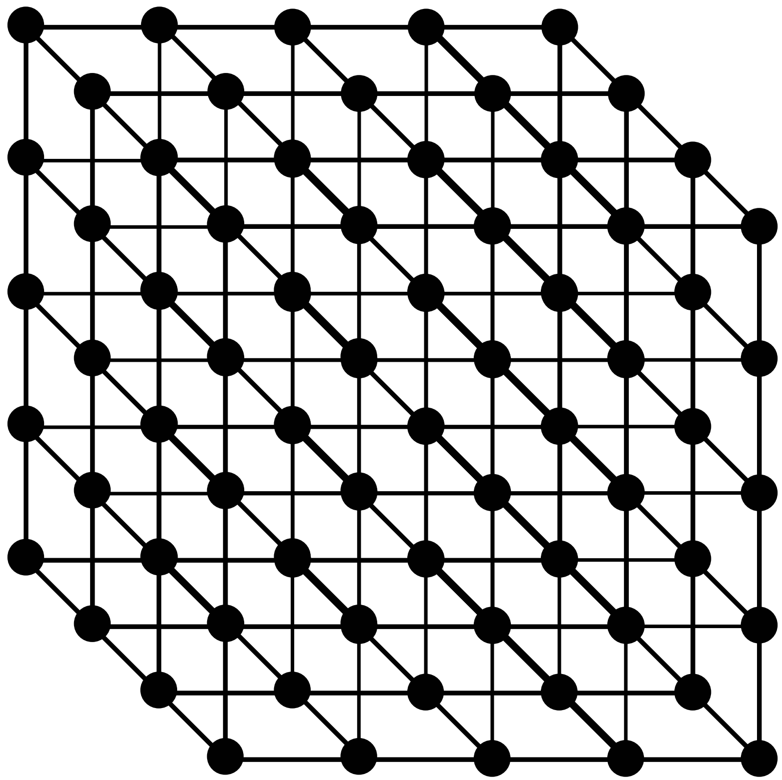
1-D Node-Based Grid / 1D knotenbasiertes Gitter



2-D Node-Based Grid / 2D knotenbasiertes Gitter



3-D Node-Based Grid / 3D knotenbasiertes Gitter



● **Nodes with Assigned Field Quantities / Knoten mit zugeordneten Feldgrößen:** Φ [V], $\underline{\mathbf{E}}$ [V/m], $\underline{\mathbf{H}}$ [A/m], $\underline{\mathbf{A}}$ [Vs/m]

FD Method – Grid Size / FD-Methode - Gittergröße

Sampling Theorem in Space / Abtastkriterium im Raum

$$\Delta x \le \frac{\lambda_{\min}}{2}$$

Δx : **Spatial grid size / Räumliche Gittergröße**

$\lambda_{\min}$: **Minimal wavelength / Minimale Wellenlänge**

$$\lambda_{\min} = \frac{c_{\min}}{f_{\max}}$$

$c_{\min}$: **Minimal phase velocity / Minimale Phasengeschwindigkeit**

$f_{\max}$: **Maximal frequency / Maximale Frequenz**

Sampling Resolution / Abtastauflösung

$$G = \frac{\lambda_{\min}}{\Delta x} \qquad G = 10, \ldots, 30$$

$$\Delta x = \frac{\lambda_{\min}}{G} = \frac{\lambda_{\min}}{10}, \ldots, \frac{\lambda_{\min}}{30}$$

Rule of thumb / Daumenregel

FD Method – Stability Condition / FD-Methode - Stabilitätsbedingung

Stability Condition for an FD algorithm of 2nd order in space and time– CFL-Condition / Stabilitätsbedingung für einen FD-Algorithmus zweiter Ordnung in Raum und Zeit– CFL-Bedingung

$$\Delta t \le \frac{1}{\sqrt{D}} \frac{\Delta x}{c}$$

$D = 1, 2, 3$: **Spatial dimension of the problem / Räumliche Dimension des Problems**

c : **Maximal Energy Propagation Velocity / Maximale Energieausbreitungsgeschwindigkeit**

CFL: Courant, Friedrichs, Lewy / CFL: Courant, Friedrichs, Lewy /

Courant, R., K. Friedrichs und H. Lewy: Über die partiellen Differenzengleichungen der mathematischen Physik. Mathematische Annalen, Vol. 100, S. 32-74, 1928.
Translation / Übersetzung:
Courant, R., K. Friedrichs, and H. Lewy: On the partial differential equations of mathematical physics. *IBM Journal*, pp. 215-324, March 1967.

1-D / 1D: $\Delta t \le \Delta t_{\max} = \frac{\Delta x}{c}$ $\qquad \widehat{\Delta t} \le 1$

2-D / 2D: $\Delta t \le \Delta t_{\max} = \frac{1}{\sqrt{2}} \frac{\Delta x}{c}$ $\qquad \widehat{\Delta t} \le \frac{1}{\sqrt{2}} \approx 0.707$

3-D / 3D: $\Delta t \le \Delta t_{\max} = \frac{1}{\sqrt{3}} \frac{\Delta x}{c}$ $\qquad \widehat{\Delta t} \le \frac{1}{\sqrt{3}} \approx 0.577$

$\widehat{\Delta t} = \frac{\Delta t}{\Delta t_{\text{ref}}}$: **Courant number / Courant - Zahl**

$\Delta t_{\text{ref}} = \frac{\Delta x}{c}$

FD Method – Normalization / FD-Methode – Normierung

Δx_{ref} = Reference cell width in m / Referenz-Zellenweite in m

c_{ref} = Reference propagation velocity in m/s / Referenz-Ausbreitungsgeschwindigkeit in m/s

ε_{ref} = Reference permittivity in As/Vm / Referenz-Permittivität in As/Vm

E_{ref} = Reference electric field strength in V/m / Elektrische Referenz-Feldstärke in V/m

$$\Delta t = \Delta t_{\text{ref}} \, \widehat{\Delta t} \qquad \Delta t_{\text{ref}} = \frac{\Delta x_{\text{ref}}}{c_{\text{ref}}}$$

$$\Delta z = \Delta x_{\text{ref}} \, \widehat{\Delta z}$$

$$c = c_{\text{ref}} \, \hat{c}$$

$$\varepsilon = \varepsilon_{\text{ref}} \, \hat{\varepsilon}$$

$$\mu = \mu_{\text{ref}} \, \hat{\mu} \qquad \mu_{\text{ref}} = c_{\text{ref}}^2 \varepsilon_{\text{ref}}$$

$$E_x = E_{\text{ref}} \, \hat{E}_x$$

$$J_{\text{e}x} = J_{\text{e ref}} \, \hat{J}_{\text{e}x} \qquad J_{\text{e ref}} = \frac{\varepsilon_{\text{ref}}}{\Delta t_{\text{ref}}} E_{\text{ref}}$$

FD Method – Normalization / FD-Methode – Normierung

$$E_x^{(n_z,n_t+1)} = 2E_x^{(n_z,n_t)} - E_x^{(n_z,n_t-1)} + \underbrace{c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}}\left[E_x^{(n_z+1,n_t)} - 2E_x^{(n_z,n_t)} + E_x^{(n_z-1,n_t)}\right] + \underbrace{c_0^2 \mu_0 \Delta t J_{\mathrm{e\,ref}}}\left[\hat{J}_{\mathrm{ex}}^{(n_z,n_t)} - \hat{J}_{\mathrm{ex}}^{(n_z,n_t-1)}\right]$$

With / Mit

$$\Delta z = \Delta x_{\mathrm{ref}}$$

$$c_{\mathrm{ref}} = c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\varepsilon_{\mathrm{ref}} = \varepsilon_0$$

$$E_{\mathrm{ref}} = 1\ \mathrm{V/m}$$

$$c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2} = c_{\mathrm{ref}}^2 \frac{(\Delta t)^2 (\Delta t_{\mathrm{ref}})^2}{(\Delta x_{\mathrm{ref}})^2} = c_{\mathrm{ref}}^2 \frac{(\widehat{\Delta t})^2 \left(\dfrac{\Delta x_{\mathrm{ref}}}{c_{\mathrm{ref}}}\right)^2}{(\Delta x_{\mathrm{ref}})^2} = (\widehat{\Delta t})^2$$

$$c_0^2 \mu_0 \Delta t J_{\mathrm{e\,ref}} = c_{\mathrm{ref}}^2 \mu_{\mathrm{ref}} \widehat{\Delta t} \Delta t_{\mathrm{ref}} \frac{\varepsilon_{\mathrm{ref}}}{\Delta t_{\mathrm{ref}}} E_{\mathrm{ref}} = \widehat{\Delta t}$$

$$\hat{E}_x^{(n_z,n_t+1)} = 2\hat{E}_x^{(n_z,n_t)} - \hat{E}_x^{(n_z,n_t-1)} + (\widehat{\Delta t})^2\left[\hat{E}_x^{(n_z+1,n_t)} - 2\hat{E}_x^{(n_z,n_t)} + \hat{E}_x^{(n_z-1,n_t)}\right] + \widehat{\Delta t}\left[\hat{J}_{\mathrm{ex}}^{(n_z,n_t)} - \hat{J}_{\mathrm{ex}}^{(n_z,n_t-1)}\right]$$

FD Solution of the 1-D Wave Equation / FD-Lösung der 1D Wellengleichung

1-D wave equation / 1D Wellengleichung

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = \mu_0 \frac{\partial}{\partial t} J_{\mathrm{ex}}(z,t) \quad \text{for / für} \quad \begin{cases} 0 \le z \le Z \\ 0 \le t \le T \end{cases}$$

Initial condition / Anfangsbedingung

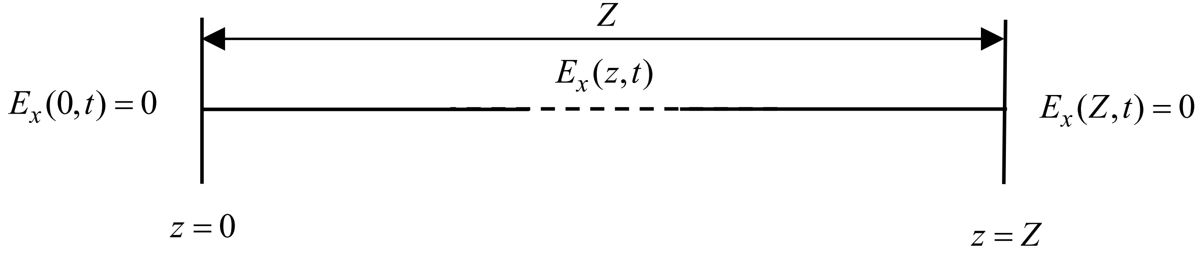
$$\begin{aligned} &E_x(z,t) = J_{\mathrm{ex}}(z,t) = 0 && t \le 0 \\ &J_{\mathrm{ex}}(z,t) = K_{\mathrm{e}0}(z_0)\,\delta(z_0)\,f(t) && t > 0 \end{aligned}$$

Causality / Kausalität

Boundary condition for a perfectly electrically conducting (PEC) material / Randbedingung für ein ideal elektrisch leitendes (IEL) Material

$$\left.\begin{aligned} E_x(0,t) &= 0 \\ E_x(Z,t) &= 0 \end{aligned}\right\} \quad \forall t$$

Hyperbolic initial-boundary-value problem / Hyperbolisches Anfangs-Randwert-Problem



FD Solution of the 1-D Wave Equation / FD-Lösung der 1D Wellengleichung

Normalized 1-D FD wave equation / Normierte 1D FD Wellengleichung

$$\hat{E}_x^{(n_z,n_t+1)} = 2\hat{E}_x^{(n_z,n_t)} - \hat{E}_x^{(n_z,n_t-1)} + (\widehat{\Delta t})^2 \left[\hat{E}_x^{(n_z+1,n_t)} - 2\hat{E}_x^{(n_z,n_t)} + \hat{E}_x^{(n_z-1,n_t)} \right] \quad \text{for / für} \quad \begin{cases} 1 \le n_z \le N_z \\ 1 \le n_t \le N_t \end{cases}$$

$$+ \widehat{\Delta t} \left[\hat{J}_{\text{ex}}^{(n_z,n_t)} - \hat{J}_{\text{ex}}^{(n_z,n_t-1)} \right]$$

Initial condition (IC) / Anfangsbedingung (AC)

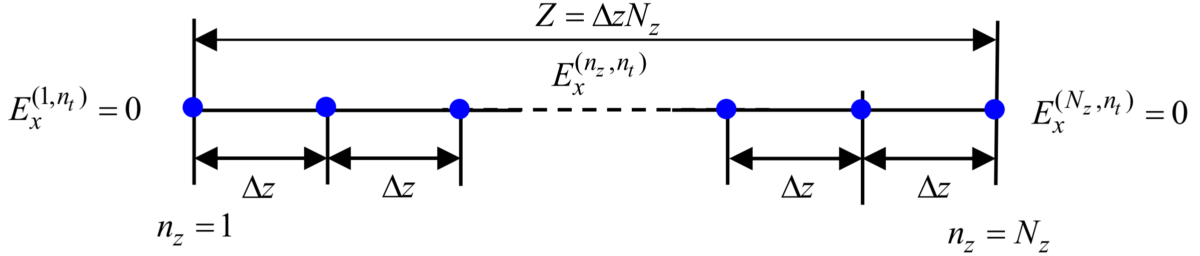
$$E_x^{(n_z,n_t)} = J_{\text{ex}}^{(n_z,n_t)} = 0 \qquad n_t \le 1$$

$$J_{\text{ex}}^{(n_z,n_t)} = K_{\text{ex}}^{(n_{z0})} \, \delta^{(n_{z0})} \, f^{(n_t)} \qquad n_t > 1$$

Causality / Kausalität

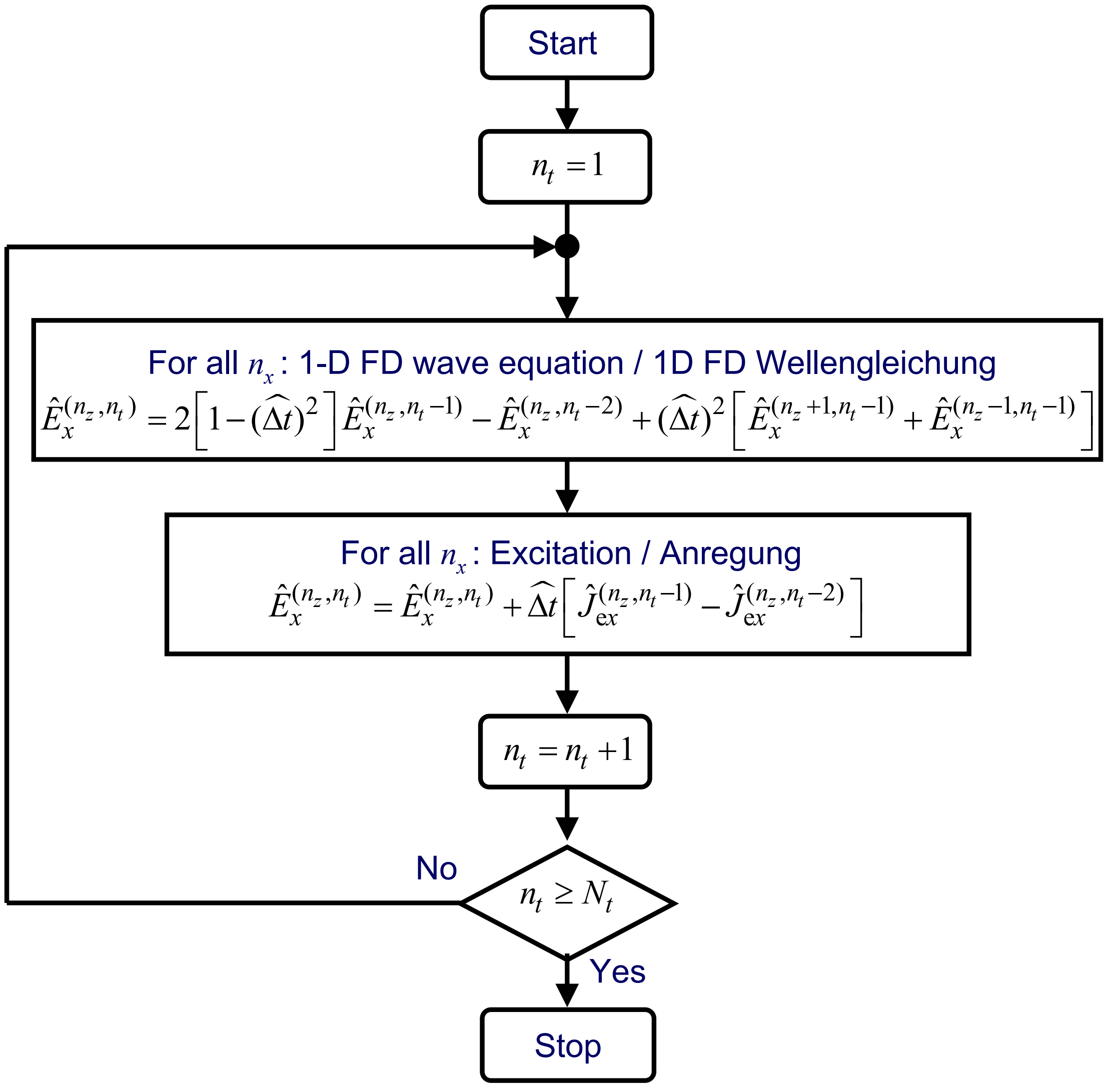
Boundary condition (BC) / Randbedingung (RB)

$$\left. \begin{aligned} E_x^{(1,n_t)} &= 0 \\ E_x^{(N_z,n_t)} &= 0 \end{aligned} \right\} \quad 1 \le n_t \le N_t$$



Discrete hyperbolic initial-boundary-value problem / Diskretes hyperbolisches Anfangs-Randwert-Problem

FD Method – 1D FD Wave Equation – Flow Chart / FD-Methode – 1D FD-Wellengleichung - Flussdiagramm



FD Method – 1D Wave Equation – Poynting Vector – Energy Flow Density / FD-Methode – 1D Wellengleichung – Poynting-Vektor – Energieflussdichte

$$\underline{\mathbf{S}}_{\mathrm{em}}(\underline{\mathbf{R}},t) = \underline{\mathbf{E}}(\underline{\mathbf{R}},t) \times \underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$
$$S_{\mathrm{em}\,z}(z,t) = E_x(z,t)\,H_y(z,t)$$

$$\frac{\partial}{\partial t} H_y(z,t) = -\frac{1}{\mu_0}\frac{\partial}{\partial z} E_x(z,t)$$

$$H_y(z,t) = \underbrace{-\frac{1}{\mu_0}\frac{\partial}{\partial z} E_x(z,t_0)}_{=H_y(z,t_0)} - \frac{1}{\mu_0}\int_{t'=t_0}^{t} \frac{\partial}{\partial z} E_x(z,t')\,dt'$$

$$= H_y(z,t_0) - \frac{1}{\mu_0}\int_{t'=t_0}^{t} \frac{\partial}{\partial z} E_x(z,t')\,dt'$$

$$H_y(z,t) = H_y(z,t_0) - \frac{1}{\mu_0}\int_{t'=t_0}^{t_0+\Delta t} \frac{\partial}{\partial z} E_x(z,t')\,dt'$$

$$= H_y(z,t_0) - \frac{1}{\mu_0}\frac{\partial}{\partial z} E_x\left(z,t_0+\frac{\Delta t}{2}\right)\underbrace{\int_{t'=t_0}^{t_0+\Delta t} dt'}_{=\Delta t}$$

$$\approx H_y(z,t_0) - \frac{\Delta t}{\mu_0} \underbrace{\frac{\partial}{\partial z} E_x\left(z,t_0+\frac{\Delta t}{2}\right)}_{=\frac{1}{2\Delta z}\left[E_x\left(z+\Delta z,t_0+\frac{\Delta t}{2}\right)-E_x\left(z-\Delta z,t_0\frac{\Delta t}{2}\right)\right]}$$

Applying the mid-point rule / Wende die Mittelpunktsregel an

FD Method – 1D Wave Equation – Poynting Vector – Energy Density Flow / FD-Methode – 1D Wellengleichung – Poynting-Vektor – Energiedichtefluss

$$H_y(z,t) \approx H_y(z,t_0) - \frac{1}{2}\frac{1}{\mu_0}\frac{\Delta t}{\Delta z}\left[E_x\left(z+\Delta z, t_0+\frac{\Delta t}{2}\right) - E_x\left(z-\Delta z, t_0\frac{\Delta t}{2}\right)\right]$$

$$\Delta z = \Delta x_{\text{ref}}$$

$$c_{\text{ref}} = c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\varepsilon_{\text{ref}} = \varepsilon_0$$

$$E_{\text{ref}} = 1\ \text{V/m}$$

$$\Delta t = \Delta t_{\text{ref}}\,\widehat{\Delta t} \qquad \Delta t_{\text{ref}} = \frac{\Delta z}{c_0}$$

$$c = c_0 \hat{c}$$

$$\mu = \mu_{\text{ref}}\,\hat{\mu} \qquad \mu_{\text{ref}} = \mu_0$$

$$E_x = E_{\text{ref}} \hat{E}_x$$

$$H_y = H_{\text{ref}} \hat{H}_y \qquad H_{\text{ref}} = \frac{E_{\text{ref}}}{c_{\text{ref}}\mu_{\text{ref}}} = \frac{\sqrt{\varepsilon_{\text{ref}}\mu_{\text{ref}}}}{\mu_{\text{ref}}}E_{\text{ref}} = \sqrt{\frac{\varepsilon_{\text{ref}}}{\mu_{\text{ref}}}}E_{\text{ref}} = \frac{E_{\text{ref}}}{Z_{\text{ref}}}$$

$$S_{\text{em z}} = S_{\text{em ref}}\hat{S}_{\text{em z}} \qquad S_{\text{em ref}} = E_{\text{ref}} H_{\text{ref}} = \frac{E_{\text{ref}}^2}{Z_{\text{ref}}}$$

$$\sqrt{\frac{\varepsilon_{\text{ref}}}{\mu_{\text{ref}}}}E_{\text{ref}} H_y(z,t) \approx \sqrt{\frac{\varepsilon_{\text{ref}}}{\mu_{\text{ref}}}}E_{\text{ref}} H_y(z,t_0) - \frac{1}{2}\frac{1}{\mu_{\text{ref}}}\frac{\widehat{\Delta t}}{\Delta z}\frac{\Delta z}{c_{\text{ref}}}E_{\text{ref}}\left[\hat{E}_x\left(z+\Delta z, t_0+\frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0\frac{\Delta t}{2}\right)\right]$$

$$H_y(z,t) \approx H_y(z,t_0) - \frac{1}{2}\sqrt{\frac{\mu_{\text{ref}}}{\varepsilon_{\text{ref}}}}\frac{1}{E_{\text{ref}}}\frac{1}{\mu_{\text{ref}}}\frac{\widehat{\Delta t}}{\Delta z}\frac{\Delta z}{c_{\text{ref}}}E_{\text{ref}}\left[\hat{E}_x\left(z+\Delta z, t_0+\frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0\frac{\Delta t}{2}\right)\right]$$

$$H_y(z,t) \approx H_y(z,t_0) - \frac{\widehat{\Delta t}}{2}\frac{1}{\sqrt{\varepsilon_{\text{ref}}\mu_{\text{ref}}}}\frac{1}{c_{\text{ref}}}\left[\hat{E}_x\left(z+\Delta z, t_0+\frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0\frac{\Delta t}{2}\right)\right]$$

FD Method – 1D Wave Equation – Poynting Vector – Energy Density Flow / FD-Methode – 1D Wellengleichung – Poynting-Vektor – Energiedichtefluss

$$\hat{H}_y(z,t) \approx \hat{H}_y(z,t_0) - \frac{\widehat{\Delta t}}{2} \frac{1}{\sqrt{\varepsilon_{\mathrm{ref}}\mu_{\mathrm{ref}}}} \frac{1}{c_{\mathrm{ref}}} \left[\hat{E}_x\left(z+\Delta z, t_0+\frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0 \frac{\Delta t}{2}\right)\right]$$

$$\hat{H}_y(z,t) \approx \hat{H}_y(z,t_0) - \frac{\widehat{\Delta t}}{2} \left[\hat{E}_x\left(z+\Delta z, t_0+\frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0 \frac{\Delta t}{2}\right)\right]$$

$$\hat{H}_y^{(n_z,n_t)} = \hat{H}_y^{(n_z,n_t-1)} - \frac{\widehat{\Delta t}}{2}\left[\hat{E}_x^{(n_z+1,n_t)} - \hat{E}_x^{(n_z-1,n_t)} \right]$$

$$\underline{\mathbf{S}}_{\mathrm{em}}(\underline{\mathbf{R}},t) = \underline{\mathbf{E}}(\underline{\mathbf{R}},t) \times \underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$S_{\mathrm{em}\,z}(z,t) = E_x(z,t)\, H_y(z,t)$$

$$\hat{S}_{\mathrm{em}z}^{(n_z,n_t)} = \hat{H}_y^{(n_z,n_t)}\,\hat{E}_x^{(n_z,n_t)}$$

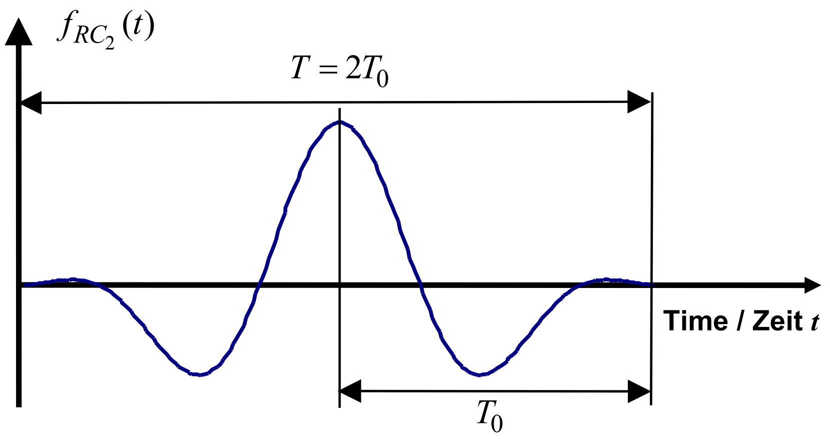
FD Method – 1D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Raised cosine pulse with n cycles / Aufsteigender Kosinus-Impuls mit n Zyklen

$$f_{RC_n}(t) = \begin{cases} \frac{(-1)^n}{2}\left[1-\cos\left(\frac{2\pi f_0}{n}t\right)\right]\cos\left(2\pi f_0 t\right) & 0 < t < \frac{n}{f_0} = nT_0 = T \\ 0 & \text{else / sonst} \end{cases}$$

Raised cosine pulse with 2 cycles / Aufsteigender Kosinus-Impuls mit 2 Zyklen

$$f_{RC_2}(t) = \begin{cases} \frac{1}{2}\left[1-\cos\left(\pi f t\right)\right]\cos\left(2\pi f_0 t\right) & 0 < t < \frac{2}{f_0} = 2T_0 = T \\ 0 & \text{else / sonst} \end{cases}$$



Frequency / Frequenz

$$f_0 = \frac{1}{T_0}$$

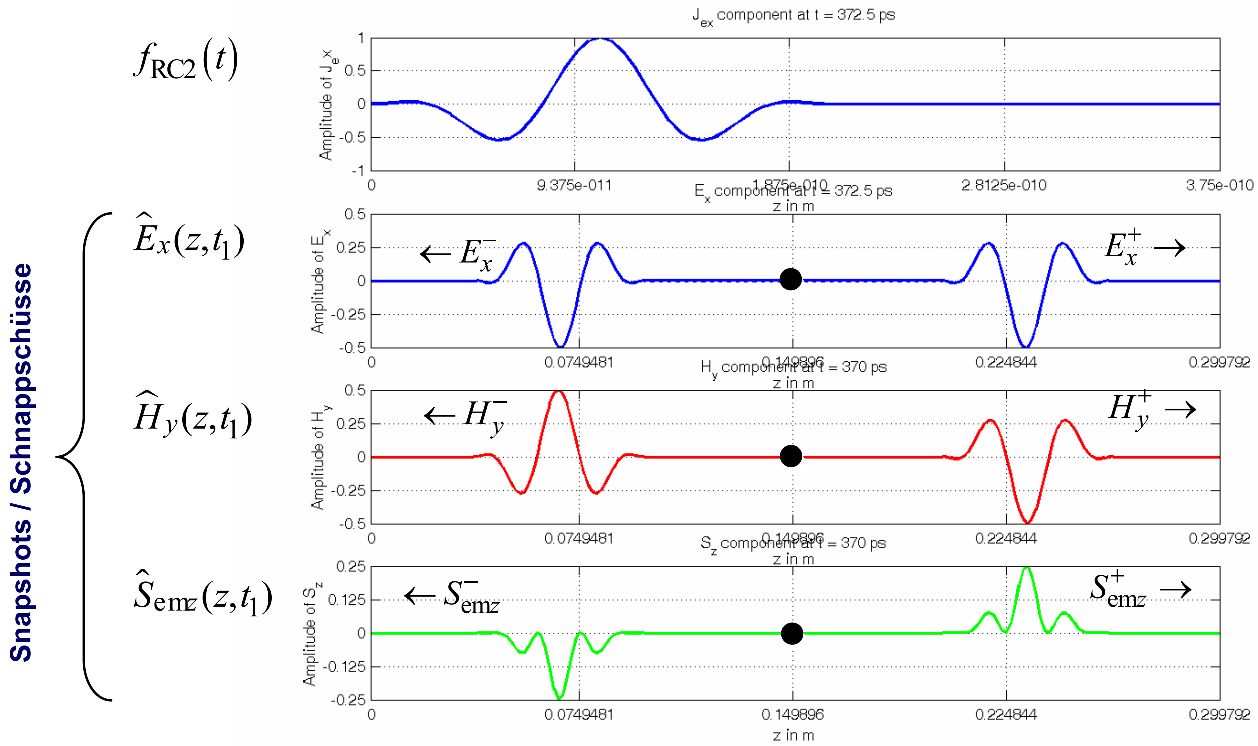
Circular Frequency / Kreisfrequenz

$$\omega_0 = \frac{2\pi}{T_0}$$

FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Electric current density excitation: broadband pulse (RC2 pulse) /
Elektrische Stromdichteanregegung: breitbandiger Impuls (RC2-Impuls)

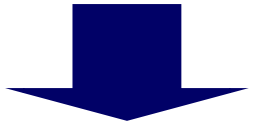
$$J_{ex}(z=z_0,t) \sim f_{\mathrm{RC2}}(t) \rightarrow E_x(z,t) \sim f_{\mathrm{RC2}}\left[t \mp \frac{z-z_0}{c_0}\right]$$



● Source point / Quellpunkt

Numerical Results – Validation / Numerische Ergebnisse – Validierung

Numerical Results / Numerische Ergebnisse



Validation / Validierung

Compare numerical results with analytical solutions or with other numerical solutions. / Vergleiche die numerischen Ergebnisse mit analytischen Lösungen oder anderen numerischen Lösungen

End of Lecture 2 /
Ende der 2. Vorlesung