Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

4th Lecture / 4. Vorlesung

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FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dissipation / Dissipation**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

Stability by the *von Neumann*'s method (Fourier series method):

Insert a complex monofrequent (monochromatic) plane wave into the discrete FD equations and analyze the spectral radius of the amplification matrix, where the spectral radius must be smaller equal one.

Stabilität durch die *von Neumann*sche Methode (Fourier-Reihen-Methode):

Setze eine komplex monofrequente (monochromatische) ebene Welle in die diskreten FD-Gleichungen ein und analysiere den spektralen Radius der Verstärkungsmatrix, wobei der spektrale Radius kleinergleich Eins sein muss.

Complex monofrequent (monochromatic) plane wave / Komplex monofrequente (monochromatische) ebene Welle

$$E_x(\underline{\mathbf{R}},t) = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\left(\omega_0 t - k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}\right)}$$

$$= E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\omega_0 t}\; \mathrm{e}^{\mathrm{j}k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}}$$

$$\{\mathbf{W}\}^{(n_t+1)} = [\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}} \{\mathbf{W}\}^{(n_t)}$$

$[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$: Amplification matrix / Verstärkungsmatrix

Spectral radius / Spektraler Radius $\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) \le 1$ of the matrix / der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

where / wobei $\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) = \max\limits_{n=1,\ldots,N} \left|\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)\right|$

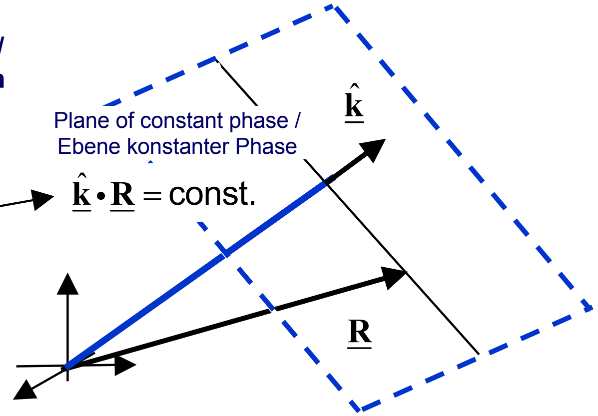
$\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)$: nth eigenvalue of the matrix / n-ter Eigenwert der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

Monofrequent (monochromatic) plane wave in the time domain / Monofrequente (monochromatische) ebene Welle im Zeitbereich

$$E_x(\underline{\mathbf{R}},t) = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\left(\omega_0 t - k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}\right)}$$

$$= E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\omega_0 t}\; \mathrm{e}^{\mathrm{j}k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}}$$



Wave vector / Wellenvektor	$\underline{\mathbf{k}} = \mathrm{k}_x\underline{\mathbf{e}}_x + \mathrm{k}_z\underline{\mathbf{e}}_z + \mathrm{k}_z\underline{\mathbf{e}}_z = \mathrm{k}_z\underline{\mathbf{e}}_z$
Magnitude of the wave vector / Betrag des Wellenvektors	$\lvert\underline{\mathbf{k}}\rvert = \sqrt{\underline{\mathbf{k}}\bullet\underline{\mathbf{k}}} = \sqrt{\mathrm{k}_x^2+\mathrm{k}_y^2+\mathrm{k}_z^2} = \sqrt{\mathrm{k}_z^2} = \lvert k_z\rvert = k$
Wavenumber / Wellenzahl	$k = \dfrac{\omega_0}{c}$
Circular frequency / Kreisfrequenz	$\omega_0 = 2\pi f_0$
Propagation direction / Ausbreitungsrichtung	$\hat{\underline{\mathbf{k}}} = \dfrac{\underline{\mathbf{k}}}{\lvert\underline{\mathbf{k}}\rvert} = \dfrac{\mathrm{k}_z\underline{\mathbf{e}}_z}{\lvert\mathrm{k}_z\rvert} = \dfrac{\mathrm{sgn}(\mathrm{k}_z)\lvert\mathrm{k}_z\rvert\underline{\mathbf{e}}_z}{k} = \dfrac{\mathrm{sgn}(\mathrm{k}_z)k\underline{\mathbf{e}}_z}{k} = \mathrm{sgn}(\mathrm{k}_z)\underline{\mathbf{e}}_z$
Phase of the plane wave / Phase der ebenen Welle	$k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}} = k\,\mathrm{sgn}(\mathrm{k}_z)\underline{\mathbf{e}}_z\bullet(x\underline{\mathbf{e}}_x + y\underline{\mathbf{e}}_z + z\underline{\mathbf{e}}_z) = k\,\mathrm{sgn}(\mathrm{k}_z)z\underline{\mathbf{e}}_z\bullet\underline{\mathbf{e}}_z = k\,\mathrm{sgn}(\mathrm{k}_z)z$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

Insert discrete plane wave / Setze die diskrete ebene Welle

$$\hat{E}_x^{(n_z,n_t)} = \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\underbrace{\mathrm{e}^{\mathrm{j}kn_z\Delta z}}_{=\exp(n_z)}\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t} = \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t}$$

into the FD scheme / in das FD-Schema ein

$$\hat{E}_x^{(n_z,n_t+1)} = 2\hat{E}_x^{(n_z,n_t)} - \hat{E}_x^{(n_z,n_t-1)} + (\widehat{\Delta t})^2\left[\hat{E}_x^{(n_z+1,n_t)} - 2\hat{E}_x^{(n_z,n_t)} + \hat{E}_x^{(n_z-1,n_t)}\right]$$

with / mit

$$\begin{aligned}\hat{E}_x^{(n_z,n_t+1)} &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\underbrace{\mathrm{e}^{\mathrm{j}kn_z\Delta z}}_{=\exp(n_z)}\,\mathrm{e}^{-\mathrm{j}\omega_0(n_t+1)\Delta t}\\ &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{-\mathrm{j}\omega_0(n_t+1)\Delta t}\\ \hat{E}_x^{(n_z,n_t)} &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t}\\ \hat{E}_x^{(n_z,n_t-1)} &= E_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{\mathrm{j}\omega_0\,\mathrm{e}^{-\mathrm{j}\omega_0(n_t-1)\Delta t}\,\Delta t}\end{aligned}$$

$$\begin{aligned}\hat{E}_x^{(n_z+1,n_t)} &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{\mathrm{j}k\Delta z}\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t}\\ \hat{E}_x^{(n_z,n_t)} &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t}\\ \hat{E}_x^{(n_z-1,n_t)} &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\exp(n_z)\,\mathrm{e}^{-\mathrm{j}k\Delta z}\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t}\end{aligned}$$

it follows / folgt

$$\begin{aligned}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z+1)\Delta t} &= 2\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}\\ &\quad + (\widehat{\Delta t})^2\left[\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{\mathrm{j}k\Delta z} - 2\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}}) + \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}k\Delta z}\right]\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}\\ &= 2\left[1-(\widehat{\Delta t})^2\right]\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}\\ &\quad + (\widehat{\Delta t})^2\left[\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{\mathrm{j}k\Delta z} + \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}k\Delta z}\right]\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}\end{aligned}$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

$$\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z+1)\Delta t} = 2\left[1-(\widehat{\Delta t})^2\right]\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}$$
$$+(\widehat{\Delta t})^2\left[\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{\mathrm{j}k\Delta z}+\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}k\Delta z}\right]\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}$$

$$= 2\left[1-(\widehat{\Delta t})^2\right]\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}$$
$$+(\widehat{\Delta t})^2\underbrace{\left[\mathrm{e}^{\mathrm{j}k\Delta z}+\mathrm{e}^{-\mathrm{j}k\Delta z}\right]}_{=2\cos(k\Delta z)}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}$$

$$= 2\left[1-(\widehat{\Delta t})^2\right]\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}$$
$$+(\widehat{\Delta t})^2\,2\cos(k\Delta z)\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}$$

$$\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z+1)\Delta t} = 2\left\{1+(\widehat{\Delta t})^2\left[\cos(k\Delta z)-1\right]\right\}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}$$

$$2\sin^2\left(\frac{\alpha}{2}\right)=1-\cos\alpha \qquad \rightarrow \qquad -2\sin^2\left(\frac{\alpha}{2}\right)=\cos\alpha-1$$

$$\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z+1)\Delta t} = 2\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t} - \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}$$
$$= -\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t} + 2\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

$$\begin{aligned}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z+1)\Delta t} &= 2\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}-\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}\\ &= -\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_z-1)\Delta t}+2\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}\hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_z\Delta t}\end{aligned}$$

Define / Definiere

$$\begin{aligned}U^{(n_t)} &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 n_t\Delta t}\\ V^{(n_t)} &= U^{(n_t-1)}\\ &= \hat{E}_{0x}(\omega_0,\underline{\hat{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0(n_t-1)\Delta t}\end{aligned}$$

which yields for the above equation / womit wir für die obere Gleichung erhalten

$$\begin{aligned}U^{(n_t+1)} &= -U^{(n_t-1)}+2\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}U^{(n_t)}\\ &= -V^{(n_t)}+2\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}U^{(n_t)}\end{aligned}$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

Define a new algebraic vector / Definiere einen neuen algebraischen Vektor

$$\{\mathbf{W}\}^{(n_t)} = \begin{Bmatrix} U^{(n_t)} \\ V^{(n_t)} \end{Bmatrix}$$

$$\underbrace{\begin{Bmatrix} U^{(n_t+1)} \\ V^{(n_t+1)} \end{Bmatrix}}_{=\{\mathbf{W}\}^{(n_t+1)}} = \underbrace{\begin{bmatrix} 2\left\{1-2(\widehat{\Delta t})^2 \sin^2\left(\dfrac{k\Delta z}{2}\right)\right\} & -1 \\ 1 & 0 \end{bmatrix}}_{=[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}} \underbrace{\begin{Bmatrix} U^{(n_t)} \\ V^{(n_t)} \end{Bmatrix}}_{=\{\mathbf{W}\}^{(n_t)}}$$

$$\{\mathbf{W}\}^{(n_t+1)} = [\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}} \{\mathbf{W}\}^{(n_t)}$$

$[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$: Amplification matrix / Verstärkungsmatrix

$$\det\left\{[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}} - \nu[\mathbf{I}]\right\} = 0$$

$\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)$: nth eigenvalue of the matrix / n-ter Eigenwert der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

$$\det\left\{[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}} - \nu[\mathbf{I}]\right\} = \begin{vmatrix} 2\left\{1-2(\widehat{\Delta t})^2 \sin^2\left(\dfrac{k\Delta z}{2}\right)\right\} - \nu & -1 \\ 1 & -\nu \end{vmatrix}$$

$$= \nu^2 - 2\nu\left\{1-2(\widehat{\Delta t})^2 \sin^2\left(\dfrac{k\Delta z}{2}\right)\right\} + 1$$

Characteristic polynomial / Charakteristisches Polynom

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

$$\nu^2 - 2\nu\left[1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right] + \left[1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right]^2 = \left[1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right]^2 - 1$$

$$\left\{\nu - \left[1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right]\right\}^2 = \left[1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right]^2 - 1$$

Eigenvalues of the amplification matrix /
Eigenwerte der Verstärkungsmatrix

$$\nu_{1/2} = \underbrace{\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}}_{=a} \pm \sqrt{\underbrace{\left\{1-2(\widehat{\Delta t})^2\sin^2\left(\frac{k\Delta z}{2}\right)\right\}^2}_{=a^2} - 1}$$

$$= a \pm \sqrt{a^2 - 1}$$

if $a^2 \leq 1$ / **falls** $a^2 \leq 1$

$$= a \pm \mathrm{j}\sqrt{1 - a^2}$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

$$\nu_{1/2} = \underbrace{\left\{1-2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}}_{=a} \pm \sqrt{\underbrace{\left\{1-2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}^2}_{=a^2} - 1}$$

$$= a \pm \sqrt{a^2 - 1}$$

$$= a \pm \mathrm{j}\sqrt{1 - a^2} \quad \text{if } a^2 \le 1 \text{ / falls } a^2 \le 1$$

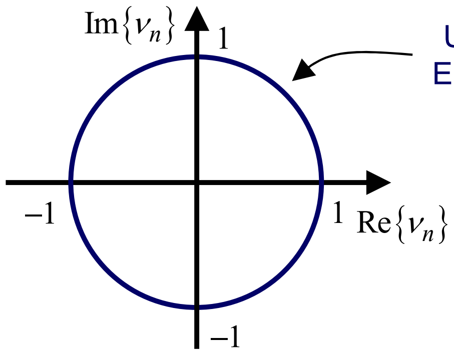
$$\nu_n = \mathrm{Re}\{\nu_n\} + \mathrm{j}\,\mathrm{Im}\{\nu_n\} \qquad n = 1,2$$

$$|\nu_{1/2}| = \left|a \pm \mathrm{j}\sqrt{1-a^2}\right| = a^2 + \left(\sqrt{1-a^2}\right)^2 = a^2 + 1 - a^2$$
$$= 1$$

Spectral radius /
Spektraler Radius

$$\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) \le 1$$

This means for, that all eigenvalues $a^2 \le 1$ are on the unit circle in the complex plane. /
Dies bedeutet, dass alle Eigenwerte für $a^2 \le 1$ auf dem Einheitskreis in der komplexen Ebene liegen.



Unit circle /
Einheitskreis

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

$$v_{1/2} = \underbrace{\left\{1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}}_{=a} \pm \mathrm{j}\sqrt{1 - \underbrace{\left\{1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}^2}_{=a^2}}$$

$$= a \pm \mathrm{j}\sqrt{1 - a^2} \quad \text{if } a^2 \le 1 \text{ / falls } a^2 \le 1$$

$$a^2 \le 1$$

$$\left\{1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}^2 \le 1$$

$$1 - 4(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right) + 4(\widehat{\Delta t})^4 \sin^4\left(\frac{k\Delta z}{2}\right) \le 1$$

$$-4(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\left[1 - (\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right] \le 0$$

$$1 - (\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right) \le 0$$

$$(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right) \le 1$$

$$(\widehat{\Delta t})^2 \le 1 \quad \textbf{because / weil } \max\left\{\sin^2\left(\frac{k\Delta z}{2}\right)\right\} = 1$$

$$\widehat{\Delta t} \le 1$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

1-D Stability Condition for an FD algorithm of 2nd order in space and time– CFL-Condition /
1D-Stabilitätsbedingung für einen FD-Algorithmus zweiter Ordnung in Raum und Zeit– CFL-Bedingung

$$\text{1-D / 1D:} \quad \Delta t \le \Delta t_{\max} = \frac{\Delta x}{c} \qquad \widehat{\Delta t} \le 1$$

2-D and 3-D Stability Condition for an FD algorithm of 2nd order in space and time– CFL-Condition /
2D- und 3D- Stabilitätsbedingung für einen FD-Algorithmus zweiter Ordnung in Raum und Zeit– CFL-Bedingung

$$\text{2-D / 2D:} \quad \Delta t \le \Delta t_{\max} = \frac{1}{\sqrt{2}} \frac{\Delta x}{c} \qquad \widehat{\Delta t} \le \frac{1}{\sqrt{2}} \approx 0.707$$

$$\text{3-D / 3D:} \quad \Delta t \le \Delta t_{\max} = \frac{1}{\sqrt{3}} \frac{\Delta x}{c} \qquad \widehat{\Delta t} \le \frac{1}{\sqrt{3}} \approx 0.577$$

$$\widehat{\Delta t} = \frac{\Delta t}{\Delta t_{\text{ref}}} \quad : \quad \textbf{Courant number / Courant-Zahl} \qquad \Delta t_{\text{ref}} = \frac{\Delta x}{c}$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

$$v_{1/2}(\widehat{\Delta t}) = a \pm \sqrt{a^2 - 1} \quad \text{if } a^2 \geq 1 \text{ / falls } a^2 \geq 1$$
$$= a \pm \mathrm{j}\sqrt{1 - a^2} \quad \text{if } a^2 \leq 1 \text{ / falls } a^2 \leq 1 \quad \text{with } a = \left\{1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}$$

$a^2 \leq 1: \quad v_{1/2} = a \pm \mathrm{j}\sqrt{1 - a^2}$

$v_1 = a + \mathrm{j}\sqrt{1 - a^2} \qquad v_2 = a - \mathrm{j}\sqrt{1 - a^2}$

$|v_1| = 1 \qquad |v_2| = 1$

Spectral radius / Spektraler Radius $\quad \rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) \leq 1$

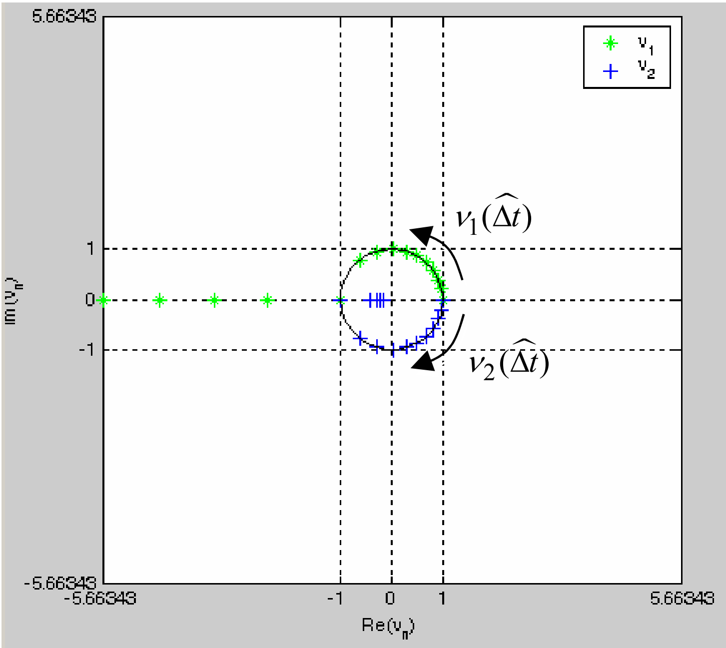
$a^2 > 1: \quad v_{1/2} = a \pm \sqrt{a^2 - 1}$

$v_1 = a + \sqrt{a^2 - 1} \qquad v_2 = a - \sqrt{a^2 - 1}$

$\lim_{a \to \infty} |v_1| \to \infty \qquad \lim_{a \to \infty} |v_2| \to 0$

Spectral radius / Spektraler Radius $\quad \rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) \geq 1$

$v_{1/2}(\widehat{\Delta t})$ as a function of / als Funktion von $(\widehat{\Delta t})$



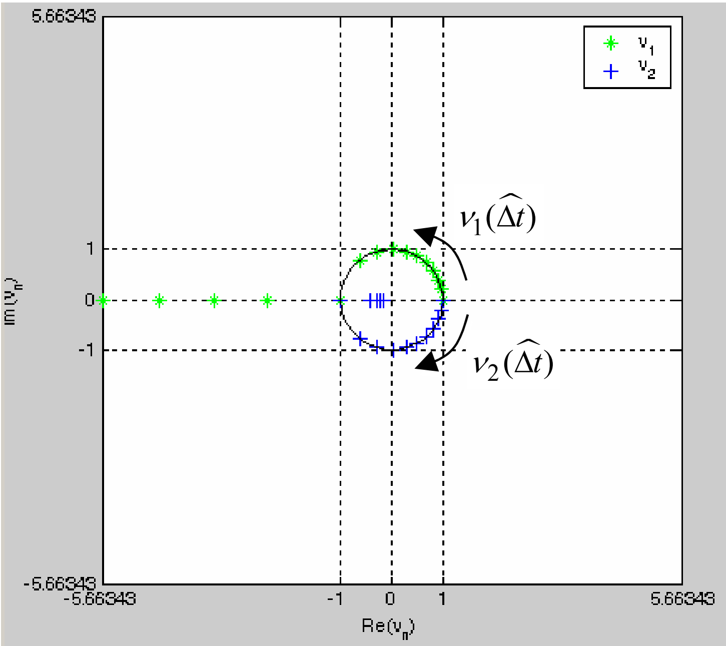
Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

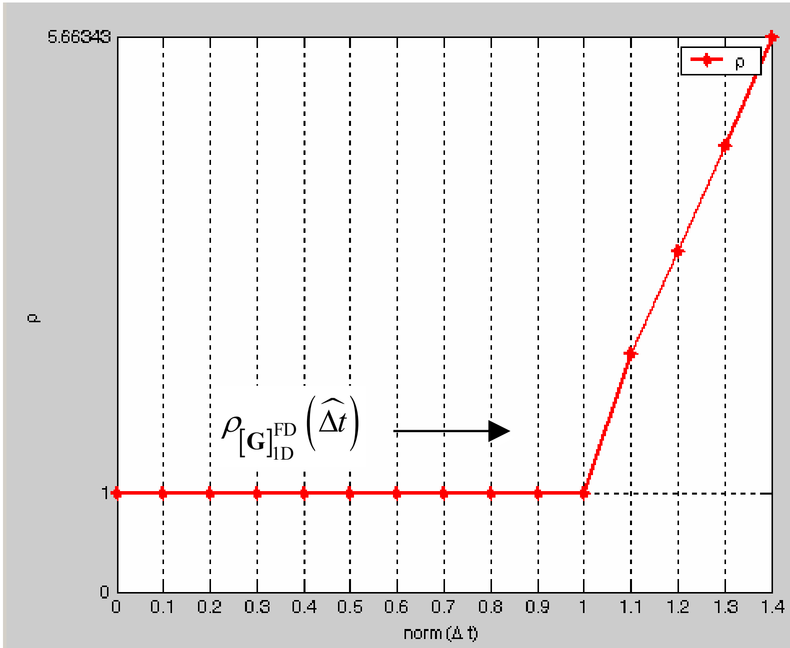
Eigenvalues / Eigenwerte

$$\nu_{1/2}(\widehat{\Delta t}) = a \pm \sqrt{a^2 - 1} \quad \text{if } a^2 \geq 1 \text{ / falls } a^2 \geq 1$$
$$= a \pm \mathrm{j}\sqrt{1 - a^2} \quad \text{if } a^2 \leq 1 \text{ / falls } a^2 \leq 1$$
$$\text{with } a = \left\{1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right)\right\}$$

Spectral radius / Spektraler Radius

$$\rho_{[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}}\left(\widehat{\Delta t}\right)$$





3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

Maxwell's equations / Maxwellsche Gleichungen

$$\left\{\begin{aligned} \frac{\partial}{\partial t}\underline{\mathbf{B}}(\underline{\mathbf{R}},t) &= -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) \\ \frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) &= \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) \end{aligned}\right.$$

$$\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

Constitutive Equations for Vacuum / Konstituierende Gleichungen (Materialgleichungen) für Vakuum

$$\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \mu_0\,\underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \varepsilon_0\,\underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

Continuity equations / Kontinuitätsgleichungen

$$\nabla\bullet\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) = -\frac{\partial}{\partial t}\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) = -\frac{\partial}{\partial t}\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\frac{1}{\mu_0}\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{\mu_0}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \frac{1}{\varepsilon_0}\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \frac{1}{\varepsilon_0}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$\frac{\partial}{\partial t}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\frac{1}{\mu_0}\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{\mu_0}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) \quad (1)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \frac{1}{\varepsilon_0}\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \frac{1}{\varepsilon_0}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) \quad (2)$$

$$\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\frac{1}{\mu_0}\nabla\times\frac{\partial}{\partial t}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{\mu_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) \quad (3)$$

$$\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \frac{1}{\varepsilon_0}\nabla\times\frac{\partial}{\partial t}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) \quad (4)$$

$$\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\frac{1}{\mu_0}\nabla\times\left[\frac{1}{\varepsilon_0}\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \frac{1}{\varepsilon_0}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\right] - \frac{1}{\mu_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) \quad (5)$$

$$\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \frac{1}{\varepsilon_0}\nabla\times\left[-\frac{1}{\mu_0}\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{\mu_0}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\right] - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) \quad (6)$$

$$\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\frac{1}{\varepsilon_0\mu_0}\nabla\times\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) + \frac{1}{\varepsilon_0\mu_0}\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) - \frac{1}{\mu_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) \quad (7)$$

$$\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = -\frac{1}{\varepsilon_0\mu_0}\nabla\times\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{\varepsilon_0\mu_0}\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) \quad (8)$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$\frac{1}{\varepsilon_0\mu_0}\nabla\times\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t)+\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=+\frac{1}{\varepsilon_0\mu_0}\nabla\times\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)-\frac{1}{\mu_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t) \qquad (1)$$

$$\frac{1}{\varepsilon_0\mu_0}\nabla\times\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t)+\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=-\frac{1}{\varepsilon_0\mu_0}\nabla\times\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)-\frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t) \qquad (2)$$

$$c_0=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$-\nabla\times\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=-\nabla\times\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t) \qquad (3)$$

$$-\nabla\times\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\nabla\times\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t) \qquad (4)$$

Vector identity / Vektoridentität

$$\nabla\times\nabla\times=\nabla\nabla\bullet-\underbrace{\nabla\bullet\nabla}_{=\nabla^2=\Delta}=\nabla\nabla\bullet-\Delta$$

Short-hand notation / Abkürzende Schreibweise

$$\nabla\bullet\nabla=\nabla^2=\Delta$$

$$-\left[\nabla\nabla\bullet-\Delta\right]\underline{\mathbf{H}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=-\nabla\times\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t) \qquad (5)$$

$$-\left[\nabla\nabla\bullet-\Delta\right]\underline{\mathbf{E}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\nabla\times\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t) \qquad (6)$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$-\left[\nabla\nabla\bullet\;-\Delta\right]\underline{\mathbf{H}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=-\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$-\left[\nabla\nabla\bullet\;-\Delta\right]\underline{\mathbf{E}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

3rd and 4th Maxwell's equations / 3. und 4. Maxwellsche Gleichung

$$\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)=\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}},t)=\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

Constitutive equations / Materialgleichungen

$$\underline{\mathbf{B}}(\underline{\mathbf{R}},t)=\mu_0\,\underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t)=\varepsilon_0\,\underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$\Rightarrow$

$$\nabla\bullet\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=\frac{1}{\mu_0}\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\frac{1}{\varepsilon_0}\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{H}}(\underline{\mathbf{R}},t)-\nabla\underbrace{\nabla\bullet\;\underline{\mathbf{H}}(\underline{\mathbf{R}},t)}_{=\frac{1}{\mu_0}\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=-\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t)-\nabla\underbrace{\nabla\bullet\;\underline{\mathbf{E}}(\underline{\mathbf{R}},t)}_{=\frac{1}{\varepsilon_0}\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$\Delta\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \nabla\left[\frac{1}{\mu_0}\rho_\mathrm{m}(\underline{\mathbf{R}},t)\right] - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t) + \varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \nabla\left[\frac{1}{\varepsilon_0}\rho_\mathrm{e}(\underline{\mathbf{R}},t)\right] - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t) + \mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t) + \varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t) + \frac{1}{\mu_0}\nabla\rho_\mathrm{m}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t) + \mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t) + \frac{1}{\varepsilon_0}\nabla\rho_\mathrm{e}(\underline{\mathbf{R}},t)$$

Laplace operator in Cartesian coordinates /
Laplace-Operator in Kartesischen Koordinaten

$$\begin{aligned}\Delta &= \nabla\bullet\nabla \\ &= \left(\underline{\mathrm{e}}_x\frac{\partial}{\partial x} + \underline{\mathrm{e}}_y\frac{\partial}{\partial y} + \underline{\mathrm{e}}_z\frac{\partial}{\partial z}\right)\bullet\left(\underline{\mathrm{e}}_x\frac{\partial}{\partial x} + \underline{\mathrm{e}}_y\frac{\partial}{\partial y} + \underline{\mathrm{e}}_z\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\end{aligned}$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \nabla \bullet \left[\nabla\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]$$
$$= \left(\underline{\mathrm{e}}_x \frac{\partial}{\partial x} + \underline{\mathrm{e}}_y \frac{\partial}{\partial y} + \underline{\mathrm{e}}_z \frac{\partial}{\partial z}\right) \bullet \left[\left(\underline{\mathrm{e}}_x \frac{\partial}{\partial x} + \underline{\mathrm{e}}_y \frac{\partial}{\partial y} + \underline{\mathrm{e}}_z \frac{\partial}{\partial z}\right)\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]$$

Short-hand notation / Abkürzende Schreibweise $\quad \frac{\partial}{\partial x} = \partial_x \quad \frac{\partial}{\partial y} = \partial_y \quad \frac{\partial}{\partial z} = \partial_z \quad \frac{\partial}{\partial t} = \partial_t$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \nabla \bullet \left[\nabla\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]$$
$$= \left(\underline{\mathrm{e}}_x\partial_x + \underline{\mathrm{e}}_y\partial_y + \underline{\mathrm{e}}_z\partial_z\right) \bullet \left[\left(\underline{\mathrm{e}}_x\partial_x + \underline{\mathrm{e}}_y\partial_y + \underline{\mathrm{e}}_z\partial_z\right)\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]$$

$$\left(\underline{\mathrm{e}}_x\partial_x + \underline{\mathrm{e}}_y\partial_y + \underline{\mathrm{e}}_z\partial_z\right)\left[E_x(\underline{\mathbf{R}},t)\underline{\mathrm{e}}_x + E_y(\underline{\mathbf{R}},t)\underline{\mathrm{e}}_y + E_z(\underline{\mathbf{R}},t)\underline{\mathrm{e}}_z\right]$$
$$= \underline{\mathrm{e}}_x\underline{\mathrm{e}}_x\partial_x E_x(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_x\underline{\mathrm{e}}_y\partial_x E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_x\underline{\mathrm{e}}_z\partial_x E_z(\underline{\mathbf{R}},t)$$
$$+ \underline{\mathrm{e}}_y\underline{\mathrm{e}}_x\partial_y E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_y\underline{\mathrm{e}}_y\partial_y E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_y\underline{\mathrm{e}}_z\partial_y E_z(\underline{\mathbf{R}},t)$$
$$+ \underline{\mathrm{e}}_z\underline{\mathrm{e}}_x\partial_x E_z(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_z\underline{\mathrm{e}}_y\partial_z E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_z\underline{\mathrm{e}}_z\partial_z E_z(\underline{\mathbf{R}},t)$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$
\begin{aligned}
\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) &= \nabla\bullet\left[\nabla\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\\
&= \left(\underline{\mathrm{e}}_x\partial_x+\underline{\mathrm{e}}_y\partial_y+\underline{\mathrm{e}}_z\partial_z\right)\bullet\left[\left(\underline{\mathrm{e}}_x\partial_x+\underline{\mathrm{e}}_y\partial_y+\underline{\mathrm{e}}_z\partial_z\right)\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\\
&= \left(\underline{\mathrm{e}}_x\partial_x+\underline{\mathrm{e}}_y\partial_y+\underline{\mathrm{e}}_z\partial_z\right)\bullet\Big[\ \underline{\mathrm{e}}_x\underline{\mathrm{e}}_x\partial_xE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_y\partial_xE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_z\partial_xE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_y\underline{\mathrm{e}}_x\partial_xE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_y\partial_yE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_z\partial_yE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_z\underline{\mathrm{e}}_x\partial_xE_z(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_y\partial_yE_z(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_z\partial_zE_z(\underline{\mathbf{R}},t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) &= \nabla\bullet\left[\nabla\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\\
&= \underline{\mathrm{e}}_x\partial_x\bullet\Big[\ \underline{\mathrm{e}}_x\underline{\mathrm{e}}_x\partial_xE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_y\partial_xE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_z\partial_xE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_y\underline{\mathrm{e}}_x\partial_yE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_y\partial_yE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_z\partial_yE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_z\underline{\mathrm{e}}_x\partial_zE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_y\partial_zE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_z\partial_zE_z(\underline{\mathbf{R}},t)\Big]\\
&\quad +\underline{\mathrm{e}}_y\partial_y\bullet\Big[\ \underline{\mathrm{e}}_x\underline{\mathrm{e}}_x\partial_xE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_y\partial_xE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_z\partial_zE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_y\underline{\mathrm{e}}_x\partial_yE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_y\partial_yE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_z\partial_yE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_z\underline{\mathrm{e}}_x\partial_zE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_y\partial_zE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_z\partial_zE_z(\underline{\mathbf{R}},t)\Big]\\
&\quad +\underline{\mathrm{e}}_z\partial_z\bullet\Big[\ \underline{\mathrm{e}}_x\underline{\mathrm{e}}_x\partial_xE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_y\partial_xE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_x\underline{\mathrm{e}}_z\partial_xE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_y\underline{\mathrm{e}}_x\partial_yE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_y\partial_yE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_y\underline{\mathrm{e}}_z\partial_yE_z(\underline{\mathbf{R}},t)\\
&\qquad +\underline{\mathrm{e}}_z\underline{\mathrm{e}}_x\partial_zE_x(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_y\partial_zE_y(\underline{\mathbf{R}},t)+\underline{\mathrm{e}}_z\underline{\mathrm{e}}_z\partial_zE_z(\underline{\mathbf{R}},t)\Big]
\end{aligned}
$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \nabla \bullet \left[\nabla \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]$$

$$= \Big[\ \underline{\mathrm{e}}_x \partial_x^2 E_x(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_y \partial_x^2 E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_z \partial_x^2 E_z(\underline{\mathbf{R}},t)$$

$$+ \Big[\ \underline{\mathrm{e}}_x \partial_y^2 E_x(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_y \partial_y^2 E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_z \partial_y^2 E_z(\underline{\mathbf{R}},t)$$

$$+ \Big[\ \underline{\mathrm{e}}_x \partial_z^2 E_x(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_y \partial_z^2 E_y(\underline{\mathbf{R}},t) + \underline{\mathrm{e}}_z \partial_z^2 E_z(\underline{\mathbf{R}},t)$$

$$= \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \underbrace{\left[E_x(\underline{\mathbf{R}},t)\underline{\mathrm{e}}_x + E_y(\underline{\mathbf{R}},t)\underline{\mathrm{e}}_y + E_z(\underline{\mathbf{R}},t)\underline{\mathrm{e}}_z\right]}_{=\underline{\mathbf{E}}(\underline{\mathbf{R}},t)}$$

$$= \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

3-D Electromagnetic Wave Propagation / 3D elektromagnetische Wellenausbreitung

$$\Delta\underline{\mathbf{H}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=-\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)+\frac{1}{\mu_0}\nabla\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)+\frac{1}{\varepsilon_0}\nabla\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\left(\partial_x^2+\partial_y^2+\partial_z^2\right)\underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\Delta\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=\left(\partial_x^2+\partial_y^2+\partial_z^2\right)\underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{H}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=-\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)+\frac{1}{\mu_0}\nabla\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{E}}(\underline{\mathbf{R}},t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)+\frac{1}{\varepsilon_0}\nabla\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)=-\frac{\partial}{\partial t}\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)=-\frac{\partial}{\partial t}\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

2-D EM Wave Propagation – 2-D TM Case and 2-D TE Case / 2D EM Wellenausbreitung – 2D-TM-Fall und 2D-TE-Fall

We consider the xz plane and assume that the field is independent of y /
Wir betrachten die xz-Ebene und nehmen an, dass das Feld unabhängig von y ist

$$\Rightarrow \quad \frac{\partial}{\partial y} \equiv 0$$

Then it follows for the 3-D wave equations / Es folgt dann für die 3D-Wellengleichungen

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{H}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(x,z,t)=-\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(x,z,t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(x,z,t)+\frac{1}{\mu_0}\nabla\rho_{\mathrm{m}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{E}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(x,z,t) = \nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(x,z,t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(x,z,t)+\frac{1}{\varepsilon_0}\nabla\rho_{\mathrm{e}}(x,z,t)$$

And we confine the current sources to / Und wir beschränken die Stromquellen auf

$$\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)=\underline{\mathbf{J}}_{\mathrm{m}}(x,y,t)=J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y$$

$$\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)=\underline{\mathbf{J}}_{\mathrm{e}}(x,y,t)=J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y$$

This yields for the above given 3-D wave equation /
Dies ergibt für die oben gegebenen 3D-Wellengleichungen

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{H}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(x,z,t)=\nabla\times\left[J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y\right]+\varepsilon_0\frac{\partial}{\partial t}J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y+\frac{1}{\mu_0}\nabla\rho_{\mathrm{m}}(x,y,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{E}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(x,z,t) = \nabla\times\left[J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y\right]+\mu_0\frac{\partial}{\partial t}J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y+\frac{1}{\varepsilon_0}\nabla\rho_{\mathrm{e}}(x,y,t)$$

2-D EM Wave Propagation – 2-D TM Case and 2-D TE Case / 2D EM Wellenausbreitung – 2D-TM-Fall und 2D-TE-Fall

Curl and divergence of the current sources /
Rotation und Divergenz der Stromquellen

$$\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) = J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y$$

$$\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) = J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y$$

$$\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) = \begin{vmatrix} \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & J_{\mathrm{e}z}(x,z,t) & 0 \end{vmatrix} = \frac{\partial}{\partial x}J_{\mathrm{e}z}(x,z,t)\underline{\mathbf{e}}_z - \frac{\partial}{\partial z}J_{\mathrm{e}z}(x,z,t)\underline{\mathbf{e}}_x = -\frac{\partial}{\partial z}J_{\mathrm{e}z}(x,z,t)\underline{\mathbf{e}}_x + \frac{\partial}{\partial x}J_{\mathrm{e}z}(x,z,t)\underline{\mathbf{e}}_z$$

$$\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) = \begin{vmatrix} \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & J_{\mathrm{m}z}(x,z,t) & 0 \end{vmatrix} = \frac{\partial}{\partial x}J_{\mathrm{m}z}(x,z,t)\underline{\mathbf{e}}_z - \frac{\partial}{\partial z}J_{\mathrm{m}z}(x,z,t)\underline{\mathbf{e}}_x = -\frac{\partial}{\partial z}J_{\mathrm{m}z}(x,z,t)\underline{\mathbf{e}}_x + \frac{\partial}{\partial x}J_{\mathrm{m}z}(x,z,t)\underline{\mathbf{e}}_z$$

$$\nabla\bullet J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y = \left(\underline{\mathbf{e}}_x\frac{\partial}{\partial x} + \underline{\mathbf{e}}_y\frac{\partial}{\partial y} + \underline{\mathbf{e}}_z\frac{\partial}{\partial z}\right)\bullet J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y = \frac{\partial}{\partial y}J_{\mathrm{m}y}(x,z,t) = 0$$

$$\nabla\bullet J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y = \left(\underline{\mathbf{e}}_x\frac{\partial}{\partial x} + \underline{\mathbf{e}}_y\frac{\partial}{\partial y} + \underline{\mathbf{e}}_z\frac{\partial}{\partial z}\right)\bullet J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y = \frac{\partial}{\partial y}J_{\mathrm{e}y}(x,z,t) = 0$$

The divergence of the current sources is in this special case zero, because the currents are constant in y direction. / Die Divergenz der Stromquellen ist in diesem speziellen Fall null, da die Ströme in y-Richtung konstant sind.

2-D EM Wave Propagation – 2-D TM Case and 2-D TE Case / 2D EM Wellenausbreitung – 2D-TM-Fall und 2D-TE-Fall

$$\nabla \bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t) = -\frac{\partial}{\partial t}\rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\nabla \bullet \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t) = -\frac{\partial}{\partial t}\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla \bullet \underline{\mathbf{J}}_{\mathrm{m}}(x,z,\omega) = \mathrm{j}\omega\rho_{\mathrm{m}}(x,z,\omega)$$

$$\nabla \bullet \underline{\mathbf{J}}_{\mathrm{e}}(x,z,\omega) = \mathrm{j}\omega\rho_{\mathrm{e}}(x,z,\omega)$$

$\Rightarrow$

$$\rho_{\mathrm{m}}(x,z,\omega) = \frac{1}{\mathrm{j}\omega}\nabla \bullet \underline{\mathbf{J}}_{\mathrm{m}}(x,z,\omega)$$

$$\rho_{\mathrm{e}}(x,z,\omega) = \frac{1}{\mathrm{j}\omega}\nabla \bullet \underline{\mathbf{J}}_{\mathrm{e}}(x,z,\omega)$$

$$\nabla \bullet J_{\mathrm{m}y}(x,z,\omega)\underline{\mathbf{e}}_y = 0$$

$$\nabla \bullet J_{\mathrm{e}y}(x,z,\omega)\underline{\mathbf{e}}_y = 0$$

$$\rho_{\mathrm{m}}(\underline{\mathbf{R}},\omega) = 0$$

$$\rho_{\mathrm{e}}(\underline{\mathbf{R}},\omega) = 0$$

$\Rightarrow$

$$\rho_{\mathrm{m}}(\underline{\mathbf{R}},t) = 0$$

$$\rho_{\mathrm{e}}(\underline{\mathbf{R}},t) = 0$$

2-D EM Wave Propagation – 2-D TM Case and 2-D TE Case / 2D EM Wellenausbreitung – 2D-TM-Fall und 2D-TE-Fall

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{H}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(x,z,t)=-\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(x,z,t)+\varepsilon_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{m}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{E}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(x,z,t)=\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(x,z,t)+\mu_0\frac{\partial}{\partial t}\underline{\mathbf{J}}_{\mathrm{e}}(x,z,t)$$

$$\nabla\times\underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)=\frac{\partial}{\partial x}J_{\mathrm{e}z}(x,z,t)\underline{\mathbf{e}}_z-\frac{\partial}{\partial z}J_{\mathrm{e}z}(x,z,t)\underline{\mathbf{e}}_x$$

$$\nabla\times\underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)=\frac{\partial}{\partial x}J_{\mathrm{m}z}(x,z,t)\underline{\mathbf{e}}_z-\frac{\partial}{\partial z}J_{\mathrm{m}z}(x,z,t)\underline{\mathbf{e}}_x$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{H}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(x,z,t)=-\frac{\partial}{\partial x}J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_z+\frac{\partial}{\partial z}J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_x+\varepsilon_0\frac{\partial}{\partial t}J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_y$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{E}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(x,z,t)=\frac{\partial}{\partial x}J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_z-\frac{\partial}{\partial z}J_{\mathrm{m}y}(x,z,t)\underline{\mathbf{e}}_x+\mu_0\frac{\partial}{\partial t}J_{\mathrm{e}y}(x,z,t)\underline{\mathbf{e}}_y$$

2-D EM Wave Propagation – 2-D TM Case and 2-D TE Case / 2D EM Wellenausbreitung – 2D-TM-Fall und 2D-TE-Fall

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{H}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{H}}(x,z,t)=-\frac{\partial}{\partial x}J_{\text{ey}}(x,z,t)\underline{\mathbf{e}}_z+\frac{\partial}{\partial z}J_{\text{ey}}(x,z,t)\underline{\mathbf{e}}_x+\varepsilon_0\frac{\partial}{\partial t}J_{\text{my}}(x,z,t)\underline{\mathbf{e}}_y$$

Decoupled equations / Entkoppelte Gleichungen ➡

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_x(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_x(x,z,t)=\frac{\partial}{\partial z}J_{\text{ey}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_y(x,z,t)=\varepsilon_0\frac{\partial}{\partial t}J_{\text{my}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_z(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_z(x,z,t)=-\frac{\partial}{\partial x}J_{\text{ey}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{E}}(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(x,z,t)=-\frac{\partial}{\partial z}J_{\text{my}}(x,z,t)\underline{\mathbf{e}}_x+\mu_0\frac{\partial}{\partial t}J_{\text{ey}}(x,z,t)\underline{\mathbf{e}}_y+\frac{\partial}{\partial x}J_{\text{my}}(x,z,t)\underline{\mathbf{e}}_z$$

Decoupled equations / Entkoppelte Gleichungen ➡

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_x(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_x(x,z,t)=-\frac{\partial}{\partial z}J_{\text{my}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t)=\mu_0\frac{\partial}{\partial t}J_{\text{ey}}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_z(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_z(x,z,t)=\frac{\partial}{\partial x}J_{\text{my}}(x,z,t)$$

2-D EM Wave Propagation – 2-D TM Case and 2-D TE Case / 2D EM Wellenausbreitung – 2D-TM-Fall und 2D-TE-Fall

Separation in 2-D → TM and TE case / Separation in 2D → TM- und TE- Fall

TM: transversal magnetic / transversal magnetisch
TE: transversal electric / transversal elektrisch

TM$_y$ case / TM$_y$-Fall

$E_y(x,z,t)$
$H_x(x,z,t)$
$H_z(x,z,t)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t) = \mu_0\frac{\partial}{\partial t}J_{ey}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_x(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_x(x,z,t) = \frac{\partial}{\partial z}J_{ey}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_z(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_z(x,z,t) = -\frac{\partial}{\partial x}J_{ey}(x,z,t)$$

TE$_y$ case / TE$_y$-Fall

$H_y(x,z,t)$
$E_x(x,z,t)$
$E_z(x,z,t)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_y(x,z,t) = \varepsilon_0\frac{\partial}{\partial t}J_{my}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_x(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_x(x,z,t) = -\frac{\partial}{\partial z}J_{my}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_z(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_z(x,z,t) = \frac{\partial}{\partial x}J_{my}(x,z,t)$$

2-D EM Wave Propagation – 2-D TM Case / 2D EM Wellenausbreitung – 2D-TM-Fall

Separation in 2-D → TM case /
Separation in 2D → TM-Fall

TM: transversal magnetic / transversal magnetisch

TM_y case / TM_y-Fall

$E_y(x,z,t)$

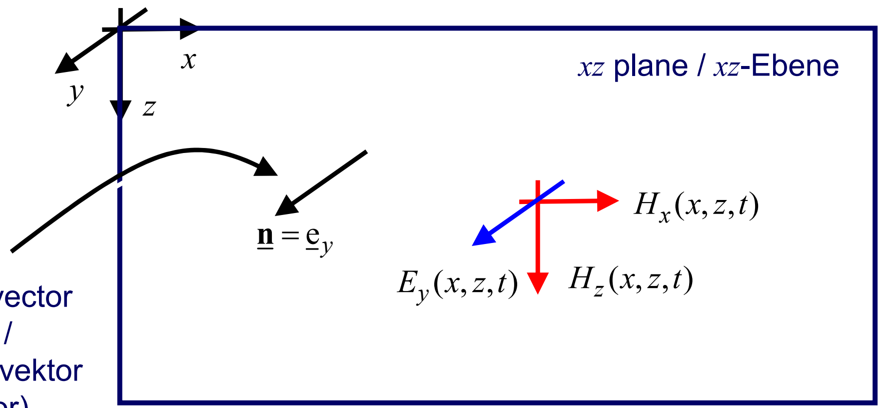
$H_x(x,z,t)$

$H_z(x,z,t)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t) = \mu_0\frac{\partial}{\partial t}J_{ey}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_x(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_x(x,z,t) = \frac{\partial}{\partial z}J_{ey}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_z(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_z(x,z,t) = -\frac{\partial}{\partial x}J_{ey}(x,z,t)$$



Surface normal vector (unit-vector) / Flächennormalenvektor (Einheitsvektor)

2-D EM Wave Propagation – 2-D TE Case / 2D EM Wellenausbreitung – 2D-TE-Fall

Separation in 2-D → TE case /
Separation in 2D → TE- Fall

TE: transversal electric / transversal elektrisch

TE$_y$ case / TE$_y$-Fall

$H_y(x,z,t)$

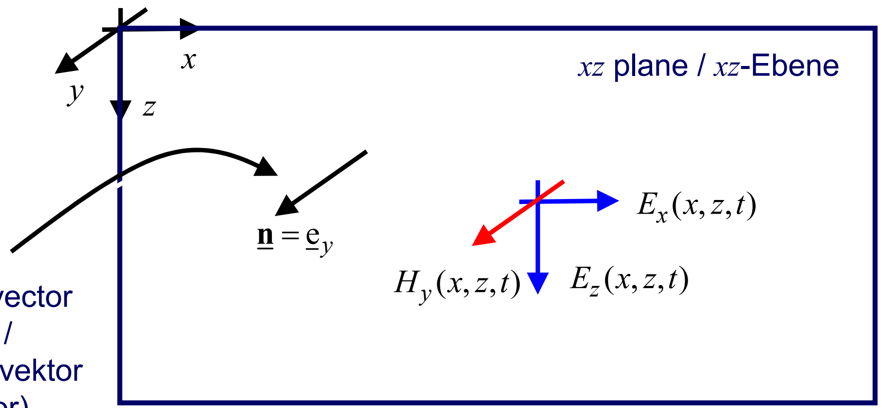
$E_x(x,z,t)$

$E_z(x,z,t)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)H_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}H_y(x,z,t)=\varepsilon_0\frac{\partial}{\partial t}J_{my}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_x(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_x(x,z,t)=-\frac{\partial}{\partial z}J_{my}(x,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_z(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_z(x,z,t)=\frac{\partial}{\partial x}J_{my}(x,z,t)$$



Surface normal vector (unit-vector) / Flächennormalenvektor (Einheitsvektor)

FD Method – 2-D TM Wave Equation / FD-Methode – 2D-TM-Wellengleichung

Central FD Operators / Zentrale FD-Operatoren

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t) = \mu_0\frac{\partial}{\partial t}J_{ey}(x,z,t)$$

Central FD Operators / Zentrale FD-Operatoren

$$\frac{\partial^2}{\partial x^2}E_y(x,z,t)=\frac{E_y(x+\Delta x,z,t)-2E_y(x,z,t)+E_y(x-\Delta x,z,t)}{(\Delta x)^2}+\bigcirc[(\Delta x)^2]$$

$$\frac{\partial^2}{\partial z^2}E_y(x,z,t)=\frac{E_y(x,z+\Delta z,t)-2E_y(x,z,t)+E_y(x,z-\Delta z,t)}{(\Delta z)^2}+\bigcirc[(\Delta z)^2]$$

$$\frac{\partial^2}{\partial t^2}E_y(x,z,t)=\frac{E_y(x,z,t+\Delta t)-2E_y(x,z,t)+E_y(x,z,t-\Delta t)}{(\Delta t)^2}+\bigcirc[(\Delta t)^2]$$

Backward FD Operator / Rückwärts-FD-Operator

$$\frac{\partial}{\partial t}J_{ey}(x,z,t)=\frac{J_{ey}(x.z,t)-J_{ey}(x,z,t-\Delta t)}{\Delta t}+\bigcirc(\Delta t)$$

FD Method – 2-D TM Wave Equation / FD-Methode – 2D-TM-Wellengleichung

2-D TM wave equation / 2D-TM-Wellengleichung

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t) = \mu_0\frac{\partial}{\partial t}J_{\mathrm{ey}}(x,z,t)$$

Explicit FD algorithm in the time domain of 2nd order in space and time / Expliziter FD-Algorithmus im Zeitbereich 2ter Ordnung in Raum und Zeit

$$\frac{E_y(x+\Delta x,z,t)-2E_y(x,z,t)+E_y(x-\Delta x,z,t)}{(\Delta x)^2}+\frac{E_y(x,z+\Delta z,t)-2E_y(x,z,t)+E_y(x,z-\Delta z,t)}{(\Delta z)^2}$$

$$-\frac{1}{c_0^2}\frac{E_y(x,z,t+\Delta t)-2E_y(x,z,t)+E_y(x,z,t-\Delta t)}{(\Delta t)^2}$$

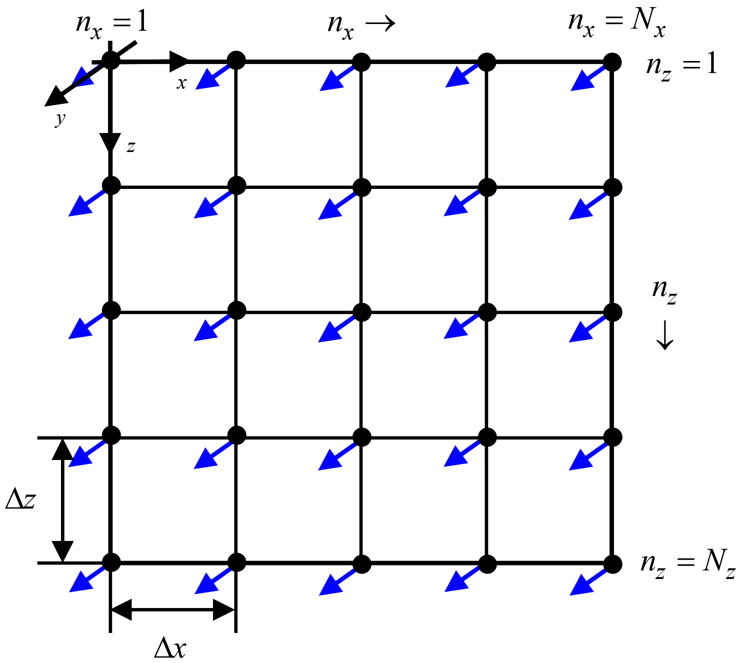
$$=\mu_0\frac{J_{\mathrm{ey}}(x,z,t)-J_{\mathrm{ey}}(x,z,t-\Delta t)}{\Delta t}+\bigcirc[(\Delta x)^2]+\bigcirc[(\Delta z)^2]+\bigcirc[(\Delta t)^2]$$

FD Method – 2-D TM Wave Equation – 2-D FD Grid / FD-Methode – 2D-TM-Wellengleichung – 2D-FD-Gitter

2-D FD grid /
2D-FD-Gitter

$$E_y(x,z,t) \rightarrow E_y^{(n_x,n_z,n_t)} \rightarrow E_y^{(n,n_t)}$$

$$E_y(x,z,t) = E_y^{(n_x,n_z,n_t)} = E_y^{(n,n_t)}$$



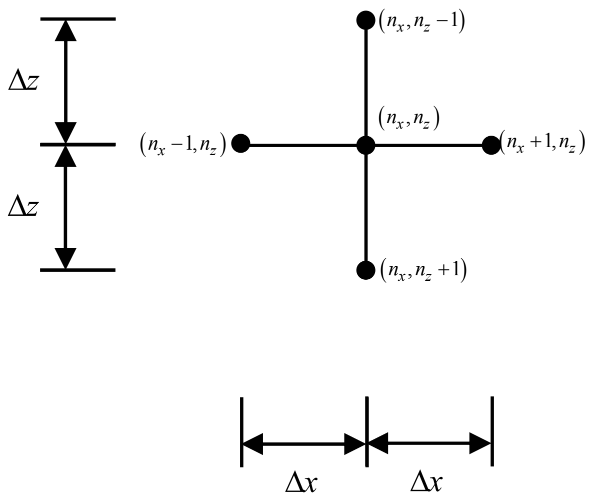
$$n_x = 1,\ldots,N_x$$
$$n_z = 1,\ldots,N_z$$

Global grid node numbering /
Globale
Gitterknotennummerierung

$$n = n_x + N_x(n_z - 1) \qquad n = 1,\ldots,N \qquad N = N_x N_z$$

FD Method – 2-D TM Wave Equation – 2-D FD Stencil / FD-Methode – 2D-TM-Wellengleichung – 2D-FD-Schablone

2-D FD stencil in space /
2D-FD-Schablone im Raum



FD Method – 2-D TM Wave Equation / FD-Methode – 2D-TM-Wellengleichung

Explicit 2-D FD algorithm in the time domain of 2nd order in space and time / Expliziter 2D-FD-Algorithmus im Zeitbereich 2ter Ordnung in Raum und Zeit

$$\begin{aligned} E_y(x,z,t+\Delta t) &= 2E_y(x,z,t) - E_y(x,z,t-\Delta t) \\ &+ c_0^2 \frac{(\Delta t)^2}{(\Delta x)^2}\left[E_y(x+\Delta x,z,t) - 2E_y(x,z,t) + E_y(x-\Delta x,x,t)\right] \\ &+ c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}\left[E_y(x,z+\Delta z,t) - 2E_y(x,z,t) + E_y(x,z-\Delta z,t)\right] \\ &+ c_0^2 \mu_0 \Delta t\left[J_{ey}(x,z,t) - J_{ey}(x,z,t-\Delta t)\right] + \bigcirc[(\Delta x)^2] + \bigcirc[(\Delta z)^2] + \bigcirc[(\Delta t)^2] \end{aligned}$$

Marching-on-in-time algorithm / „Marschieren in der Zeit"-Algorithmus

$$x \to n_x \Delta x, \quad n_x = 1,\ldots,N_x$$

$$z \to n_z \Delta z, \quad n_z = 1,\ldots,N_z$$

$$t \to n_t \Delta t, \quad n_t = 1,\ldots,N_t$$

$$E_y(x,z,t) \to E_y^{(n_x,n_z,n_t)}$$

$$J_{ey}(x,z,t) \to J_{ey}^{(n_x,n_z,n_t)}$$

FD Method – 2-D TM Wave Equation / FD-Methode – 2D-TM-Wellengleichung

Explicit 2-D FD algorithm in the time domain of 2nd order in space and time / Expliziter 2D FD-Algorithmus im Zeitbereich 2ter Ordnung in Raum und Zeit

$$\begin{aligned} E_y^{(n_x,n_z,n_t+1)} &= 2E_y^{(n_x,n_z,n_t)} - E_y^{(n_x,n_z,n_t-1)} \\ &+ \left(\frac{c_0\Delta t}{\Delta x}\right)^2 \left[E_y^{(n_x+1,n_z,n_t)} - 2E_y^{(n_x,n_z,n_t)} + E_y^{(n_x-1,n_z,n_t)}\right] \\ &+ \left(\frac{c_0\Delta t}{\Delta z}\right)^2 \left[E_y^{(n_x,n_z+1,n_t)} - 2E_y^{(n_x,n_z,n_t)} + E_y^{(n_x,n_z-1,n_t)}\right] \\ &+ c_0^2\mu_0\Delta t\left[J_{\mathrm{ey}}^{(n_x,n_z,n_t)} - J_{\mathrm{ey}}^{(n_x,n_z,n_t-1)}\right] + \bigcirc[(\Delta x)^2] + \bigcirc[(\Delta z)^2] + \bigcirc[(\Delta t)^2] \end{aligned}$$

Homogeneous 2-D FD grid of quadratic cells / Homogenes 2D- FD-Gitter aus quadratischen Zellen

$$\Delta x = \Delta z$$

$$\begin{aligned} E_y^{(n_x,n_z,n_t+1)} &= 2E_y^{(n_x,n_z,n_t)} - E_y^{(n_x,n_z,n_t-1)} \\ &+ \left(\frac{c_0\Delta t}{\Delta x}\right)^2 \left[E_y^{(n_x,n_z+1,n_t)} + E_y^{(n_x+1,n_z,n_t)} - 4E_y^{(n_x,n_z,n_t)} + E_y^{(n_x-1,n_z,n_t)} + E_y^{(n_x,n_z-1,n_t)}\right] \\ &+ c_0^2\mu_0\Delta t\left[J_{\mathrm{ey}}^{(n_x,n_z,n_t)} - J_{\mathrm{ey}}^{(n_x,n_z,n_t-1)}\right] + \bigcirc[(\Delta x)^2] + \bigcirc[(\Delta t)^2] \end{aligned}$$

FD Method – 2-D TM Wave Equation / FD-Methode – 2D-TM-Wellengleichung

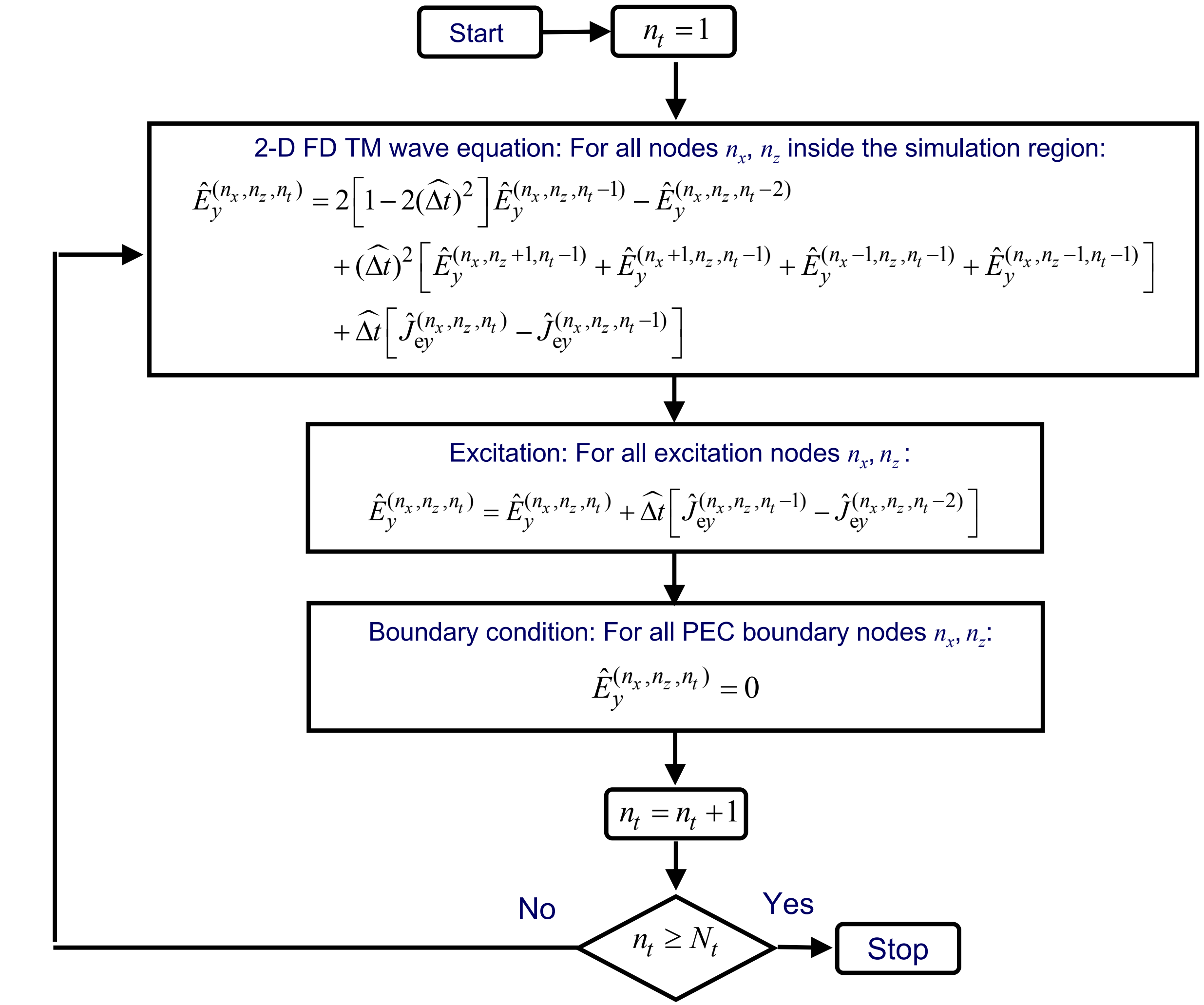
Explicit FD algorithm in the time domain of 2nd order in space and time / Expliziter FD-Algorithmus im Zeitbereich 2ter Ordnung in Raum und Zeit

$$\hat{E}_y^{(n_x,n_z,n_t+1)} = 2\hat{E}_y^{(n_x,n_z,n_t)} - \hat{E}_y^{(n_x,n_z,n_t-1)} + (\widehat{\Delta t})^2 \left[\hat{E}_y^{(n_x,n_z+1,n_t)} + \hat{E}_y^{(n_x+1,n_z,n_t)} - 4\hat{E}_y^{(n_x,n_z,n_t)} + \hat{E}_y^{(n_x-1,n_z,n_t)} + \hat{E}_y^{(n_x,n_z-1,n_t)} \right] + \widehat{\Delta t}\left[\hat{J}_{\mathrm{e}y}^{(n_x,n_z,n_t)} - \hat{J}_{\mathrm{e}y}^{(n_x,n_z,n_t-1)} \right]$$

$$\text{for / für} \quad \begin{cases} 1 \le n_x \le N_x \\ 1 \le n_z \le N_z \\ 1 \le n_t \le N_t \end{cases}$$

$$\hat{E}_y^{(n_x,n_z,n_t)} = 2\left[1 - 2(\widehat{\Delta t})^2\right]\hat{E}_y^{(n_x,n_z,n_t-1)} - \hat{E}_y^{(n_x,n_z,n_t-2)} + (\widehat{\Delta t})^2 \left[\hat{E}_y^{(n_x,n_z+1,n_t)} + \hat{E}_y^{(n_x+1,n_z,n_t)} + \hat{E}_y^{(n_x-1,n_z,n_t)} + \hat{E}_y^{(n_x,n_z-1,n_t)} \right] + \widehat{\Delta t}\left[\hat{J}_{\mathrm{e}y}^{(n_x,n_z,n_t)} - \hat{J}_{\mathrm{e}y}^{(n_x,n_z,n_t-1)} \right]$$

FD Method – 2-D FD Wave Equation – TM Case – Flow Chart / FD-Methode – 2D FD-Wellengleichung – TM-Fall – Flussdiagramm



FD Method – 2-D TM Wave Equation – Example / FD-Methode – 2D-TM-Wellengleichung – Beispiel

Scalar 2-D TM wave equation / Skalare 2D-TM-Wellengleichung

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)E_y(x,z,t)-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}E_y(x,z,t)=\mu_0\frac{\partial}{\partial t}J_{\text{ey}}(x,z,t)\quad \text{for / für}\quad \begin{cases}0\le x\le X\\ 0\le z\le Z\\ 0\le t\le T\end{cases}$$

Initial condition / Anfangsbedingung

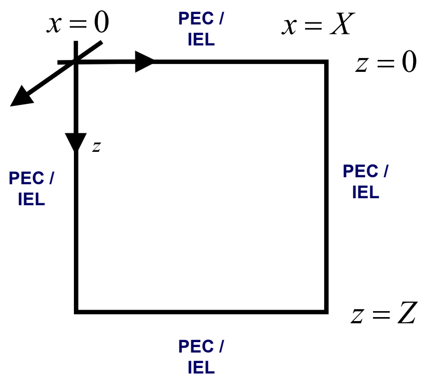
$$E_y(x,z,t)=J_{\text{ey}}(x,z,t)=0 \qquad t\le 0$$

$$J_{\text{ey}}(x,z,t)=\delta(x-x_0)\,\delta(z-z_0)\,f(t) \qquad t>0$$

Causality / Kausalität

Boundary conditions for a perfectly electrically conducting (PEC) boundary / Randbedingung für einen ideal elektrisch leitenden (IEL) Rand

$$\left.\begin{aligned}E_y(0,z,t)&=0\\ E_y(X,z,t)&=0\end{aligned}\right\}\ \forall z,t\forall t \quad \text{and / und} \quad \left.\begin{aligned}E_y(x,0,t)&=0\\ E_y(x,Z,t)&=0\end{aligned}\right\}\ \forall x,t\forall t$$



Hyperbolic initial-boundary-value problem / Hyperbolisches Anfangs-Randwert-Problem

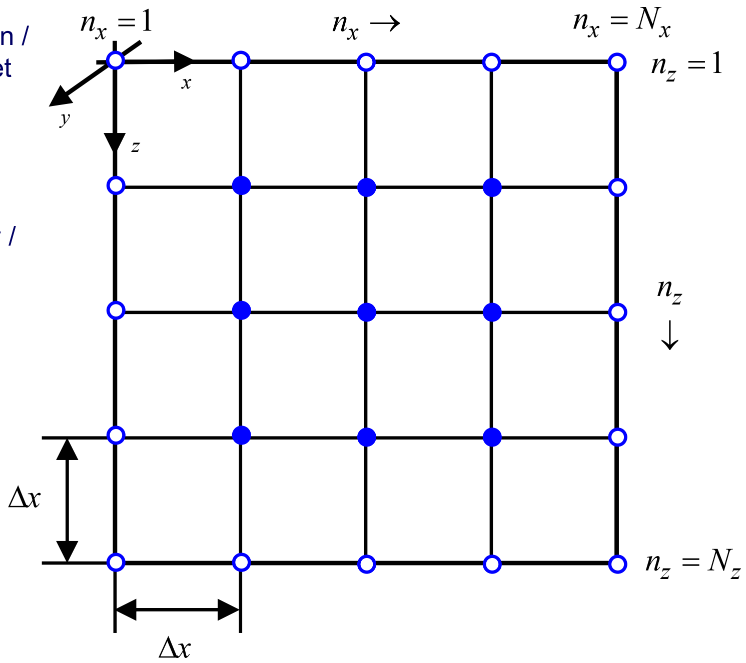
FD Method – 2-D TM Wave Equation – Example / FD-Methode – 2D-TM-Wellengleichung – Beispiel

Nodes in the simulation region / Knoten im Simulationsgebiet

● $E_y^{(n,n_t)} \neq 0$

Nodes at the PEC boundary / Knoten auf dem IEL-Rand

○ $E_y^{(n,n_t)} = 0$



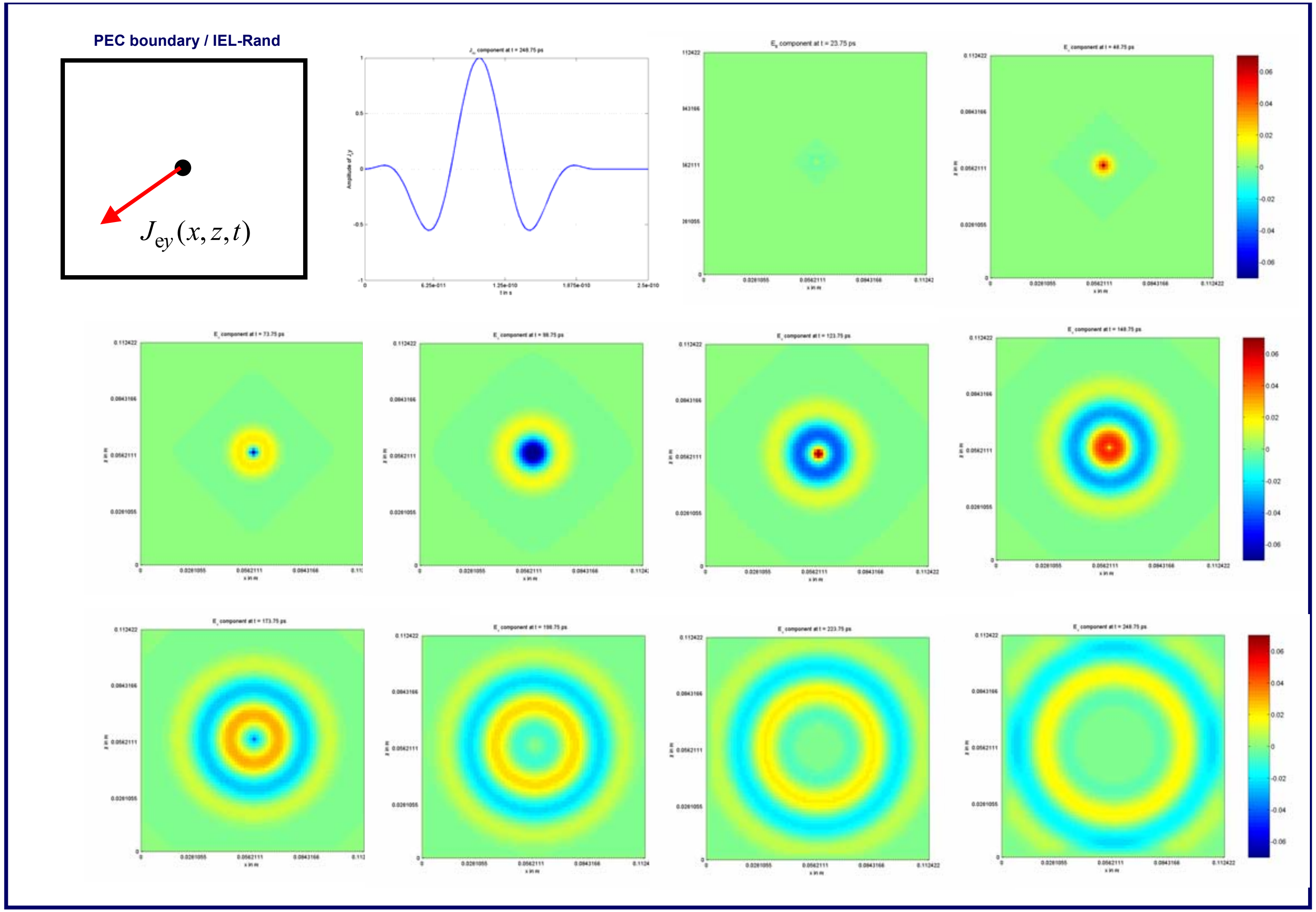
$n_x = 1, \dots, N_x$

$n_z = 1, \dots, N_z$

Global grid node numbering / Globale Gitterknotennummerierung

$n = n_x + N_x(n_z - 1)$ $\quad n = 1, \dots, N$ $\quad N = N_x N_z$

FD Method – 2-D TM Wave Equation – Example/
FD-Methode – 2D-TM-Wellengleichung – Beispiel



FD Method – 2-D FD Wave Equation – TM Case – Validation / FD-Methode – 2D FD-Wellengleichung – TM-Fall – Validierung

Domain integral representation /
(Gebiets-) Integraldarstellung

1-D case / 1D-Fall

$$E_x(z,\omega) = \mathrm{j}\omega\mu_0 \int_{z'=-\infty}^{\infty} G(z-z',\omega)\, J_{\mathrm{e}x}(z',\omega)\, \mathrm{d}z'$$

2-D case / 2D-Fall

$$E_y(x,z,\omega) = \mathrm{j}\omega\mu_0 \int_{z'=-\infty}^{\infty} \int_{x'=-\infty}^{\infty} G(x-x',z-z',\omega)\, J_{\mathrm{e}y}(x',z',\omega)\, \mathrm{d}x'\mathrm{d}z'$$

Green's function / Greensche Funktion

$$G(z,\omega) = \frac{c_0}{2}\left[\mathrm{j}\,\mathrm{PV}\frac{1}{\omega_0} + \pi\delta(z)\right]\mathrm{e}^{\mathrm{j}k_0|z|}$$

$$G(z,t) = \frac{c_0}{2} u\left(t - \frac{|z|}{c_0}\right)$$

$$G(x,z,\omega) = \frac{\mathrm{j}}{4}\mathrm{H}_0^{(1)}\left(\frac{\omega}{c_0}\sqrt{x^2+z^2}\right)$$

$$G(x,z,t) = \frac{c_0}{2\pi}\frac{1}{\sqrt{c_0^2 t^2 - (x^2+z^2)}}\, u\left(t - \frac{\sqrt{x^2+z^2}}{c_0}\right)$$

FD Method – 2-D FD Wave Equation – TM Case – Validation / FD-Methode – 2D FD-Wellengleichung – TM-Fall – Validierung

2-D Domain integral representation /
2D-(Gebiets-)Integraldarstellung

$$E_y(r,\omega) = \mathrm{j}\omega\mu_0 \int_{r'=0}^{\infty} G(r-r',\omega)\, J_{ey}(r',\omega)\, \mathrm{d}r'$$

$$G(r,\omega) = \frac{\mathrm{j}}{4}\mathrm{H}_0^{(1)}\left(\frac{\omega}{c_0} r\right)$$

$$G(r,t) = \frac{c_0}{2\pi}\frac{1}{\sqrt{c_0^2 t^2 - r^2}}\, u\left(t - \frac{r}{c_0}\right)$$

$$E_y(\underline{\mathbf{r}},\omega) = \mathrm{j}\omega\mu_0 \int_{\underline{\mathbf{r}}'} G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',\omega)\, J_{ey}(\underline{\mathbf{r}}',\omega)\, \mathrm{d}\underline{\mathbf{r}}'$$

$$G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',\omega) = \frac{\mathrm{j}}{4}\mathrm{H}_0^{(1)}\left(\frac{\omega}{c_0}\left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|\right)$$

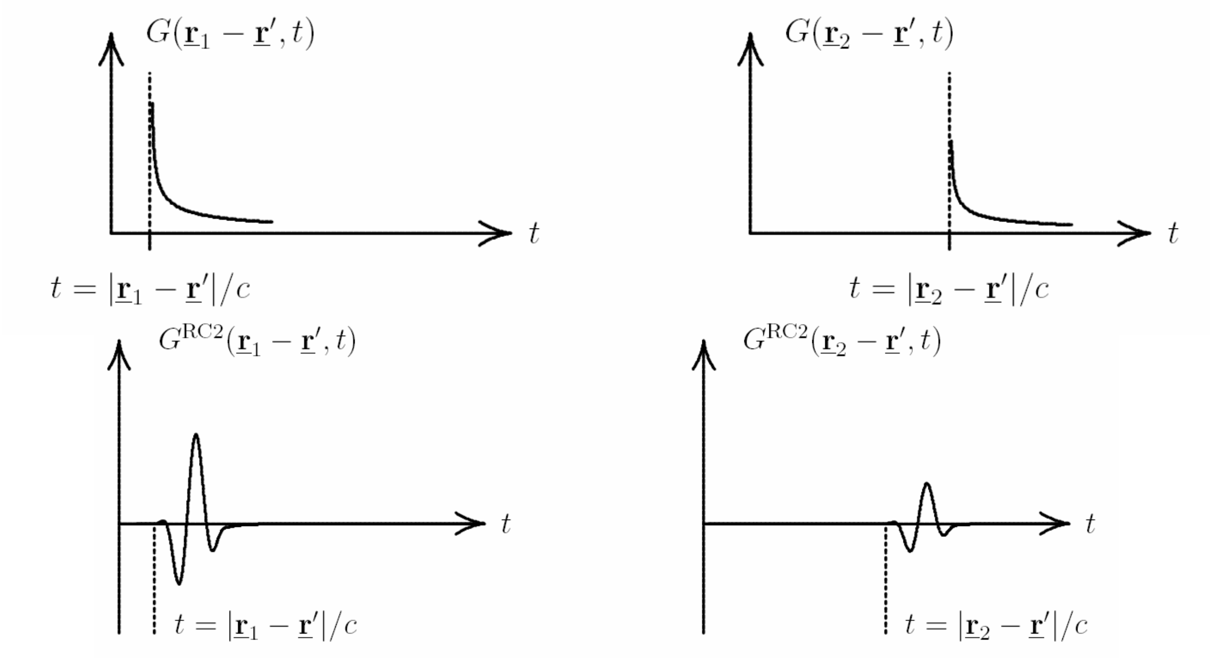
$$G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t) = \frac{c_0}{2\pi}\frac{1}{\sqrt{c_0^2 t^2 - \left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|^2}}\, u\left(t - \frac{\left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|}{c_0}\right)$$

FD Method – 2-D FD Wave Equation – TM Case – Validation / FD-Methode – 2D FD-Wellengleichung – TM-Fall – Validierung

2-D Domain integral representation / 2D-(Gebiets-)Integraldarstellung

$$G^{\mathrm{RC2}}(\underline{\mathbf{r}}-\underline{\mathbf{r}}',\omega) = \mathrm{RC2}(\omega)\,\frac{\mathrm{j}}{4}\,\mathrm{H}_0^{(1)}\left(\frac{\omega}{c_0}\left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|\right)$$

$$G^{\mathrm{RC2}}(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t) = \mathrm{RC2}(t) *_t \frac{c_0}{2\pi}\frac{1}{\sqrt{c_0^2 t^2 - \left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|^2}}\,u\left(t-\frac{\left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|}{c_0}\right)$$



$$G^{\mathrm{RC2}}(\underline{\mathbf{r}}-\underline{\mathbf{r}}',\omega) \simeq \frac{1}{4}\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\sqrt{\frac{2c}{\pi}}\,\frac{\mathrm{RC2}(\omega)}{\sqrt{\omega}}\,\frac{\mathrm{e}^{\mathrm{j}k\left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|}}{\sqrt{\left|\underline{\mathbf{r}}-\underline{\mathbf{r}}'\right|}}$$

End of Lecture 4 /
Ende der 4. Vorlesung