Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

6th Lecture / 6. Vorlesung

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FDTD and FIT / FDTD und FIT

FDTD : Finite Difference Time Domain / Finite Differenzen im Zeitbereich
FIT : Finite Integration Technique / Finite Integrationstechnik

FDTD

Maxwell's equations in differential form /
Maxwellsche Gleichungen in Differentialform

$$\frac{\partial}{\partial t}\mathbf{B}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

⇨ FD approximation of spatial and temporal derivatives / FD-Approximation von räumlichen und zeitlichen Ableitungen

Central difference approximation /
Zentrale Differenzen Approximation

$$\left.\frac{\partial}{\partial z}f(z,t)\right|_{z=z_0} \approx \frac{f\left(z_0+\frac{\Delta z}{2},t\right)-f\left(z_0-\frac{\Delta z}{2},t\right)}{\Delta z}$$

FIT

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

⇨ FIT approximation of spatial and temporal integrals / FIT-Approximation von räumlichen und zeitlichen Integralen

Mid point rule approximation of a 1-D integral /
Mittelpunktsregel-Approximation eines 1D-Integrals

$$\int_{z=z_0}^{z_0+\Delta z} f(z,t)\,\mathrm{d}z \approx f\left(z_0+\frac{\Delta z}{2},t\right)\Delta z$$

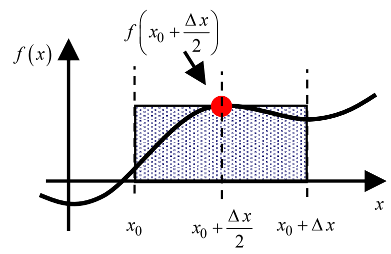
FIT Approximations Based on the Mid Point Rule in 1-D, 2-D, and 3-D / FIT-Approximationen basierend auf der Mittelpunktsregel in 1D, 2D und 3D

Cartesian coordinate system; homogeneous dual-orthogonal grid system with **rectangular grid cells /** Kartesisches Koordinatensystem; homogenes dual-orthogonales Gittersystem mit **rechteckförmigen Gitterzellen**

1-D:

$$\int_{x=x_0}^{x_0+\Delta x} f(x)\,\mathrm{d}x = f\left(x_0 + \frac{\Delta x}{2}\right)\Delta x + \bigcirc\left[(\Delta x)^3\right]$$

$$\approx f\left(x_0 + \frac{\Delta x}{2}\right)\Delta x$$



2-D:

$$\int_{y=y_0}^{y_0+\Delta y} \int_{x=x_0}^{x_0+\Delta x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = f\left(x_0 + \frac{\Delta x}{2}, y_0 + \frac{\Delta y}{2}\right)\Delta x \Delta y + \bigcirc\left[(\Delta x)^3 \Delta y + \Delta x (\Delta y)^3\right]$$

$$\approx f\left(x_0 + \frac{\Delta x}{2}, y_0 + \frac{\Delta y}{2}\right)\Delta x \Delta y$$

3-D:

$$\int_{z=z_0}^{z_0+\Delta x} \int_{y=y_0}^{y_0+\Delta x} \int_{x=x_0}^{x_0+\Delta x} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = f\left(x_0 + \frac{\Delta x}{2}, y_0 + \frac{\Delta y}{2}, z_0 + \frac{\Delta z}{2}\right)\Delta x \Delta y \Delta z$$

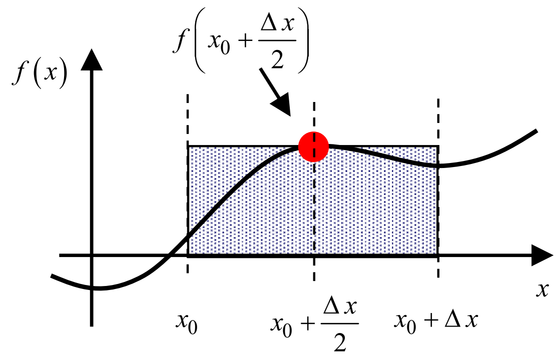
$$+\bigcirc\left[(\Delta x)^3 \Delta y \Delta z + \Delta x (\Delta y)^3 \Delta z + \Delta x \Delta y (\Delta z)^3\right]$$

$$\approx f\left(x_0 + \frac{\Delta x}{2}, y_0 + \frac{\Delta y}{2}, z_0 + \frac{\Delta z}{2}\right)\Delta x \Delta y \Delta z$$

FIT Approximation of a 1-D Integral Based on the Mid Point Rule / FIT-Approximation eines 1D-Integrals basierend auf der Mittelpunktsregel

Cartesian coordinate system; homogeneous dual-orthogonal grid system with **rectangular grid cells /** Kartesisches Koordinatensystem; homogenes dual-orthogonales Gittersystem mit **rechteckförmigen Gitterzellen**

$$\int_{x=x_0}^{x_0+\Delta x} f(x)\,\mathrm{d}x = f\left(x_0+\frac{\Delta x}{2}\right)\Delta x + \bigcirc\left[(\Delta x)^3\right]$$
$$\approx f\left(x_0+\frac{\Delta x}{2}\right)\Delta x$$



Taylor series are expansions of a function *f(x)* in a finite distance *Δx: f(x+Δx/2)* / Taylor-Reihen sind Entwicklungen einer Funktion *f(x)* in einer endlichen Distanz *Δx: f(x+Δx/2)*

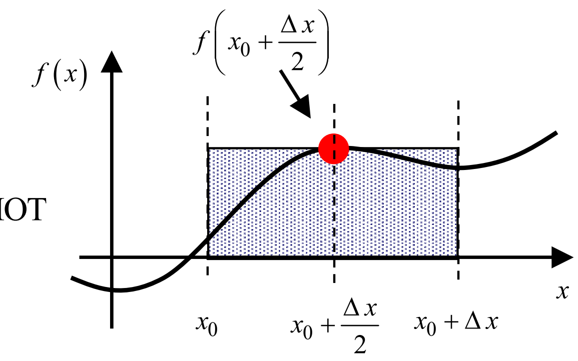
$$f(x\pm\Delta x) = \sum_{n=0}^{\infty}\frac{1}{n!}(\pm\Delta x)^n \frac{d^n}{dx^n} f(x)$$

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d}{dx}\right)^n f(x)\bigg|_{x=x_1} (x-x_1)^n$$

FIT Approximation of a 1-D Integral Based on the Mid Point Rule / FIT-Approximation eines 1D-Integrals basierend auf der Mittelpunktsregel

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d}{dx} \right)^n f(x) \Big|_{x=x_1} (x - x_1)^n$$

$$= f(x)\big|_{x=x_1} + \frac{d}{dx} f(x)\Big|_{x=x_1} (x - x_1) + \frac{1}{2!} \frac{d^2}{dx^2} f(x)\Big|_{x=x_1} (x - x_1)^2 + \mathrm{HOT}$$



$$x_1 = x_0 + \frac{\Delta x}{2}$$

$$f(x) = f\left(x_0 - \frac{\Delta x}{2}\right) + \underbrace{\frac{d}{dx} f(x)\Big|_{x=x_0-\frac{\Delta x}{2}}}_{=f^{(x)}\left(x_0-\frac{\Delta x}{2}\right)} \left[x - \left(x_0 - \frac{\Delta x}{2}\right)\right] + \frac{1}{2!} \underbrace{\frac{d^2}{dx^2} f(x)\Big|_{x=x_0-\frac{\Delta x}{2}}}_{=f^{(xx)}\left(x_0-\frac{\Delta x}{2}\right)} \left[x - \left(x_0 - \frac{\Delta x}{2}\right)\right]^2 + \mathrm{HOT}$$

$$= f\left(x_0 + \frac{\Delta x}{2}\right) + f^{(x)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right] + \frac{1}{2!} f^{(xx)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right]^2 + \mathrm{HOT}$$

FIT Approximation of a 1-D Integral Based on the Mid Point Rule / FIT-Approximation eines 1D-Integrals basierend auf der Mittelpunktsregel

$$f(x) = f\left(x_0 + \frac{\Delta x}{2}\right) + f^{(x)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right] + \frac{1}{2!} f^{(xx)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right]^2 + \mathrm{HOT}$$

$$\int_{x=x_0}^{x_0+\Delta x} f(x)\,\mathrm{d}x$$

$$= \int_{x=x_0}^{x_0+\Delta x} \left\{ f\left(x_0 + \frac{\Delta x}{2}\right) + f^{(x)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right] + \frac{1}{2!} f^{(xx)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 - \frac{\Delta x}{2}\right)\right]^2 + \mathrm{HOT} \right\} \mathrm{d}x$$

$$= \underbrace{\int_{x=x_0}^{x_0+\Delta x} f\left(x_0 + \frac{\Delta x}{2}\right)\mathrm{d}x}_{f\left(x_0 + \frac{\Delta x}{2}\right)\int_{x=x_0}^{x_0+\Delta x} \mathrm{d}x}$$

$$+ \underbrace{\int_{x=x_0}^{x_0+\Delta x} f^{(x)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right]\mathrm{d}x}_{= f^{(x)}\left(x_0 + \frac{\Delta x}{2}\right)\int_{x=x_0}^{x_0+\Delta x}\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right]\mathrm{d}x}$$

$$+ \underbrace{\int_{x=x_0}^{x_0+\Delta x} \frac{1}{2!} f^{(xx)}\left(x_0 + \frac{\Delta x}{2}\right)\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right]^2 \mathrm{d}x}_{= \frac{1}{2} f^{(xx)}\left(x_0 + \frac{\Delta x}{2}\right)\int_{x=x_0}^{x_0+\Delta x}\left[x - \left(x_0 + \frac{\Delta x}{2}\right)\right]^2 \mathrm{d}x} + \mathrm{HOT}$$

FIT Approximation of a 1-D Integral Based on the Mid Point Rule / FIT-Approximation eines 1D-Integrals basierend auf der Mittelpunktsregel

$$\int_{x=x_0}^{x_0+\Delta x} f(x)\,\mathrm{d}x = f\left(x_0+\frac{\Delta x}{2}\right)\underbrace{\int_{x=x_0}^{x_0+\Delta x}\mathrm{d}x}_{=\Delta x}$$

$$+f^{(x)}\left(x_0+\frac{\Delta x}{2}\right)\underbrace{\int_{x=x_0}^{x_0+\Delta x}\left[x-\left(x_0+\frac{\Delta x}{2}\right)\right]\mathrm{d}x}_{=0}$$

$$+\frac{1}{2}f^{(xx)}\left(x_0+\frac{\Delta x}{2}\right)\underbrace{\int_{x=x_0}^{x_0+\Delta x}\left[x-\left(x_0+\frac{\Delta x}{2}\right)\right]^2\mathrm{d}x}_{=\frac{1}{12}(\Delta x)^3}+\mathrm{HOT}$$

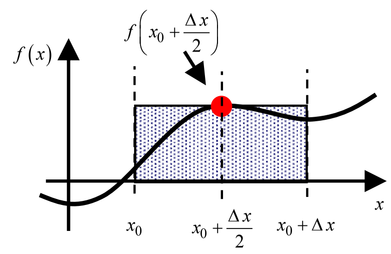
$$=f\left(x_0+\frac{\Delta x}{2}\right)\Delta x+\underbrace{\frac{1}{24}f^{(xx)}\left(x_0+\frac{\Delta x}{2}\right)(\Delta x)^3+\mathrm{HOT}}_{=\bigcirc\left[(\Delta x)^3\right]}$$

$$=f\left(x_0+\frac{\Delta x}{2}\right)\Delta x+\bigcirc\left[(\Delta x)^3\right]$$

FIT Approximations Based on the Mid Point Rule in 1-D, 2-D, and 3-D / FIT-Approximationen basierend auf der Mittelpunktsregel in 1D, 2D und 3D

Cartesian coordinate system; homogeneous dual-orthogonal grid system with **cubic grid cells** / Kartesisches Koordinatensystem; homogenes dual-orthogonales Gittersystem mit **kubischen Gitterzellen**

1-D:



$$\int_{x_0}^{x_0+\Delta x} f(x)\,\mathrm{d}x = f\left(x_0+\frac{\Delta x}{2}\right)\Delta x + O\left[(\Delta x)^3\right] \approx f\left(x_0+\frac{\Delta x}{2}\right)\Delta x$$

2-D:

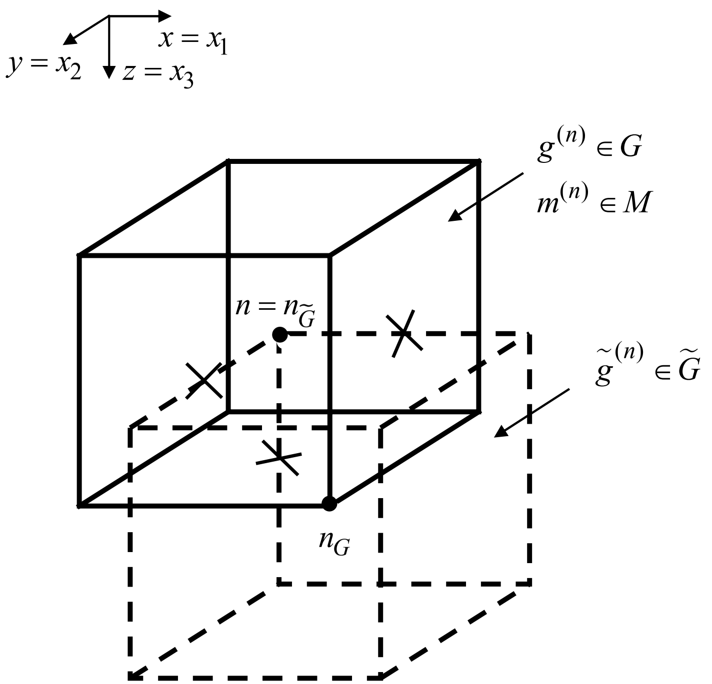
$$\int_{y_0}^{y_0+\Delta x}\int_{x_0}^{x_0+\Delta x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta x}{2}\right)(\Delta x)^2 + O\left[(\Delta x)^4\right] \approx f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta x}{2}\right)(\Delta x)^2$$

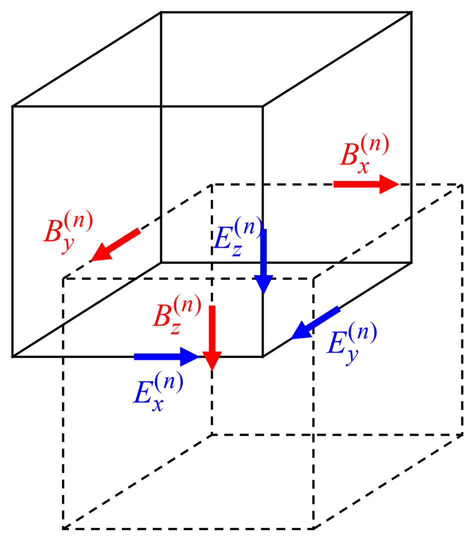
3-D:

$$\int_{z_0}^{z_0+\Delta x}\int_{y_0}^{y_0+\Delta x}\int_{x_0}^{x_0+\Delta x} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta x}{2}, z_0+\frac{\Delta x}{2}\right)(\Delta x)^3 + O\left[(\Delta x)^5\right] \approx f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta x}{2}, z_0+\frac{\Delta x}{2}\right)(\Delta x)^3$$

Dual-Orthogonal Grid System in Space / Dual-orthogonales Gittersystem im Raum

3-D / 3D





Primary grid / Primäres Gitter	$G \perp \tilde{G}$	Secondary (dual) grid Sekundäres (duales) Gitter
Primary grid / Primäres Gitter	$G = M$	Material grid Materialgitter

Global node numbering / Globale Gitternummerierung

$$n = 1 + M_x(n_x - 1) + M_y(n_y - 1) + M_z(n_z - 1)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

FIT – 3-D Electromagnetic Wave Propagation / FIT – 3D elektromagnetische Wellenausbreitung

Continuous spatial and temporal space / Kontinuierlicher räumlicher und zeitlicher Raum

Maxwell's equations in integral form / Maxwellsche Gleichungen in Integralform

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

Continuity equations in integral form / Kontinuitätsgleichungen in Integralform

$$\iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\frac{d}{dt}\iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

$$\iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\frac{d}{dt}\iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

Discrete spatial and temporal space / Diskreter räumlicher und zeitlicher Raum

Maxwell's grid equations / Maxwellsche Gittergleichungen

$$[\mathbf{S}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) - [\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$[\widetilde{\mathbf{S}}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{D}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{R}}]\{\mathbf{H}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$[\mathbf{S}][\mathbf{div}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\rho_{\mathrm{m}}\}(t)$$

$$[\widetilde{\mathbf{S}}][\widetilde{\mathbf{div}}]\{\mathbf{D}\}(t) = [\widetilde{\mathbf{V}}]\{\rho_{\mathrm{e}}\}(t)$$

Continuity grid equations / Kontinuitätsgittergleichungen

$$[\mathbf{S}][\mathbf{div}]\{\mathbf{J}_{\mathrm{m}}\}(t) = -[\mathbf{V}]\frac{d}{dt}\{\rho_{\mathrm{m}}\}(t)$$

$$[\widetilde{\mathbf{S}}][\widetilde{\mathbf{div}}]\{\mathbf{J}_{\mathrm{e}}\}(t) = -[\widetilde{\mathbf{V}}]\frac{d}{dt}\{\rho_{\mathrm{e}}\}(t)$$

FIT Formulation / FIT-Formulierung

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \oint_{C=\partial S} \underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

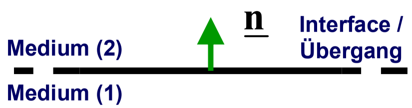
$$\oiint_{S=\partial V} \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

$$\left.\begin{array}{l}\underline{\mathbf{E}}(\underline{\mathbf{R}},t),\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\\ \underline{\mathbf{E}}(\underline{\mathbf{R}},t),\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\\ \underline{\mathbf{D}}(\underline{\mathbf{R}},t),\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\\ \underline{\mathbf{D}}(\underline{\mathbf{R}},t),\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\end{array}\right\}$$

Which field quantities should we use in our formulation /
Welche Feldgrößen sollten wir in unserer FIT-Formulierung verwenden **?**

⇨ $\underline{\mathbf{E}}(\underline{\mathbf{R}},t),\underline{\mathbf{B}}(\underline{\mathbf{R}},t)$ Formulation / Formulierung **!**

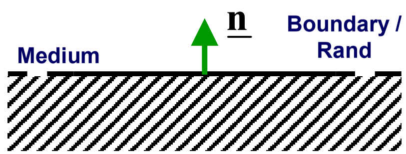
Transition conditions for a source-free interface /
Übergangsbedingungen für einen quellenfreien Übergang



$$\underline{\mathbf{n}}\times\left[\underline{\mathbf{E}}^{(2)}(\underline{\mathbf{R}},t)-\underline{\mathbf{E}}^{(1)}(\underline{\mathbf{R}},t)\right]=\underline{\mathbf{0}} \;\rightarrow\; E_t^{(2)}(\underline{\mathbf{R}},t)=E_t^{(2)}(\underline{\mathbf{R}},t) \quad \underline{\mathbf{R}}\in \text{Interface / Übergang}$$

$$\underline{\mathbf{n}}\bullet\left[\underline{\mathbf{B}}^{(2)}(\underline{\mathbf{R}},t)-\underline{\mathbf{B}}^{(1)}(\underline{\mathbf{R}},t)\right]=0 \;\rightarrow\; B_n^{(2)}(\underline{\mathbf{R}},t)=B_n^{(2)}(\underline{\mathbf{R}},t) \quad \underline{\mathbf{R}}\in \text{Interface / Übergang}$$

Boundary conditions for a PEC boundary /
Randbedingungen für einen IEL-Rand



$$\underline{\mathbf{n}}\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\underline{\mathbf{0}} \;\rightarrow\; E_t(\underline{\mathbf{R}},t)=0 \quad \underline{\mathbf{R}}\in \text{Boundary / Rand}$$

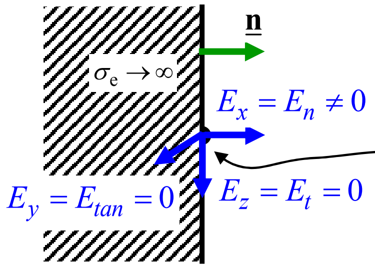
$$\underline{\mathbf{n}}\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)=0 \;\rightarrow\; B_n(\underline{\mathbf{R}},t)=0 \quad \underline{\mathbf{R}}\in \text{Boundary / Rand}$$

Problems of a Node-Based Formulation – Boundary Conditions / Probleme einer knotenbasierten Formulierung – Randbedingungen

Node-Based Grid System / Knotenbasiertes Gittersystem

Perfectly electrically conducting (PEC) material in the xz plane /
Ideal elektrisch leitendes (IEL) Material in der xz-Ebene

Boundary conditions for perfectly electrically conducting (PEC) material /
Randbedingungen für ein ideal elektrisch leitendes (IEL) Material

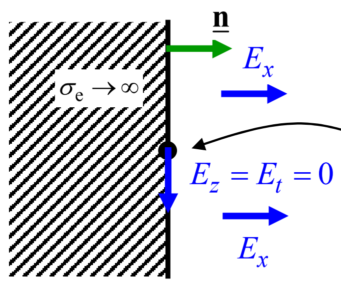


$$\underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \underline{\mathbf{0}} \quad \to E_{tan}(\underline{\mathbf{R}},t) = E_t(\underline{\mathbf{R}},t) = 0$$

$$\underline{\mathbf{n}} \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \underline{\mathbf{0}} \quad \to B_n(\underline{\mathbf{R}},t) = 0$$

Node-based formulation: all vector components are placed at the same one node /
Knotenbasierte Formulierung: alle Vektorkomponenten sind an dem selben Knoten platziert

Staggered Grid System / Versetztes Gittersystem

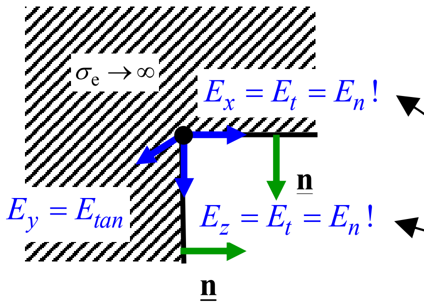


Only tangential components are allocated at the PEC boundary /
Nur Tangentialkomponenten sind am IEL-Rand allokiert

Problems of a Node-Based Formulation – Boundary Conditions / Probleme einer knotenbasierten Formulierung – Randbedingungen

Node-Based Grid System / Knotenbasiertes Gittersystem

Perfectly electrically conducting (PEC) material in the xz plane / Ideal elektrisch leitendes (IEL) Material in der xz-Ebene



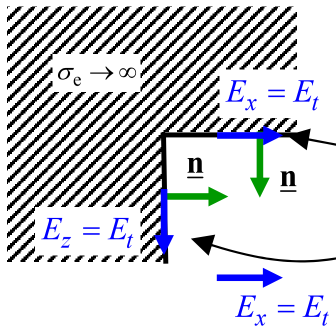
Boundary conditions for perfectly electrically conducting (PEC) material / Randbedingungen für ein ideal elektrisch leitendes (IEL) Material

$$\underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}, t) = \underline{\mathbf{0}} \quad \to E_{tan}(\underline{\mathbf{R}}, t) = E_t(\underline{\mathbf{R}}, t) = 0$$

$$\underline{\mathbf{n}} \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}}, t) = \underline{\mathbf{0}} \quad \to B_n(\underline{\mathbf{R}}, t) \quad = 0$$

These components are tangential and normal either to the vertical or to the horizontal boundary / Diese Komponenten sind tangential und normal entweder zum vertikalen oder zum horizontalen Rand

Staggered Grid System / Versetztes Gittersystem



In this case only tangential components are allocated at the PEC boundary / In diesem Fall sind nur Tangentialkomponenten am IEL-Rand allokiert

FIT – 3-D Electromagnetic Wave Propagation / FIT – 3D elektromagnetische Wellenausbreitung

Constitutive equations for inhomogeneous, anisotropic material / Konstituierende Gleichungen (Materialgleichungen) für ein inhomogenes, anisotropes Material

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}},t)$$

Diagonal permittivity and impermeability tensor of second rank / Diagonaler Permittivitäts- und Impermeabilitätstensor zweiter Stufe

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) = \begin{bmatrix} \varepsilon_{xx}(\underline{\mathbf{R}}) & 0 & 0 \\ 0 & \varepsilon_{yy}(\underline{\mathbf{R}}) & 0 \\ 0 & 0 & \varepsilon_{zz}(\underline{\mathbf{R}}) \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}(\underline{\mathbf{R}}) & & \\ & \varepsilon_{yy}(\underline{\mathbf{R}}) & \\ \mathrm{sym} & & \varepsilon_{zz}(\underline{\mathbf{R}}) \end{bmatrix}$$

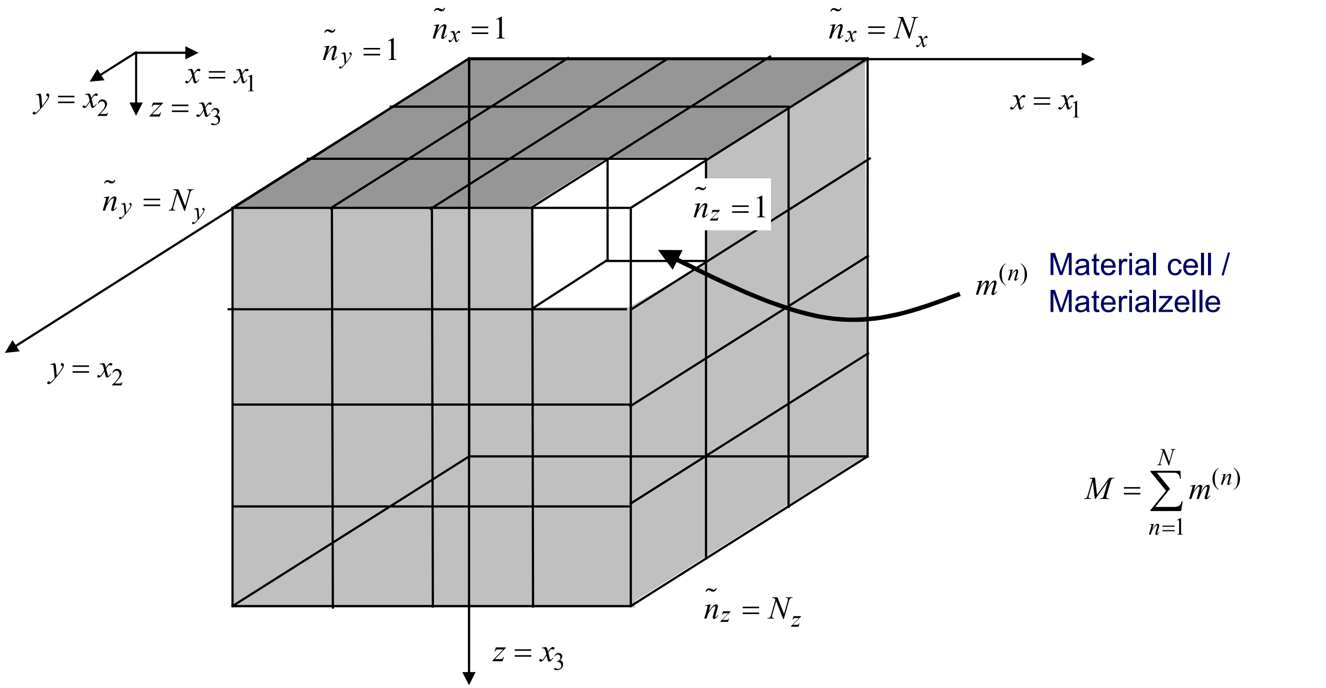
$$\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) = \begin{bmatrix} \nu_{xx}(\underline{\mathbf{R}}) & 0 & 0 \\ 0 & \nu_{yy}(\underline{\mathbf{R}}) & 0 \\ 0 & 0 & \nu_{zz}(\underline{\mathbf{R}}) \end{bmatrix} = \begin{bmatrix} \nu_{xx}(\underline{\mathbf{R}}) & & \\ & \nu_{yy}(\underline{\mathbf{R}}) & \\ \mathrm{sym} & & \nu_{zz}(\underline{\mathbf{R}}) \end{bmatrix}$$

FIT – 3-D Electromagnetic Wave Propagation / FIT – 3D elektromagnetische Wellenausbreitung

$$
\begin{aligned}
\underline{\mathbf{D}}(\underline{\mathbf{R}},t) &= \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t) \\
&= \begin{bmatrix} \varepsilon_{xx}(\underline{\mathbf{R}}) & 0 & 0 \\ 0 & \varepsilon_{yy}(\underline{\mathbf{R}}) & 0 \\ \text{sym} & 0 & \varepsilon_{zz}(\underline{\mathbf{R}}) \end{bmatrix} \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t) \\
&= \left[\varepsilon_{xx}(\underline{\mathbf{R}}) \underline{\mathbf{e}}_x \underline{\mathbf{e}}_x + \varepsilon_{yy}(\underline{\mathbf{R}}) \underline{\mathbf{e}}_y \underline{\mathbf{e}}_y + \varepsilon_{zz}(\underline{\mathbf{R}}) \underline{\mathbf{e}}_z \underline{\mathbf{e}}_z \right] \bullet \left[E_x(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_x + E_y(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_y + E_z(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_z \right] \\
&= \underbrace{\varepsilon_{xx}(\underline{\mathbf{R}}) E_x(\underline{\mathbf{R}},t)}_{D_x(\underline{\mathbf{R}},t)} \underline{\mathbf{e}}_x + \underbrace{\varepsilon_{yy}(\underline{\mathbf{R}}) E_y(\underline{\mathbf{R}},t)}_{D_y(\underline{\mathbf{R}},t)} \underline{\mathbf{e}}_y + \underbrace{\varepsilon_{zz}(\underline{\mathbf{R}}) E_z(\underline{\mathbf{R}},t)}_{D_z(\underline{\mathbf{R}},t)} \underline{\mathbf{e}}_z \\
&= D_x(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_x + D_y(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_y + D_z(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_z
\end{aligned}
$$

$$
\begin{aligned}
\underline{\mathbf{H}}(\underline{\mathbf{R}},t) &= \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}},t) \\
&= \begin{bmatrix} \nu_{xx}(\underline{\mathbf{R}}) & & \\ & \nu_{yy}(\underline{\mathbf{R}}) & \\ \text{sym} & & \nu_{zz}(\underline{\mathbf{R}}) \end{bmatrix} \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}},t) \\
&== \left[\nu_{xx}(\underline{\mathbf{R}}) \underline{\mathbf{e}}_x \underline{\mathbf{e}}_x + \nu_{yy}(\underline{\mathbf{R}}) \underline{\mathbf{e}}_y \underline{\mathbf{e}}_y + \nu_{zz}(\underline{\mathbf{R}}) \underline{\mathbf{e}}_z \underline{\mathbf{e}}_z \right] \bullet \left[B_x(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_x + B_y(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_y + B_z(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_z \right] \\
&= \underbrace{\nu_{xx}(\underline{\mathbf{R}}) B_x(\underline{\mathbf{R}},t)}_{H_x(\underline{\mathbf{R}},t)} \underline{\mathbf{e}}_x + \underbrace{\nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)}_{H_y(\underline{\mathbf{R}},t)} \underline{\mathbf{e}}_y + \underbrace{\nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)}_{H_z(\underline{\mathbf{R}},t)} \underline{\mathbf{e}}_z \\
&= H_x(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_x + H_y(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_y + H_z(\underline{\mathbf{R}},t) \underline{\mathbf{e}}_z
\end{aligned}
$$

Definition of Material Cells / Definition der Materialzellen



$$M = \sum_{n=1}^{N} m^{(n)}$$

$$n = 1 + M_x\left(n_x - 1\right) + M_y\left(n_y - 1\right) + M_z\left(n_z - 1\right)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \rightarrow \underline{\underline{\boldsymbol{\varepsilon}}}^{(n)} \in m^{(n)} \quad n \in \mathbb{R}^N$$

$$\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \rightarrow \underline{\underline{\boldsymbol{\nu}}}^{(n)} \in m^{(n)} \quad n \in \mathbb{R}^N$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}S$$

$$\underline{\mathbf{dR}} = \underline{\mathbf{s}}\,\mathrm{d}R$$

$$\iint_S \underline{\mathbf{n}}\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\,dS = \iint_S \underline{\mathbf{e}}_x\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$

$$= \iint_S B_x(\underline{\mathbf{R}},t)\,\mathrm{d}S$$

$$= B_x^{(m)}(t)\underbrace{\iint_S \mathrm{d}S}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$

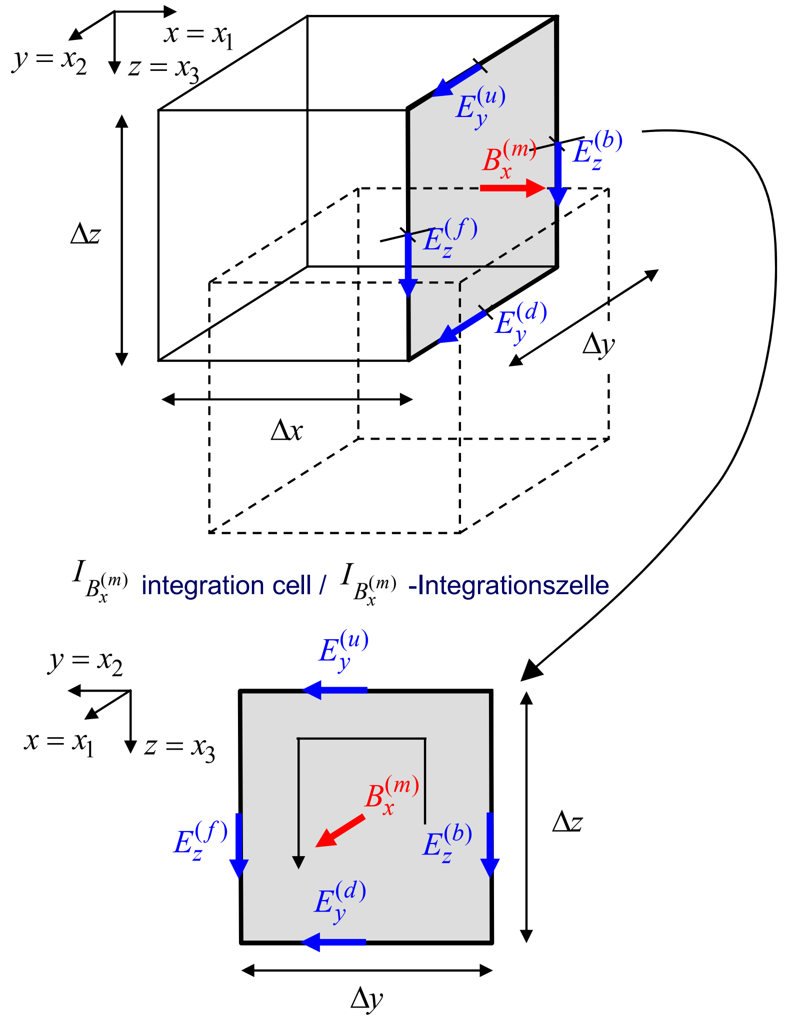
$$= B_x^{(m)}(t)\Delta y\Delta z + \underbrace{\mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]}$$

Field component in the middle / Feldkomponente in der Mitte

Approximation error / Approximationsfehler

$$\iint_S f(\underline{\mathbf{R}},t)\,\mathrm{d}S = f^{(m)}(t)\underbrace{\iint_S \mathrm{d}y\,\mathrm{d}z}_{=\Delta y\Delta z} + \overbrace{\mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]}$$

$$= f^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$



3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

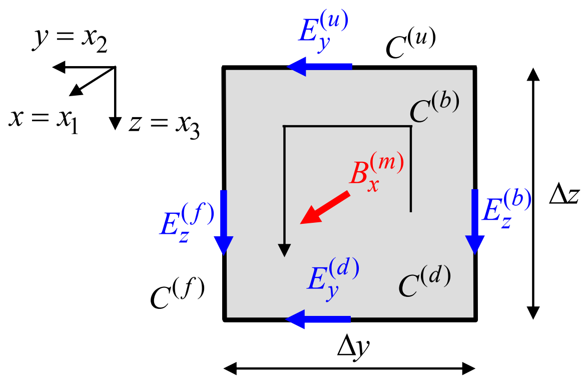
$$\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} = ?$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

$$\underline{\mathbf{dR}}_y = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_y\,\mathrm{d}y$$

$$\underline{\mathbf{dR}}_z = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_z\,\mathrm{d}z$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} = \int_{C^{(u)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(f)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(d)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(b)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}}$$

$$= \int_{C^{(u)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_y\mathrm{d}y + \int_{C^{(f)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_z\mathrm{d}z - \int_{C^{(d)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_y\mathrm{d}y - \int_{C^{(b)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_z\mathrm{d}z$$

$$= \int_{C^{(u)}}E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}}E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}}E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}}E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = \int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$\int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = E_y^{(u)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = E_y^{(u)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$

$$\int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = E_z^{(f)}(t)\underbrace{\int_{C^{(f)}} \mathrm{d}z}_{=\Delta z} + \bigcirc\left[(\Delta z)^3\right] = E_z^{(f)}(t)\Delta z + \bigcirc\left[(\Delta z)^3\right]$$

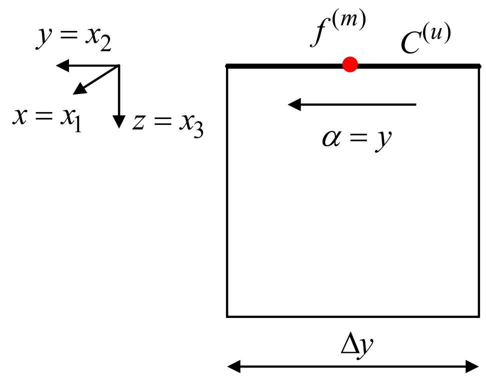
$$\int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = E_y^{(d)}(t)\underbrace{\int_{C^{(d)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = E_y^{(d)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$

$$\int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = E_z^{(b)}(t)\underbrace{\int_{C^{(b)}} \mathrm{d}z}_{=\Delta z} + \bigcirc\left[(\Delta z)^3\right] = E_z^{(b)}(t)\Delta z + \bigcirc\left[(\Delta z)^3\right]$$

Field component in the middle / Feldkomponente in der Mitte

Approximation error / Approximationsfehler

$$\int_{C^{(u)}} f(\underline{\mathbf{R}},t)\,\mathrm{d}R = f^{(m)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = f^{(m)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$



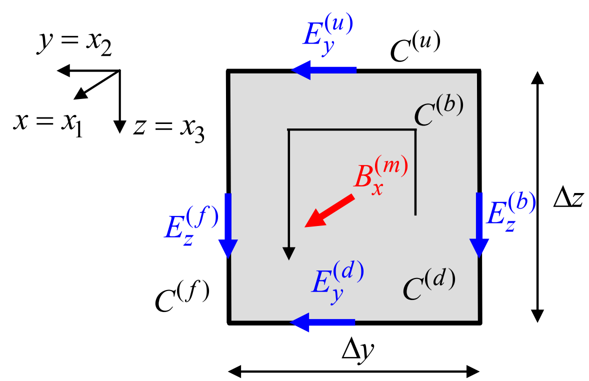
3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \mathrm{d}\underline{\mathbf{R}} = \int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$= E_y^{(u)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + E_z^{(f)}(t)\underbrace{\int_{C^{(f)}} \mathrm{d}z}_{=\Delta z} - E_y^{(d)}(t)\underbrace{\int_{C^{(d)}} \mathrm{d}y}_{=\Delta y} - E_z^{(b)}(t)\underbrace{\int_{C^{(b)}} \mathrm{d}z}_{=\Delta z}$$

$$+\bigcirc\left[(\Delta y)^3\right] + \bigcirc\left[(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \mathrm{d}\underline{\mathbf{R}} = E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z + \bigcirc\left[(\Delta y)^3\right] + \bigcirc\left[(\Delta z)^3\right]$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z$$
$$+\mathrm{O}\left[(\Delta y)^3\right]+\mathrm{O}\left[(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle

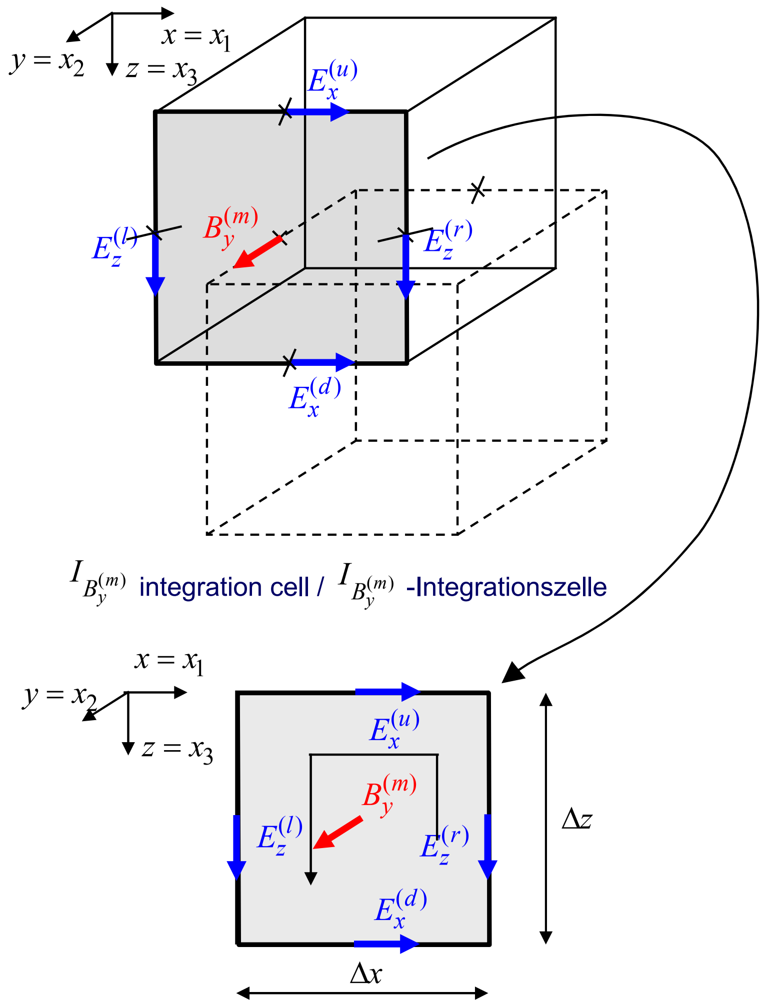
$$\iint_S \underline{\mathbf{n}}\bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\, dS = \iint_S \underline{\mathbf{e}}_x \bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$
$$= \iint_S J_{\mathrm{m}x}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$
$$= J_{\mathrm{m}x}^{(m)}(t)\underbrace{\iint_S \mathrm{d}S}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$
$$= J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle

$$\frac{d}{dt}B_x^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

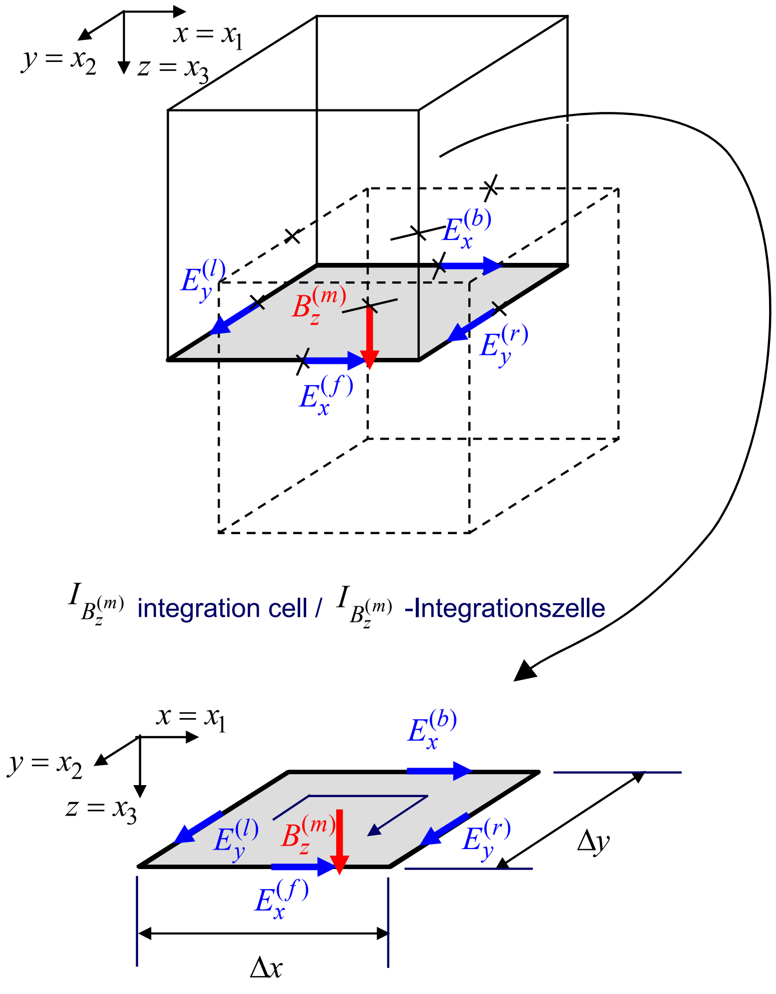


$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$I_{B_y^{(m)}}$ integration cell / $I_{B_y^{(m)}}$ -Integrationszelle

$$\frac{d}{dt}B_y^{(m)}(t)\Delta y\Delta z$$
$$= -\left[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\right]$$
$$-J_{\mathrm{my}}^{(m)}(t)\Delta y\Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen



$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$I_{B_z^{(m)}}$ integration cell / $I_{B_z^{(m)}}$ -Integrationszelle

$$\frac{d}{dt}B_z^{(m)}(t)\Delta x\Delta y$$
$$= -\left[E_x^{(b)}(t)\Delta x + E_y^{(r)}(t)\Delta y - E_x^{(f)}(t)\Delta x - E_y^{(l)}(t)\Delta y\right]$$
$$- J_{\mathrm{m}z}^{(m)}(t)\Delta x\Delta y$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$\Downarrow$$

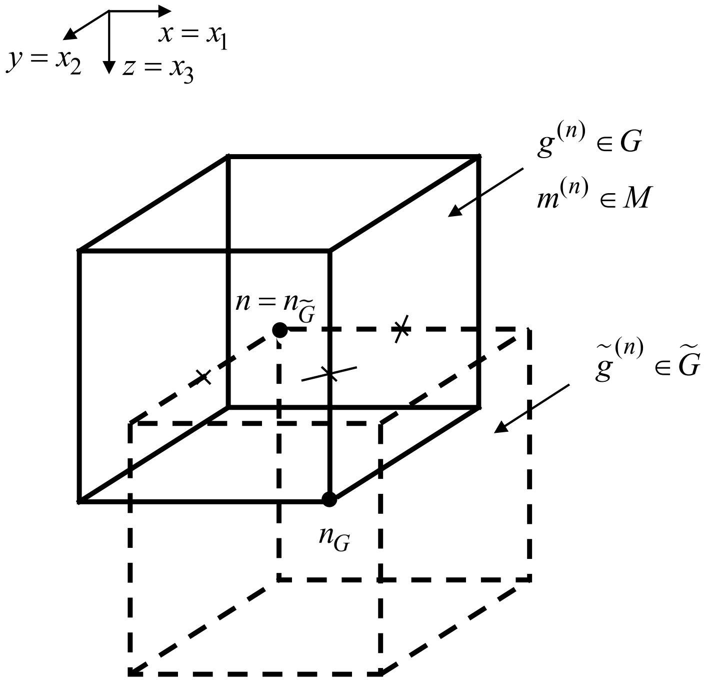
$$\frac{d}{dt}B_x^{(m)}(t)\Delta y\Delta z = -\Big[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\Big] - J_{\mathrm{mx}}^{(m)}(t)\Delta y\Delta z$$

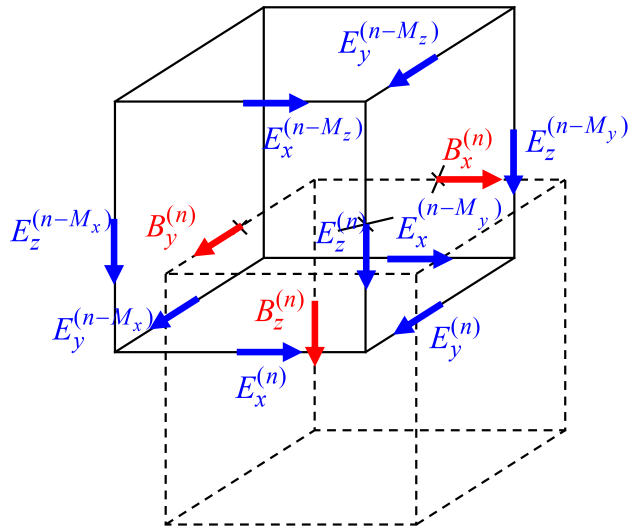
$$\frac{d}{dt}B_y^{(m)}(t)\Delta y\Delta z = -\Big[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\Big] - J_{\mathrm{my}}^{(m)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_z^{(m)}(t)\Delta y\Delta z = -\Big[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y + E_z^{(b)}(t)\Delta z\Big] - J_{\mathrm{mz}}^{(m)}(t)\Delta y\Delta z$$

Dual-Orthogonal Grid System in Space / Dual-orthogonales Gittersystem im Raum

3-D / 3D





Primary grid / Primäres Gitter	$G \perp \tilde{G}$	Secondary (dual) grid Sekundäres (duales) Gitter
Primary grid / Primäres Gitter	$G = M$	Material grid Materialgitter

Global node numbering / Globale Gitternummerierung

$$n = 1 + M_x(n_x - 1) + M_y(n_y - 1) + M_z(n_z - 1)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

Local grid equations in local notation / Lokale Gittergleichungen in lokaler Notation

$$\frac{d}{dt}B_x^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{mx}}^{(m)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(m)}(t)\Delta x\Delta z = -\left[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\right] - J_{\mathrm{my}}^{(m)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(m)}(t)\Delta x\Delta y = -\left[E_x^{(b)}(t)\Delta x + E_y^{(r)}(t)\Delta y - E_x^{(f)}(t)\Delta x - E_y^{(l)}(t)\Delta y\right] - J_{\mathrm{mz}}^{(m)}(t)\Delta x\Delta y$$

Local grid equations in global grid node notation / Lokale Gittergleichungen in globaler Gitterknotennotation

$$\frac{d}{dt}B_x^{(n)}(t)\Delta y\Delta z = -\left\{\left[E_y^{(n-M_z)}(t) - E_y^{(n)}(t)\right]\Delta y + \left[E_z^{(n)}(t) - E_z^{(n-M_y)}(t)\right]\Delta z\right\} - J_{\mathrm{mx}}^{(n)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(n)}(t)\Delta x\Delta z = -\left\{\left[E_x^{(n)}(t) - E_x^{(n-M_z)}(t)\right]\Delta x + \left[E_z^{(n-M_x)}(t) - E_z^{(n)}(t)\right]\Delta z\right\} - J_{\mathrm{my}}^{(n)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(n)}(t)\Delta x\Delta y = -\left\{\left[E_x^{(n-M_y)}(t) - E_x^{(n)}(t)\right]\Delta x + \left[E_y^{(n)}(t) - E_y^{(n-M_x)}(t)\right]\Delta y\right\} - J_{\mathrm{mz}}^{(n)}(t)\Delta x\Delta y$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

Local grid equations in global grid node notation / Lokale Gittergleichungen in globaler Gitterknotennotation

$$\frac{d}{dt}B_x^{(n)}(t)\Delta y\Delta z = -\left\{\left[E_y^{(n-M_z)}(t) - E_y^{(n)}(t)\right]\Delta y + \left[E_z^{(n)}(t) - E_z^{(n-M_y)}(t)\right]\Delta z\right\} - J_{\mathrm{mx}}^{(n)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(n)}(t)\Delta x\Delta z = -\left\{\left[E_x^{(n)}(t) - E_x^{(n-M_z)}(t)\right]\Delta x + \left[E_z^{(n-M_x)}(t) - E_z^{(n)}(t)\right]\Delta z\right\} - J_{\mathrm{my}}^{(n)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(n)}(t)\Delta x\Delta y = -\left\{\left[E_x^{(n-M_y)}(t) - E_x^{(n)}(t)\right]\Delta x + \left[E_y^{(n)}(t) - E_y^{(n-M_x)}(t)\right]\Delta y\right\} - J_{\mathrm{mz}}^{(n)}(t)\Delta x\Delta y$$

Local spatial shift operators / Lokale räumliche Schiebeoperatoren

$$S_{\pm M_i} f^{(n)} = f^{(n\pm M_i)}$$

$$S_0 f^{(n)} = f^{(n)}$$

$$S_0 = I$$

$$I\, f^{(n)} = f^{(n)}$$

Local grid equations with local spatial shift operators in global grid node notation / Lokale Gittergleichungen mit lokalen räumlichen Schiebeoperatoren in globaler Gitterknotennotation

$$\frac{d}{dt}B_x^{(n)}(t)\Delta y\Delta z = -\left\{\left[S_{-M_z} - I\right]E_y^{(n)}(t)\Delta y + \left[I - S_{M_y}\right]E_z^{(n)}(t)\Delta z\right\} - J_{\mathrm{mx}}^{(n)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(n)}(t)\Delta x\Delta z = -\left\{\left[I - S_{-M_z}\right]E_x^{(n)}(t)\Delta x + \left[S_{-M_x} - I\right]E_z^{(n)}(t)\Delta z\right\} - J_{\mathrm{my}}^{(n)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(n)}(t)\Delta x\Delta y = -\left\{\left[S_{-M_y} - I\right]E_x^{(n)}(t)\Delta x + \left[I - S_{-M_x}\right]E_y^{(n)}(t)\Delta y\right\} - J_{\mathrm{mz}}^{(n)}(t)\Delta x\Delta y$$

3-D FIT – Local Spatial Shift Operators / 3D-FIT – Lokale räumliche Schiebeoperatoren

1. Simple spatial shift operation / Einfache räumliche Schiebeoperation

$$S_{\pm M_i} f^{(n)} = f^{(n \pm M_i)}$$

2. Identity operation / Identitätsoperation

$$I\, f^{(n)} = f^{(n)}$$

3. Multiple shift operations / Zusammengesetzte Schiebeoperationen

$$S_{\pm M_i} S_{\pm M_j} f^{(n)} = S_{\pm M_j} S_{\pm M_i} f^{(n)} = f^{(n \pm M_i \pm M_j)}$$

Special case for $M_j = -M_i$ / Speziell folgt für $M_j = -M_i$

$$S_{\pm M_i} S_{\mp M_i} = I$$

4. Local difference operator / Lokaler Differenzoperator

$$P_{\pm M_i} = \mp I \pm S_{\pm M_i}$$

5. Local averaging operator / Lokaler Mittelungsoperator

$$A_{\pm M_i} = \frac{1}{2}\left(I + S_{\pm M_i} \right)$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

Local grid equations with local spatial shift operators in global grid node notation /
Lokale Gittergleichungen mit lokalen räumlichen Schiebeoperatoren in globaler Gitterknotennotation

$$\frac{d}{dt}B_x^{(n)}(t)\Delta y\Delta z = -\left\{\left[S_{-M_z} - I\right]E_y^{(n)}(t)\Delta y + \left[I - S_{M_y}\right]E_z^{(n)}(t)\Delta z\right\} - J_{\mathrm{mx}}^{(n)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(n)}(t)\Delta x\Delta z = -\left\{\left[I - S_{-M_z}\right]E_x^{(n)}(t)\Delta x + \left[S_{-M_x} - I\right]E_z^{(n)}(t)\Delta z\right\} - J_{\mathrm{my}}^{(n)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(n)}(t)\Delta x\Delta y = -\left\{\left[S_{-M_y} - I\right]E_x^{(n)}(t)\Delta x + \left[I - S_{-M_x}\right]E_y^{(n)}(t)\Delta y\right\} - J_{\mathrm{mz}}^{(n)}(t)\Delta x\Delta y$$

… in local matrix form / … in lokaler Matrixform

$$\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[S]}\frac{d}{dt}\underbrace{\begin{Bmatrix}B_x^{(n)}(t)\\ B_y^{(n)}(t)\\ B_z^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{B\}^{(n)}(t)} = -\underbrace{\begin{bmatrix}0 & S_{-M_z} - I & I - S_{M_y}\\ I - S_{-M_z} & 0 & S_{-M_x} - I\\ S_{-M_y} - I & I - S_{-M_x} & 0\end{bmatrix}}_{=[\mathrm{curl}]}\underbrace{\begin{bmatrix}\Delta x & & \\ & \Delta y & \\ & & \Delta z\end{bmatrix}}_{=[R]}\underbrace{\begin{Bmatrix}E_x^{(n)}(t)\\ E_y^{(n)}(t)\\ E_z^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}(t)}$$

$$-\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[S]}\underbrace{\begin{Bmatrix}J_{\mathrm{mx}}^{(n)}(t)\\ J_{\mathrm{mx}}^{(n)}(t)\\ J_{\mathrm{mx}}^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{J_{\mathrm{m}}\}^{(n)}(t)}$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

$$\underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=[S]} \frac{d}{dt} \underbrace{\begin{Bmatrix} B_x^{(n)}(t) \\ B_y^{(n)}(t) \\ B_z^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{B\}^{(n)}(t)} = - \underbrace{\begin{bmatrix} 0 & S_{-M_z} - I & I - S_{M_y} \\ I - S_{-M_z} & 0 & S_{-M_x} - I \\ S_{-M_y} - I & I - S_{-M_x} & 0 \end{bmatrix}}_{=[\mathrm{curl}]} \underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}(t)}$$

$$- \underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=[S]} \underbrace{\begin{Bmatrix} J_{\mathrm{mx}}^{(n)}(t) \\ J_{\mathrm{mx}}^{(n)}(t) \\ J_{\mathrm{mx}}^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{J_{\mathrm{m}}\}^{(n)}(t)}$$

$$\begin{bmatrix} 0 & S_{-M_z} - I & I - S_{-M_y} \\ I - S_{-M_z} & 0 & S_{-M_x} - I \\ S_{-M_y} - I & I - S_{-M_x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -P_{-M_z} & P_{-M_y} \\ P_{-M_z} & 0 & -P_{-M_x} \\ -P_{-M_y} & P_{-M_x} & 0 \end{bmatrix} = [\mathrm{curl}]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Faraday's induction law in local matrix form / Faradaysches Induktionsgesetz in lokaler Matrixform

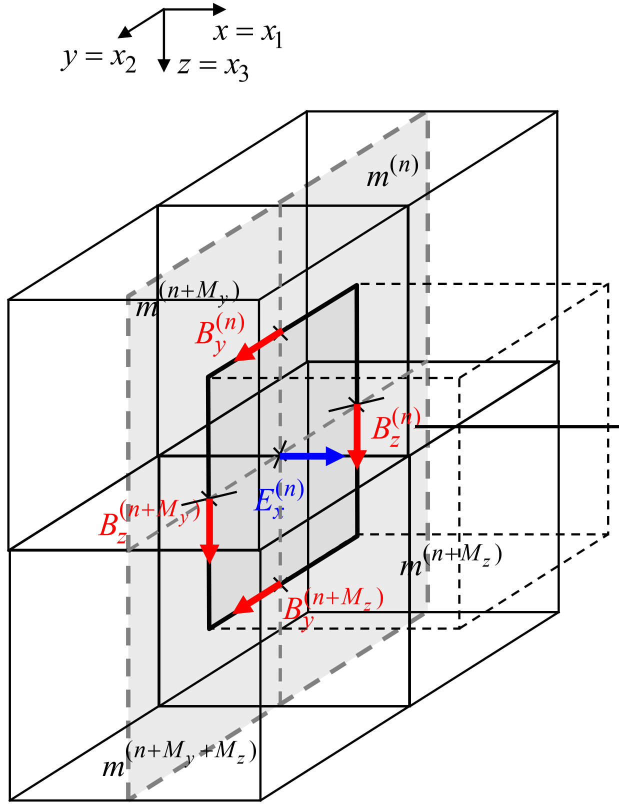
$$\underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y \end{bmatrix}}_{=[S]} \frac{d}{dt} \underbrace{\begin{Bmatrix} B_x^{(n)}(t) \\ B_y^{(n)}(t) \\ B_z^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{B\}^{(n)}(t)} = - \underbrace{\begin{bmatrix} 0 & -P_{-M_z} & P_{-M_y} \\ P_{-M_z} & 0 & -P_{-M_x} \\ -P_{-M_y} & P_{-M_x} & 0 \end{bmatrix}}_{=[\mathrm{curl}]} \underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}(t)} - \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y \end{bmatrix}}_{=[S]} \underbrace{\begin{Bmatrix} J_{\mathrm{mx}}^{(n)}(t) \\ J_{\mathrm{mx}}^{(n)}(t) \\ J_{\mathrm{mx}}^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{J_\mathrm{m}\}^{(n)}(t)}$$

$$[S]\frac{d}{dt}\{B\}^{(n)}(t) = -[\mathrm{curl}][R]\{E\}^{(n)}(t) - [S]\{J_\mathrm{m}\}^{(n)}(t)$$

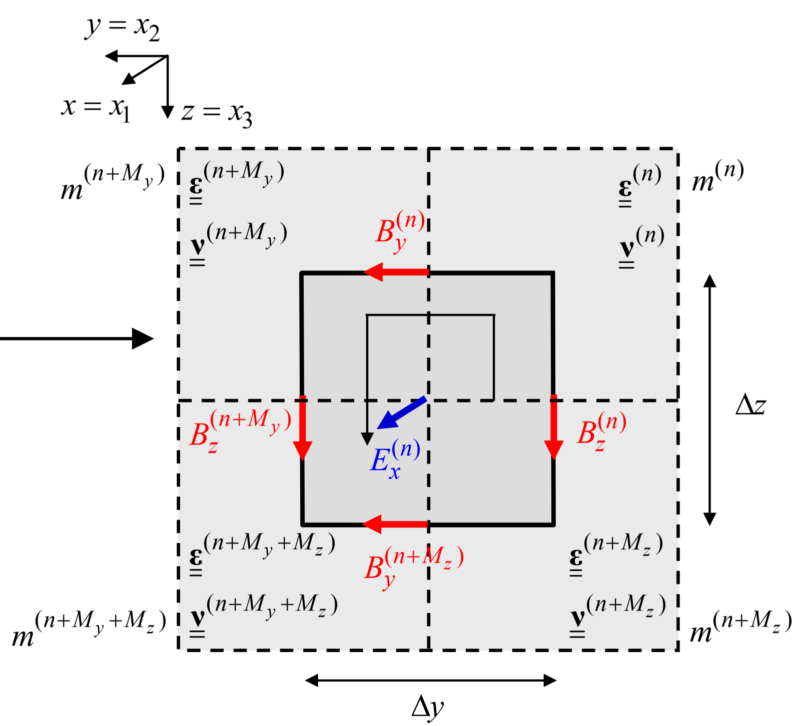
$[S]$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\{B\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$[\mathrm{curl}]$	$\in \mathbb{R}^{3\times 3}$	Topological curl operator in matrix form on the grid G / Topologischer Rotationsoperator in Matrixform auf dem Gitter G
$[R]$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of elementary lines on the grid G / Diagonalmatrix der Elementarstrecken auf dem Gitter G
$\{E\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic electric field strength vector / Algebraischer elektrische Feldstärkevektor
$\{J_\mathrm{m}\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic magnetic current density vector / Algebraischer magnetischer Stromdichtevektor

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\mathbf{v}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$



$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle

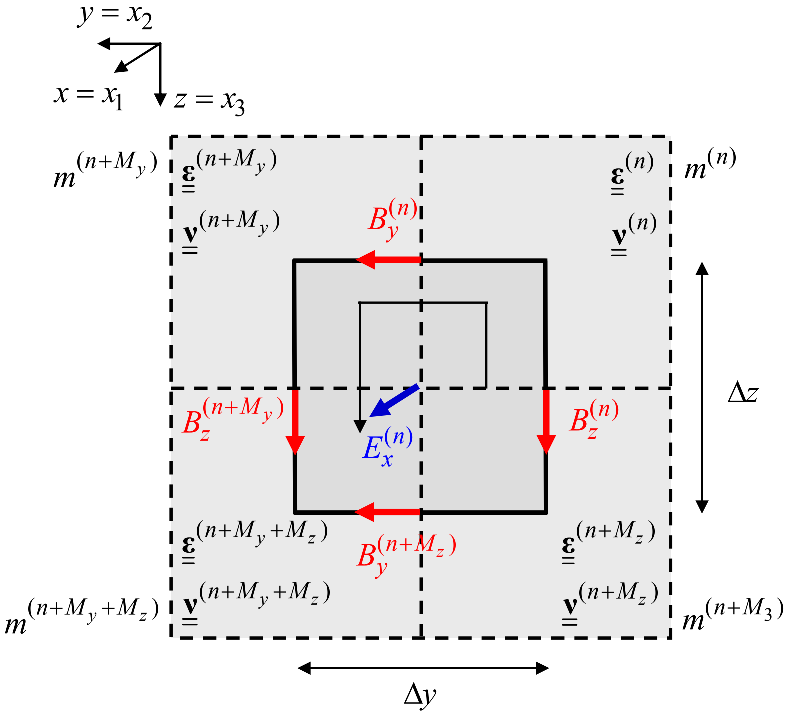


3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\underbrace{\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\underline{\mathbf{dS}} = \iint_S \underline{\mathbf{e}}_x\bullet\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\mathrm{d}S$$
$$= \iint_S \varepsilon_{xx}(\underline{\mathbf{R}})\,E_x(\underline{\mathbf{R}},t)\,\mathrm{d}S$$
$$= E_x^{(n)}(t)\iint_S \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S$$
$$+\mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle



$$\iint_S \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S$$
$$= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n)} + \varepsilon_{xx}^{(n+M_y)} + \varepsilon_{xx}^{(n+M_z)} + \varepsilon_{xx}^{(n+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n)}}\Delta y\Delta z$$
$$= \tilde{\varepsilon}_{xx}^{(n)}\Delta y\Delta z$$

$$\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\underline{\mathbf{dS}}$$
$$= E_x^{(n)}(t)\tilde{\varepsilon}_{xx}^{(n)}\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

$$\iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$= J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$

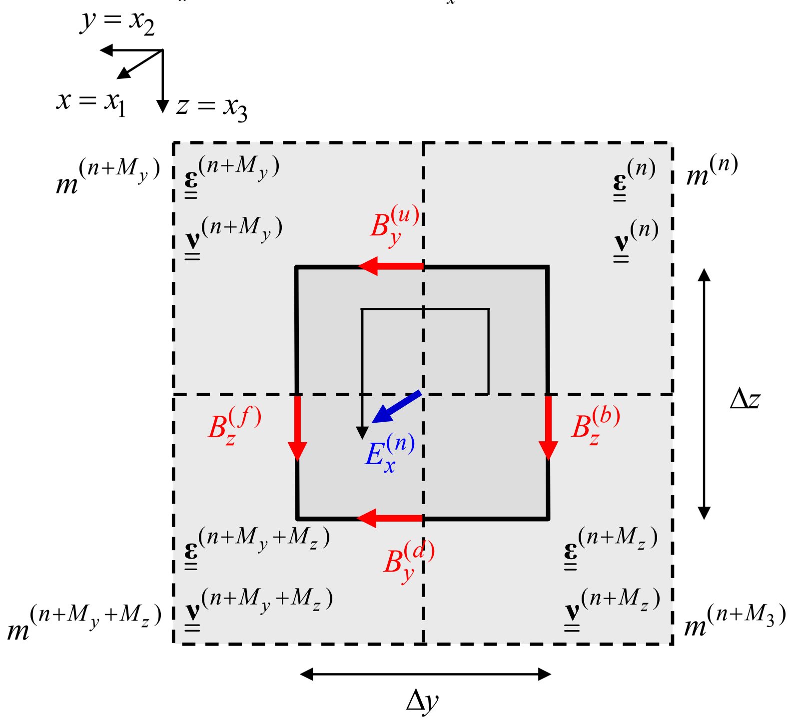
$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} = ?$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

$$\underline{\mathbf{dR}}_y = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_y\,\mathrm{d}y$$

$$\underline{\mathbf{dR}}_z = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_z\,\mathrm{d}z$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} = \int_{C^{(u)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} + \int_{C^{(f)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} - \int_{C^{(d)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} - \int_{C^{(b)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}}$$

$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dR}} = \int_{C^{(u)}}\nu_{yy}(\underline{\mathbf{R}})B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}}\nu_{zz}(\underline{\mathbf{R}})B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}}\nu_{yy}(\underline{\mathbf{R}})B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}}\nu_{zz}(\underline{\mathbf{R}})B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \left[\underline{\underline{\nu}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{B}}(\underline{\mathbf{R}},t) \right] \cdot \mathbf{d}\underline{\mathbf{R}}$$

$$= \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$\int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = B_y^{(u)}(t) \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y + \bigcirc\left[(\Delta y)^3 \right] \qquad \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y = \frac{1}{2}\left[\nu_{yy}^{(n)} + \nu_{yy}^{(n+M_y)} \right] \Delta y$$

$$\int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = B_z^{(f)}(t) \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z + \bigcirc\left[(\Delta z)^3 \right] \qquad \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z = \frac{1}{2}\left[\nu_{yy}^{(n+M_z)} + \nu_{yy}^{(n+M_y+M_z)} \right] \Delta z$$

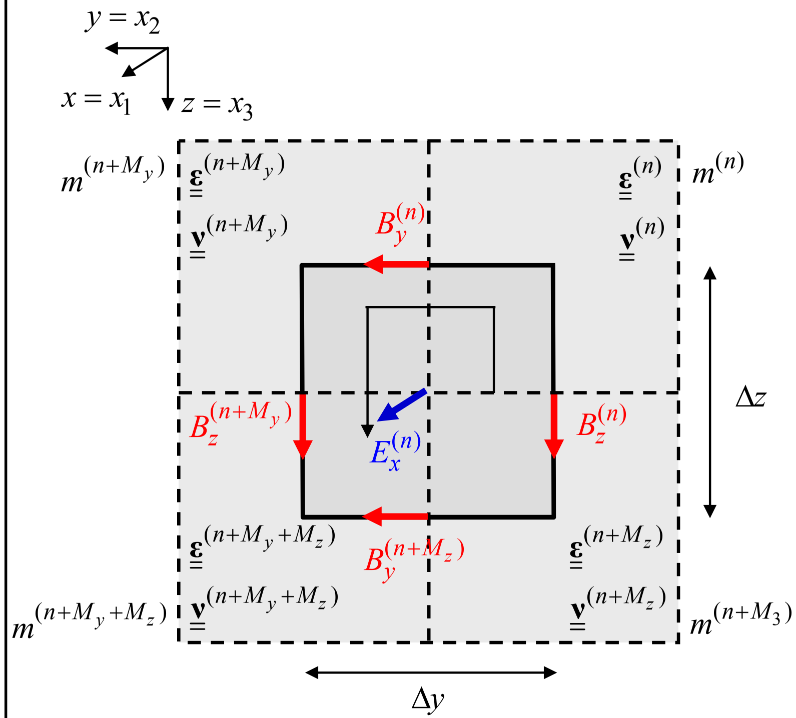
$$\int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = B_y^{(d)}(t) \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y + \bigcirc\left[(\Delta y)^3 \right] \qquad \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y = \frac{1}{2}\left[\nu_{zz}^{(n+M_y)} + \nu_{zz}^{(n+M_y+M_z)} \right] \Delta y$$

$$\int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = B_z^{(b)}(t) \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z + \bigcirc\left[(\Delta z)^3 \right] \qquad \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z = \frac{1}{2}\left[\nu_{zz}^{(n)} + \nu_{zz}^{(n+M_z)} \right] \Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}}$$

$$= \underbrace{\frac{1}{2}\left[\nu_{yy}^{(n)} + \nu_{yy}^{(n+M_y)}\right]}_{=\tilde{\nu}_{yy}^{(n)}} B_y^{(n)}(t)\Delta y$$

$$- \underbrace{\frac{1}{2}\left[\nu_{yy}^{(n+M_z)} + \nu_{yy}^{(n+M_y+M_z)}\right]}_{=\tilde{\nu}_{yy}^{(n+M_y)}} B_y^{(n+M_z)}(t)\Delta y$$

$$+ \underbrace{\frac{1}{2}\left[\nu_{zz}^{(n+M_y)} + \nu_{zz}^{(n+M_y+M_z)}\right]}_{=\tilde{\nu}_{zz}^{(n+M_z)}} B_z^{(n+M_y)}(t)\Delta z$$

$$- \underbrace{\frac{1}{2}\left[\nu_{zz}^{(n)} + \nu_{zz}^{(n+M_z)}\right]}_{=\tilde{\nu}_{zz}^{(n)}} B_z^{(n)}(t)\Delta z$$

$$= \tilde{\nu}_{yy}^{(n)} B_y^{(n)}(t)\Delta y - \tilde{\nu}_{yy}^{(n+M_y)} B_y^{(n+M_y)}(t)\Delta y$$

$$+ \tilde{\nu}_{zz}^{(n+M_z)} B_z^{(n+M_y)}(t)\Delta z - \tilde{\nu}_{zz}^{(n)} B_z^{(n)}(t)\Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

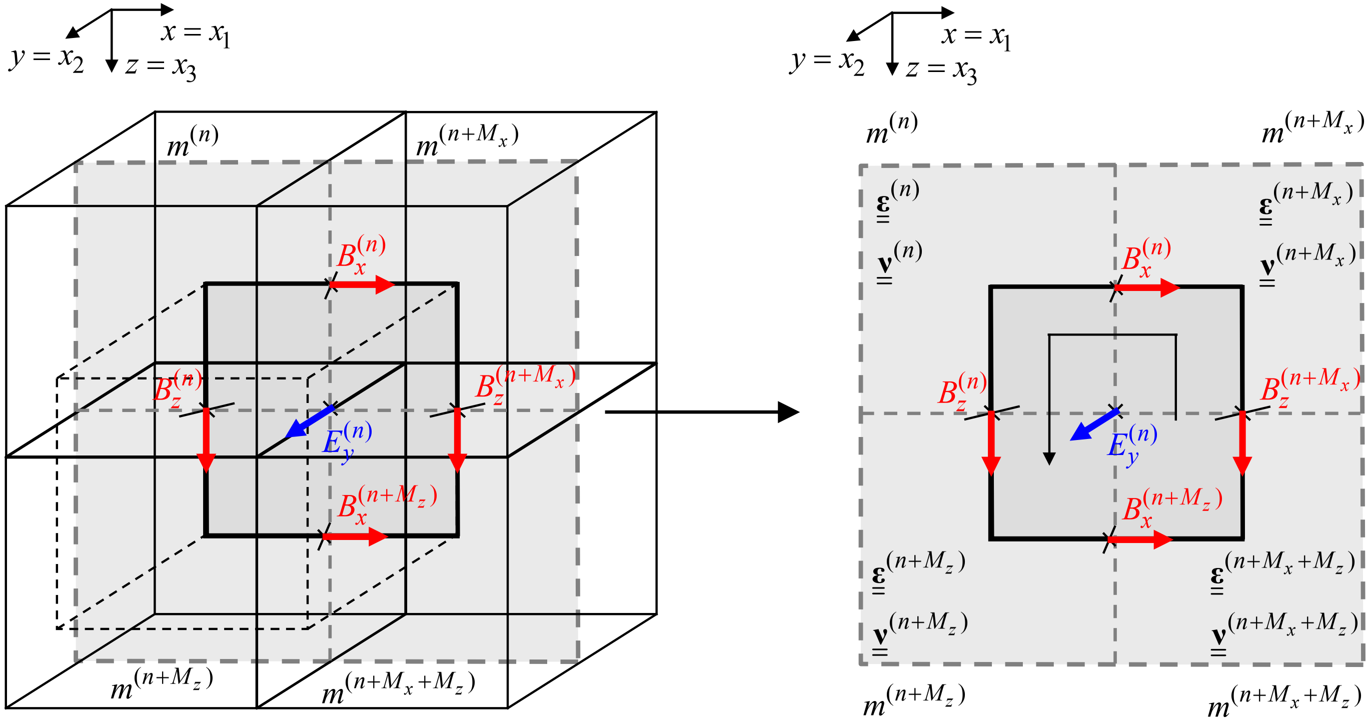
$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle

$$\begin{aligned}\tilde{\varepsilon}_{xx}^{(n)}\frac{d}{dt}E_x^{(n)}(t)\Delta y\Delta z &= \tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - \tilde{\nu}_{yy}^{(n+M_z)}B_y^{(n+M_z)}(t)\Delta y \\ &\quad + \tilde{\nu}_{zz}^{(n+M_y)}B_z^{(n+M_y)}(t)\Delta z - \tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z \\ &= \left(I - S_{M_z}\right)\tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y + \left(S_{M_y} - I\right)\tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z\end{aligned}$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

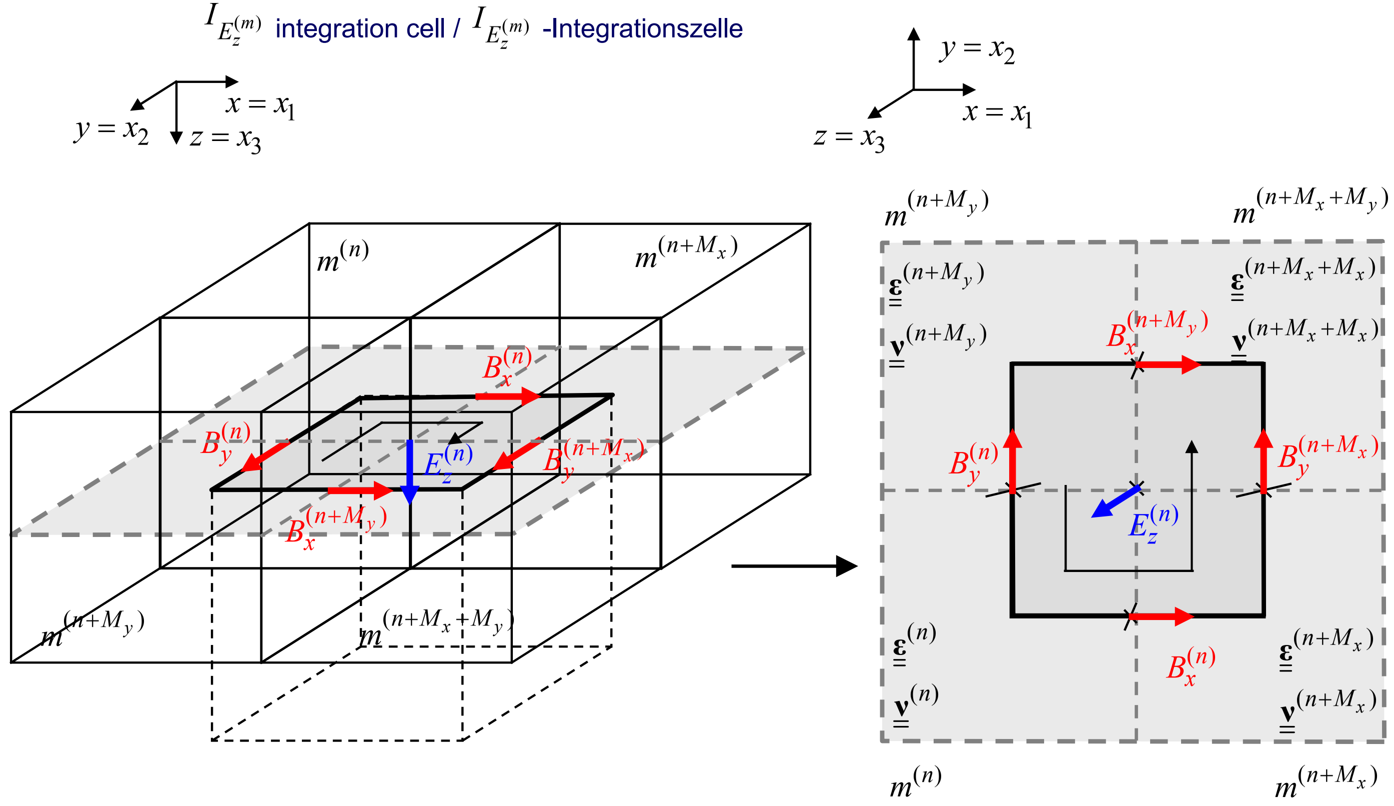
$I_{E_y^{(m)}}$ integration cell / $I_{E_y^{(m)}}$ -Integrationszelle



3-D FIT – Derivation of the Discrete Grid Equations /
3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_z^{(m)}}$ integration cell / $I_{E_z^{(m)}}$ -Integrationszelle



3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\Downarrow$$

$$\begin{aligned}
\tilde{\varepsilon}_{xx}^{(n)}\frac{d}{dt}E_x^{(n)}(t)\Delta y\Delta z &= \tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - \tilde{\nu}_{yy}^{(n+M_z)}B_y^{(n+M_z)}(t)\Delta y \\
&\quad + \tilde{\nu}_{zz}^{(n+M_y)}B_z^{(n+M_y)}(t)\Delta z - \tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z \\
&= \left(I - S_{M_z}\right)\tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y + \left(S_{M_y} - I\right)\tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z \\
\tilde{\varepsilon}_{yy}^{(n)}\frac{d}{dt}E_y^{(n)}(t)\Delta x\Delta z &= \tilde{\nu}_{xx}^{(n+M_3)}B_x^{(n+M_3)}(t)\Delta x - \tilde{\nu}_{xx}^{(n_z)}B_x^{(n)}(t)\Delta x \\
&\quad + \tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - \tilde{\nu}_{zz}^{(n+M_x)}B_z^{(n+M_x)}(t)\Delta z - J_{\mathrm{ey}}^{(n)}(t)\Delta x\Delta z \\
&= \left(S_{M_z} - I\right)\tilde{\nu}_{xx}^{(n)}B_x^{(n)}(t)\Delta x + \left(I - S_{M_x}\right)\tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ey}}^{(n)}(t)\Delta y\Delta z \\
\tilde{\varepsilon}_{zz}^{(n)}\frac{d}{dt}E_z^{(n)}(t)\Delta x\Delta y &= \tilde{\nu}_{xx}^{(n)}B_x^{(n)}(t)\Delta x - \tilde{\nu}_{xx}^{(n+M_y)}B_x^{(n+M_y)}(t)\Delta x \\
&\quad + \tilde{\nu}_{yy}^{(n+M_x)}B_y^{(n+M_x)}(t)\Delta y - \tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - J_{\mathrm{ez}}^{(n)}(t)\Delta x\Delta y \\
&= \left(I - S_{M_y}\right)\tilde{\nu}_{xx}^{(n)}B_x^{(n)}(t)\Delta x + \left(S_{M_x} - I\right)\tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - J_{\mathrm{ex}}^{(n)}(t)\Delta x\Delta y
\end{aligned}$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

$$\underbrace{\begin{bmatrix} \tilde{\varepsilon}_{xx}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)} \end{bmatrix}}_{=[\tilde{\varepsilon}]^{(n)}} \underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=[\tilde{S}]} \frac{d}{dt} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}}_{=\{E\}^{(n)}(t)}$$

$$= \underbrace{\begin{bmatrix} 0 & I - S_{M_z} & S_{M_y} - I \\ S_{M_z} - I & 0 & i - S_{M_x} \\ I - S_{M_y} & S_{M_x} - I & 0 \end{bmatrix}}_{=[\widetilde{\mathrm{curl}}]} \underbrace{\begin{bmatrix} \tilde{\nu}_{xx}^{(n)} & & \\ & \tilde{\nu}_{yy}^{(n)} & \\ & & \tilde{\nu}_{zz}^{(n)} \end{bmatrix}}_{=[\tilde{\nu}]^{(n)}} \underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[\tilde{R}]} \underbrace{\begin{Bmatrix} B_x^{(n)}(t) \\ B_y^{(n)}(t) \\ B_z^{(n)}(t) \end{Bmatrix}}_{=\{B\}^{(n)}(t)} - \underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=[\tilde{S}]} \underbrace{\begin{Bmatrix} J_{\mathrm{ex}}^{(n)}(t) \\ J_{\mathrm{ex}}^{(n)}(t) \\ J_{\mathrm{ex}}^{(n)}(t) \end{Bmatrix}}_{=\{J_{\mathrm{e}}\}^{(n)}(t)}$$

$$\begin{bmatrix} 0 & I - S_{M_z} & S_{M_y} - I \\ S_{M_z} - I & 0 & i - S_{M_x} \\ I - S_{M_y} & S_{M_x} - I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -P_{M_z} & P_{M_y} \\ P_{M_z} & 0 & -P_{M_x} \\ -P_{M_y} & P_{M_x} & 0 \end{bmatrix} = [\widetilde{\mathrm{curl}}]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

$$\underbrace{\begin{bmatrix}\tilde{\varepsilon}_{xx}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)}\end{bmatrix}}_{=[\tilde{\varepsilon}]^{(n)}}\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[\tilde{S}]}\frac{d}{dt}\underbrace{\begin{Bmatrix}E_x^{(n)}(t)\\ E_y^{(n)}(t)\\ E_z^{(n)}(t)\end{Bmatrix}}_{=\{E\}^{(n)}(t)}=\underbrace{\begin{bmatrix}0 & -P_{M_z} & P_{M_y}\\ P_{M_z} & 0 & -P_{M_x}\\ -P_{M_y} & P_{M_x} & 0\end{bmatrix}}_{=[\widetilde{\mathrm{curl}}]}\underbrace{\begin{bmatrix}\tilde{\nu}_{xx}^{(n)} & & \\ & \tilde{\nu}_{yy}^{(n)} & \\ & & \tilde{\nu}_{zz}^{(n)}\end{bmatrix}}_{=[\tilde{\nu}]^{(n)}}\underbrace{\begin{bmatrix}\Delta x & & \\ & \Delta y & \\ & & \Delta z\end{bmatrix}}_{=[\tilde{R}]}\underbrace{\begin{Bmatrix}B_x^{(n)}(t)\\ B_y^{(n)}(t)\\ B_z^{(n)}(t)\end{Bmatrix}}_{=\{B\}^{(n)}(t)}-\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[\tilde{S}]}\underbrace{\begin{Bmatrix}J_{\mathrm{ex}}^{(n)}(t)\\ J_{\mathrm{ex}}^{(n)}(t)\\ J_{\mathrm{ex}}^{(n)}(t)\end{Bmatrix}}_{=\{J_{\mathrm{e}}\}^{(n)}(t)}$$

$$[\tilde{\varepsilon}]^{(n)}[\tilde{S}]\frac{d}{dt}\{E\}^{(n)}(t)=[\widetilde{\mathrm{curl}}][\tilde{\nu}]^{(n)}[\tilde{R}]\{B\}^{(n)}(t)-[\tilde{S}]\{J_{\mathrm{e}}\}^{(n)}(t)$$

$[\tilde{\varepsilon}]^{(n)}$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of permittivities on the grid $\tilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\tilde{G}$
$[\tilde{S}]$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of elementary surfaces on the grid $\tilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\tilde{G}$
$\{E\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$[\widetilde{\mathrm{curl}}]$	$\in \mathbb{R}^{3\times 3}$	Topological curl operator in matrix form on the grid $\tilde{G}$ / Topologischer Rotationsoperator in Matrixform auf dem Gitter $\tilde{G}$
$[\tilde{\nu}]^{(n)}$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of impermeabilities on the grid $\tilde{G}$ / Diagonalmatrix der Impermeabilitäten auf dem Gitter $\tilde{G}$
$[\tilde{R}]$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of elementary lines on the grid $\tilde{G}$ / Diagonalmatrix der Elementarstrecken auf dem Gitter $\tilde{G}$
$\{B\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$\{J_{\mathrm{e}}\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic electric current density vector / Algebraischer elektrischer Stromdichtevektor

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler und globaler Matrixform

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{R}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}}$$

$$\frac{d}{dt}\iint_S \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}} = \oint_{C=\partial S} \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{R}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}}$$

Discrete grid equations in local matrix form / Diskrete Gittergleichungen in lokaler Matrixform

$$[S]\frac{d}{dt}\{B\}^{(n)}(t) = -[\mathrm{curl}][R]\{E\}^{(n)}(t) \qquad -[S]\{J_{\mathrm{m}}\}^{(n)}(t)$$

$$\widetilde{[\varepsilon]}^{(n)}\widetilde{[S]}\frac{d}{dt}\{E\}^{(n)}(t) = \widetilde{[\mathrm{curl}]}\widetilde{[\nu]}^{(n)}\widetilde{[R]}\{B\}^{(n)}(t) - \widetilde{[S]}\{J_{\mathrm{e}}\}^{(n)}(t)$$

Discrete grid equations in global matrix form / Diskrete Gittergleichungen in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \qquad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\boldsymbol{\nu}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_{\mathrm{e}}\}(t)$$

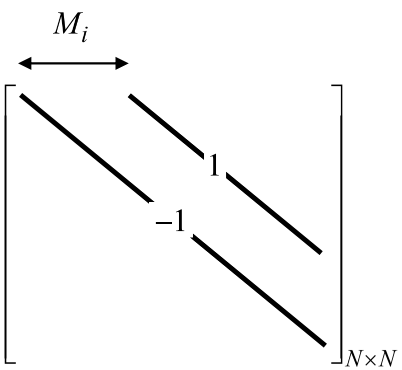
Elementary Difference Matrix $[\mathbf{P}_i]$ (P Matrix) / Elementare Differenzmatrix $[\mathbf{P}_i]$ (P-Matrix)

Elementary difference operator in global matrix form (P matrix) /
Elementarer Differenzoperator in globaler Matrixform (P-Matrix)

$$\left[\mathbf{P}_{\pm i}\right] := \left(\left[\mathbf{P}_{\pm i}\right]\right)_{jk}, \quad j,k \in \{1,2,\ldots,N\}$$

$$\left(\left[\mathbf{P}_{\pm i}\right]\right)_{jk} = \begin{cases} \mp 1 & j = k \\ \pm 1 & j = k \mp M_i \text{ or / bzw. } k = j \pm M_i; \quad i = x,y,z; \quad j,k \in \{1,2,\ldots,N\} \\ 0 & \text{else / sonst} \end{cases}$$

$\left[\mathbf{P}_i\right] =$

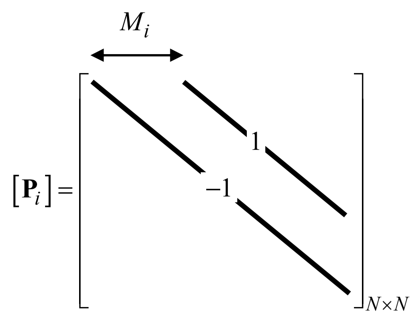


The P matrix has only two bands /
Die P-Matrix hat nur zwei Bänder

Elementary Difference Matrix $[\mathbf{P}_i]$ (P Matrix) (…) / Elementare Differenzmatrix $[\mathbf{P}_i]$ (P-Matrix) (…)

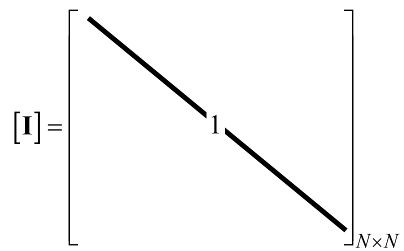
The P matrix can be represented by a sum of an identity matrix $[\mathbf{I}]$ and a band matrix $[\mathbf{B}]$ /
Die P-Matrix kann als Summe aus einer Einheitsmatrix (Identitätsmatrix) $[\mathbf{I}]$ und Bandmatrix $[\mathbf{B}]$ dargestellt werden

$$[\mathbf{P}_{\pm i}] := \mp[\mathbf{I}] \pm [\mathbf{B}_{\pm i}], \qquad i = \{x, y, z\}$$



Identity matrix / Einheitsmatrix (Identitätsmatrix)

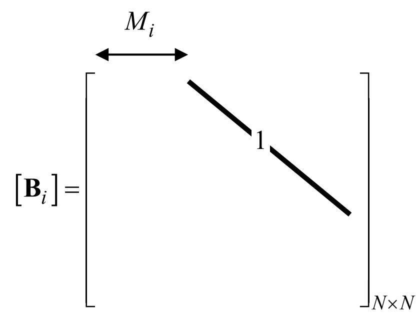
$$([\mathbf{I}])_{ij} = \delta_{ij} \qquad i, j \in \{1, 2, \ldots, N\}$$



Band matrix / Bandmatrix

$$([\mathbf{B}]_{\pm i})_{jk} = \begin{cases} 1 & j = k \mp M_i \text{ or / bzw. } k = j \pm M_i \\ 0 & \text{else / sonst} \end{cases}$$

$$i = x, y, z; \;\; j, k \in \{1, 2, \ldots, N\}$$

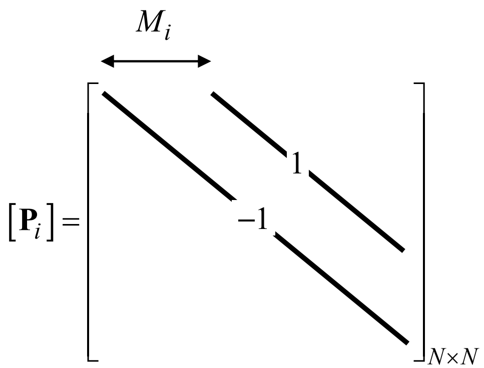


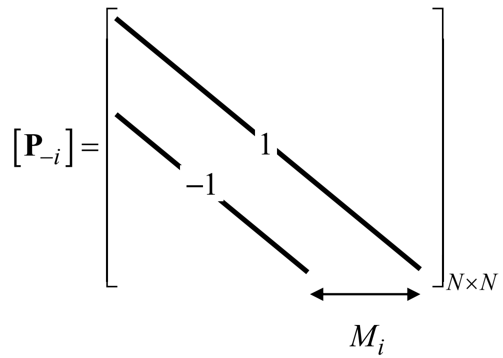
Properties of the Difference Matrix $[\mathrm{P}_i]$ (P Matrix) / Eigenschaften der Differenzmatrix $[\mathrm{P}_i]$ (P-Matrix)

$$[\mathbf{P}_{\pm i}] := \mp[\mathbf{I}] \pm [\mathbf{B}_{\pm i}], \quad i = \{x, y, z\}$$

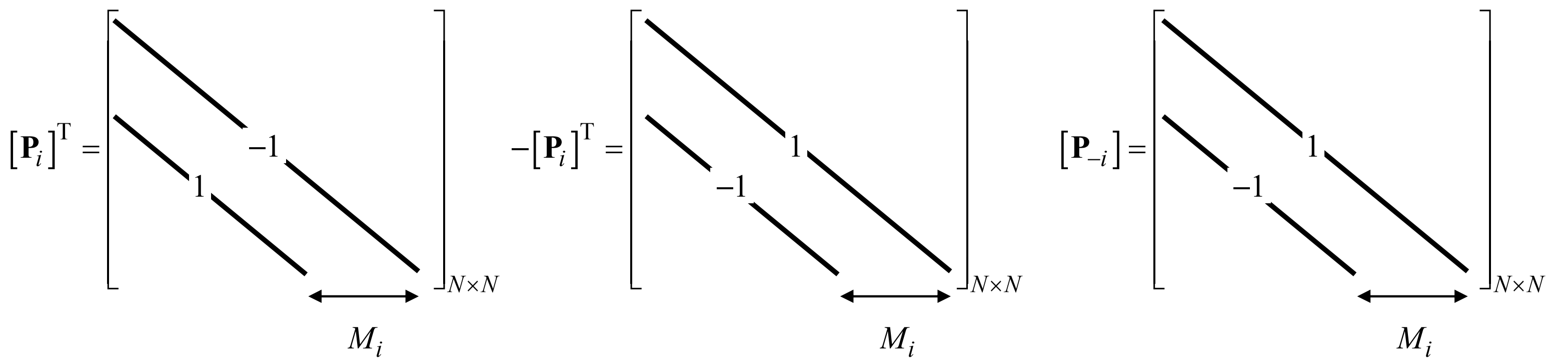
$$[\mathbf{P}_i] := -[\mathbf{I}] + [\mathbf{B}_i], \quad i = \{x, y, z\}$$

$$[\mathbf{P}_{-i}] := [\mathbf{I}] - [\mathbf{B}_{-i}], \quad i = \{x, y, z\}$$





Property / Eigenschaft $\quad -[\mathbf{P}_i]^{\mathrm{T}} = [\mathbf{P}_{-i}]$



Discrete Global Gradient, Divergence, and Curl Operator / Diskreter globaler Gradienten-, Divergenz- und Rotationsoperator

Discrete gradient operator /
Diskreter Gradientenoperator

$$[\mathbf{grad}] = \begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}_{3N\times N}$$

$$[\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix}_{3N\times N}$$

Discrete curl operator /
Diskreter Rotationsoperator

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^{\mathrm{T}} & -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} & [\mathbf{0}] & [\mathbf{P}_x]^{\mathrm{T}} \\ [\mathbf{P}_y]^{\mathrm{T}} & -[\mathbf{P}_x]^{\mathrm{T}} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

Discrete divergence operator /
Diskreter Divergenzoperator

$$[\mathbf{div}] := \left[-[\mathbf{P}_x]^{\mathrm{T}}, -[\mathbf{P}_y]^{\mathrm{T}}, -[\mathbf{P}_z]^{\mathrm{T}} \right]_{N\times 3N}$$

$$[\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x], [\mathbf{P}_y], [\mathbf{P}_z] \right]_{N\times 3N}$$

The matrix operators /
Die Matrixoperatoren

$$[\mathbf{grad}] \quad [\widetilde{\mathbf{grad}}]$$
$$[\mathbf{div}] \quad [\widetilde{\mathbf{div}}]$$
$$[\mathbf{curl}] \quad [\widetilde{\mathbf{curl}}]$$

are **global** matrix operators /
sind **globale** Matrixoperatoren

Properties of the Global Matrix Operators / Eigenschaften der globalen Matrixoperatoren

Some properties of the global matrix operators of the dual grid system / Einige Eigenschaften der globalen Matrixoperatoren des dualen Gittersystems

$$-\left[\widetilde{\mathbf{div}}\right] = \left[\mathbf{grad}\right]^{\mathrm{T}}$$

$$\left[\widetilde{\mathbf{grad}}\right]^{\mathrm{T}} = \left[\mathbf{div}\right]$$

$$\left[\mathbf{curl}\right] = \left[\widetilde{\mathbf{curl}}\right]^{\mathrm{T}}$$

Conservation of important vector identities / Erhaltung von wichtigen Vektoridentitäten

Vector identities / Vektoridentitäten

$$\mathrm{curl\ grad} = \nabla \times \nabla = \underline{\mathbf{0}}$$

$$\mathrm{div\ curl} = \nabla \bullet \nabla = 0$$

$$\Updownarrow$$

are conserved in the dual grid system / bleiben im dualen Gittersystem erhalten

$$\left[\mathbf{curl}\right]\left[\mathbf{grad}\right] = \left[\mathbf{0}\right]$$

$$\left[\widetilde{\mathbf{curl}}\right]\left[\widetilde{\mathbf{grad}}\right] = \left[\mathbf{0}\right]$$

$$\left[\mathbf{div}\right]\left[\mathbf{curl}\right] = \left[\mathbf{0}\right]$$

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\mathbf{curl}}\right] = \left[\mathbf{0}\right]$$

Properties of the Global Matrix Operators / Eigenschaften der globalen Matrixoperatoren

Consistency test / Konsistenztest

$$[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix} = \begin{bmatrix} [\mathbf{P}_y][\mathbf{P}_z] - [\mathbf{P}_z][\mathbf{P}_y] \\ [\mathbf{P}_z][\mathbf{P}_x] - [\mathbf{P}_x][\mathbf{P}_z] \\ [\mathbf{P}_x][\mathbf{P}_y] - [\mathbf{P}_y][\mathbf{P}_x] \end{bmatrix}$$

$$\begin{aligned} [\mathbf{P}_i][\mathbf{P}_j] - [\mathbf{P}_j][\mathbf{P}_i] &= \left(-[\mathbf{I}] + [\mathbf{B}_i]\right)\left(-[\mathbf{I}] + [\mathbf{B}_j]\right) - \left(-[\mathbf{I}] + [\mathbf{B}_j]\right)\left(-[\mathbf{I}] + [\mathbf{B}_i]\right) \\ &= \left(-[\mathbf{I}][\mathbf{I}] - [\mathbf{I}][\mathbf{B}_j] - [\mathbf{B}_i][\mathbf{I}] + [\mathbf{B}_i][\mathbf{B}_j]\right) \\ &\quad - \left(-[\mathbf{I}][\mathbf{I}] - [\mathbf{I}][\mathbf{B}_i] - [\mathbf{B}_j][\mathbf{I}] + [\mathbf{B}_j][\mathbf{B}_i]\right) \\ &= \left(-[\mathbf{I}] - [\mathbf{B}_j] - [\mathbf{B}_i] + [\mathbf{B}_i][\mathbf{B}_j]\right) \\ &\quad - \left(-[\mathbf{I}] - [\mathbf{B}_i] - [\mathbf{B}_j] + [\mathbf{B}_j][\mathbf{B}_i]\right) \\ &= -[\mathbf{I}] - [\mathbf{B}_j] - [\mathbf{B}_i] + [\mathbf{B}_i][\mathbf{B}_j] + [\mathbf{I}] + [\mathbf{B}_i] + [\mathbf{B}_j] - [\mathbf{B}_j][\mathbf{B}_i] \\ &= [\mathbf{B}_i][\mathbf{B}_j] - [\mathbf{B}_j][\mathbf{B}_i] \end{aligned}$$

Properties of the Global Matrix Operators / Eigenschaften der globalen Matrixoperatoren

With the property /
Mit der Eigenschaft

$$\left([\mathbf{B}_{\pm i}][\mathbf{B}_{\pm j}]\right)_{kl} = \begin{cases} 1 & k = l \mp M_i \mp M_j \\ 0 & \text{else / sonst} \end{cases}$$

⇨ i and j can be arbitrarily interchanged /
i und j können beliebig vertauscht werden

⇨ This means that the matrices $[\mathbf{B}_{\pm i}]$ and $[\mathbf{B}_{\pm j}]$ as well as $[\mathbf{P}_{\pm i}]$ and $[\mathbf{P}_{\pm j}]$ are commutative!
Das bedeutet, dass die Matrizen $[\mathbf{B}_{\pm i}]$ und $[\mathbf{B}_{\pm j}]$ als auch $[\mathbf{P}_{\pm i}]$ und $[\mathbf{P}_{\pm j}]$ kommutativ sind!

⇨

$$[\mathbf{B}_{\pm i}][\mathbf{B}_{\pm j}] = [\mathbf{B}_{\pm j}][\mathbf{B}_{\pm i}]$$

$$[\mathbf{P}_{\pm i}][\mathbf{P}_{\pm j}] = [\mathbf{P}_{\pm j}][\mathbf{P}_{\pm i}]$$

⇨

$$\begin{aligned}[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{grad}}] &= [\mathbf{P}_i][\mathbf{P}_j] - [\mathbf{P}_j][\mathbf{P}_i] \\ &= [\mathbf{B}_i][\mathbf{B}_j] - \underbrace{[\mathbf{B}_j][\mathbf{B}_i]}_{=[\mathbf{B}_i][\mathbf{B}_j]} \\ &= [\mathbf{B}_i][\mathbf{B}_j] - [\mathbf{B}_i][\mathbf{B}_j] \\ &= [\mathbf{0}]\end{aligned}$$

End of Lecture 6 /
Ende der 6. Vorlesung