Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

7th Lecture / 7. Vorlesung

Dr.-Ing. René Marklein
marklein@uni-kassel.de
http://www.tet.e-technik.uni-kassel.de
http://www.uni-kassel.de/fb16/tet/marklein/index.html

Universität Kassel
Fachbereich Elektrotechnik / Informatik
(FB 16)
Fachgebiet Theoretische Elektrotechnik
(FG TET)
Wilhelmshöher Allee 71
Büro: Raum 2113 / 2115
D-34121 Kassel

University of Kassel
Dept. Electrical Engineering / Computer Science (FB 16)
Electromagnetic Field Theory
(FG TET)
Wilhelmshöher Allee 71
Office: Room 2113 / 2115
D-34121 Kassel

FDTD and FIT / FDTD und FIT

FDTD : Finite Difference Time Domain / Finite Differenzen im Zeitbereich
FIT : Finite Integration Technique / Finite Integrationstechnik

FDTD

Maxwell's equations in differential form /
Maxwellsche Gleichungen in Differentialform

$$\frac{\partial}{\partial t}\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

⇨ FD approximation of spatial and temporal derivatives / FD-Approximation von räumlichen und zeitlichen Ableitungen

Central difference approximation /
Zentrale Differenzen Approximation

$$\left.\frac{\partial}{\partial z}f(z,t)\right|_{z=z_0} \approx \frac{f\left(z_0+\frac{\Delta z}{2},t\right)-f\left(z_0-\frac{\Delta z}{2},t\right)}{\Delta z}$$

FIT

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

⇨ FIT approximation of spatial and temporal integrals / FIT-Approximation von räumlichen und zeitlichen Integralen

Mid point rule approximation of a 1-D integral /
Mittelpunktsregel-Approximation eines 1D-Integrals

$$\int_{z=z_0}^{z_0+\Delta z} f(z,t)\,\mathrm{d}z \approx f\left(z_0+\frac{\Delta z}{2},t\right)\Delta z$$

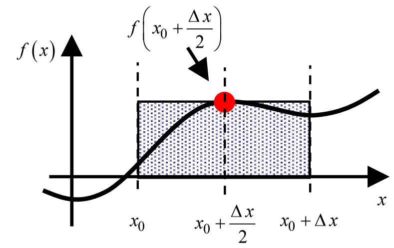
FIT Approximations Based on the Mid Point Rule in 1-D, 2-D, and 3-D / FIT-Approximationen basierend auf der Mittelpunktsregel in 1D, 2D und 3D

Cartesian coordinate system; homogeneous dual-orthogonal grid system with **rectangular grid cells /** Kartesisches Koordinatensystem; homogenes dual-orthogonales Gittersystem mit **rechteckförmigen Gitterzellen**

1-D:

$$\int_{x=x_0}^{x_0+\Delta x} f(x)\,\mathrm{d}x = f\left(x_0+\frac{\Delta x}{2}\right)\Delta x + \bigcirc\left[(\Delta x)^3\right]$$

$$\approx f\left(x_0+\frac{\Delta x}{2}\right)\Delta x$$



2-D:

$$\int_{y=y_0}^{y_0+\Delta y}\int_{x=x_0}^{x_0+\Delta x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta y}{2}\right)\Delta x\Delta y + \bigcirc\left[(\Delta x)^3\Delta y + \Delta x(\Delta y)^3\right]$$

$$\approx f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta y}{2}\right)\Delta x\Delta y$$

3-D:

$$\int_{z=z_0}^{z_0+\Delta x}\int_{y=y_0}^{y_0+\Delta x}\int_{x=x_0}^{x_0+\Delta x} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta y}{2}, z_0+\frac{\Delta z}{2}\right)\Delta x\Delta y\Delta z$$

$$+\bigcirc\left[(\Delta x)^3\Delta y\Delta z + \Delta x(\Delta y)^3\Delta z + \Delta x\Delta y(\Delta z)^3\right]$$

$$\approx f\left(x_0+\frac{\Delta x}{2}, y_0+\frac{\Delta y}{2}, z_0+\frac{\Delta z}{2}\right)\Delta x\Delta y\Delta z$$

FIT Formulation / FIT-Formulierung

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

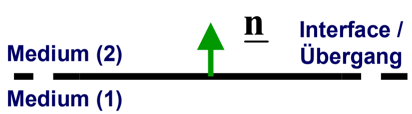
$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{m}(\underline{\mathbf{R}},t)\,\mathrm{d}V$$

$\underline{\mathbf{E}}(\underline{\mathbf{R}},t),\underline{\mathbf{B}}(\underline{\mathbf{R}},t)$
$\underline{\mathbf{E}}(\underline{\mathbf{R}},t),\underline{\mathbf{H}}(\underline{\mathbf{R}},t)$
$\underline{\mathbf{D}}(\underline{\mathbf{R}},t),\underline{\mathbf{H}}(\underline{\mathbf{R}},t)$
$\underline{\mathbf{D}}(\underline{\mathbf{R}},t),\underline{\mathbf{B}}(\underline{\mathbf{R}},t)$

Which field quantities should we use in our formulation / Welche Feldgrößen sollten wir in unserer FIT-Formulierung verwenden ?

⇨ $\underline{\mathbf{E}}(\underline{\mathbf{R}},t),\underline{\mathbf{B}}(\underline{\mathbf{R}},t)$ Formulation / Formulierung !

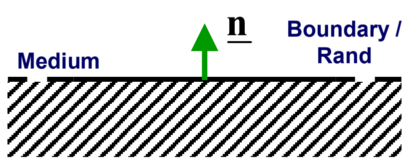
Transition conditions for a source-free interface /
Übergangsbedingungen für einen quellenfreien Übergang



$$\underline{\mathbf{n}}\times\left[\underline{\mathbf{E}}^{(2)}(\underline{\mathbf{R}},t)-\underline{\mathbf{E}}^{(1)}(\underline{\mathbf{R}},t)\right]=\underline{\mathbf{0}} \;\rightarrow\; E_t^{(2)}(\underline{\mathbf{R}},t)=E_t^{(2)}(\underline{\mathbf{R}},t) \quad \underline{\mathbf{R}}\in \text{Interface / Übergang}$$

$$\underline{\mathbf{n}}\bullet\left[\underline{\mathbf{B}}^{(2)}(\underline{\mathbf{R}},t)-\underline{\mathbf{B}}^{(1)}(\underline{\mathbf{R}},t)\right]=0 \;\rightarrow\; B_n^{(2)}(\underline{\mathbf{R}},t)=B_n^{(2)}(\underline{\mathbf{R}},t) \quad \underline{\mathbf{R}}\in \text{Interface / Übergang}$$

Boundary conditions for a PEC boundary /
Randbedingungen für einen IEL-Rand



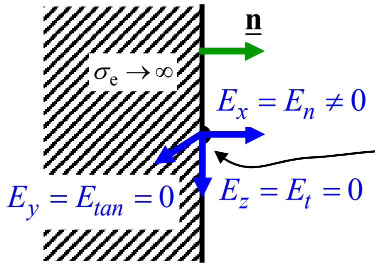
$$\underline{\mathbf{n}}\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t)=\underline{\mathbf{0}} \;\rightarrow\; E_t(\underline{\mathbf{R}},t)=0 \quad \underline{\mathbf{R}}\in\text{Boundary / Rand}$$

$$\underline{\mathbf{n}}\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)=0 \;\rightarrow\; B_n(\underline{\mathbf{R}},t)=0 \quad \underline{\mathbf{R}}\in\text{Boundary / Rand}$$

Problems of a Node-Based Formulation – Boundary Conditions / Probleme einer knotenbasierten Formulierung – Randbedingungen

Node-Based Grid System / Knotenbasiertes Gittersystem

Perfectly electrically conducting (PEC) material in the xz plane / Ideal elektrisch leitendes (IEL) Material in der xz-Ebene



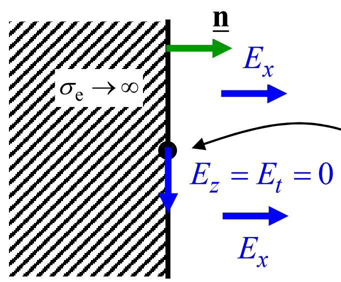
Boundary conditions for perfectly electrically conducting (PEC) material / Randbedingungen für ein ideal elektrisch leitendes (IEL) Material

$$\underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \underline{\mathbf{0}} \quad \to E_{tan}(\underline{\mathbf{R}},t) = E_t(\underline{\mathbf{R}},t) = 0$$

$$\underline{\mathbf{n}} \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \underline{\mathbf{0}} \quad \to B_n(\underline{\mathbf{R}},t) = 0$$

Node-based formulation: all vector components are placed at the same one node / Knotenbasierte Formulierung: alle Vektorkomponenten sind an dem selben Knoten platziert

Staggered Grid System / Versetztes Gittersystem

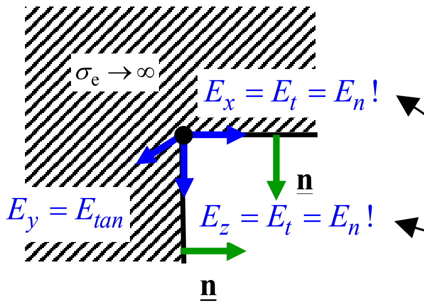


Only tangential components are allocated at the PEC boundary / Nur Tangentialkomponenten sind am IEL-Rand allokiert

Problems of a Node-Based Formulation – Boundary Conditions / Probleme einer knotenbasierten Formulierung – Randbedingungen

Node-Based Grid System / Knotenbasiertes Gittersystem

Perfectly electrically conducting (PEC) material in the xz plane /
Ideal elektrisch leitendes (IEL) Material in der xz-Ebene



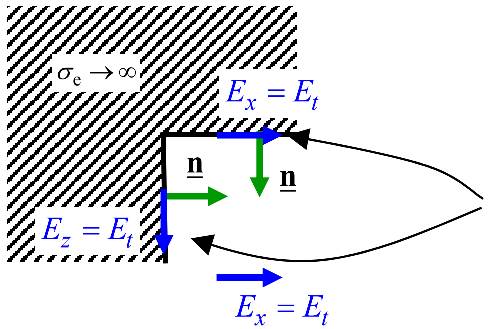
Boundary conditions for perfectly electrically conducting (PEC) material /
Randbedingungen für ein ideal elektrisch leitendes (IEL) Material

$$\underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}, t) = \underline{\mathbf{0}} \qquad \to E_{tan}(\underline{\mathbf{R}}, t) = E_t(\underline{\mathbf{R}}, t) = 0$$

$$\underline{\mathbf{n}} \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}}, t) = \underline{\mathbf{0}} \qquad \to B_n(\underline{\mathbf{R}}, t) = 0$$

These components are tangential and normal either to the vertical or to the horizontal boundary /
Diese Komponenten sind tangential und normal entweder zum vertikalen oder zum horizontalen Rand

Staggered Grid System / Versetztes Gittersystem



In this case only tangential components are allocated at the PEC boundary /
In diesem Fall sind nur Tangentialkomponenten am IEL-Rand allokiert

FIT – 3-D Electromagnetic Wave Propagation / FIT – 3D elektromagnetische Wellenausbreitung

Constitutive equations for inhomogeneous, anisotropic material /
Konstituierende Gleichungen (Materialgleichungen) für ein
inhomogenes, anisotropes Material

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{B}}(\underline{\mathbf{R}},t)$$

Diagonal permittivity and impermeability tensor of second rank /
Diagonaler Permittivitäts- und Impermeabilitätstensor zweiter Stufe

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) = \begin{bmatrix} \varepsilon_{xx}(\underline{\mathbf{R}}) & 0 & 0 \\ 0 & \varepsilon_{yy}(\underline{\mathbf{R}}) & 0 \\ 0 & 0 & \varepsilon_{zz}(\underline{\mathbf{R}}) \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}(\underline{\mathbf{R}}) & & \\ & \varepsilon_{yy}(\underline{\mathbf{R}}) & \\ \mathrm{sym} & & \varepsilon_{zz}(\underline{\mathbf{R}}) \end{bmatrix}$$

$$\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) = \begin{bmatrix} \nu_{xx}(\underline{\mathbf{R}}) & 0 & 0 \\ 0 & \nu_{yy}(\underline{\mathbf{R}}) & 0 \\ 0 & 0 & \nu_{zz}(\underline{\mathbf{R}}) \end{bmatrix} = \begin{bmatrix} \nu_{xx}(\underline{\mathbf{R}}) & & \\ & \nu_{yy}(\underline{\mathbf{R}}) & \\ \mathrm{sym} & & \nu_{zz}(\underline{\mathbf{R}}) \end{bmatrix}$$

FIT – 3-D Electromagnetic Wave Propagation / FIT – 3D elektromagnetische Wellenausbreitung

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t)=\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$=\begin{bmatrix}\varepsilon_{xx}(\underline{\mathbf{R}}) & 0 & 0\\ 0 & \varepsilon_{yy}(\underline{\mathbf{R}}) & 0\\ \text{sym} & 0 & \varepsilon_{zz}(\underline{\mathbf{R}})\end{bmatrix}\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$=\left[\varepsilon_{xx}(\underline{\mathbf{R}})\underline{\mathbf{e}}_x\underline{\mathbf{e}}_x+\varepsilon_{yy}(\underline{\mathbf{R}})\underline{\mathbf{e}}_y\underline{\mathbf{e}}_y+\varepsilon_{zz}(\underline{\mathbf{R}})\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z\right]\bullet\left[E_x(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x+E_y(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y+E_z(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z\right]$$

$$=\underbrace{\varepsilon_{xx}(\underline{\mathbf{R}})E_x(\underline{\mathbf{R}},t)}_{D_x(\underline{\mathbf{R}},t)}\underline{\mathbf{e}}_x+\underbrace{\varepsilon_{yy}(\underline{\mathbf{R}})E_y(\underline{\mathbf{R}},t)}_{D_y(\underline{\mathbf{R}},t)}\underline{\mathbf{e}}_y+\underbrace{\varepsilon_{zz}(\underline{\mathbf{R}})E_z(\underline{\mathbf{R}},t)}_{D_z(\underline{\mathbf{R}},t)}\underline{\mathbf{e}}_z$$

$$=D_x(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x+D_y(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y+D_z(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}},t)=\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)$$

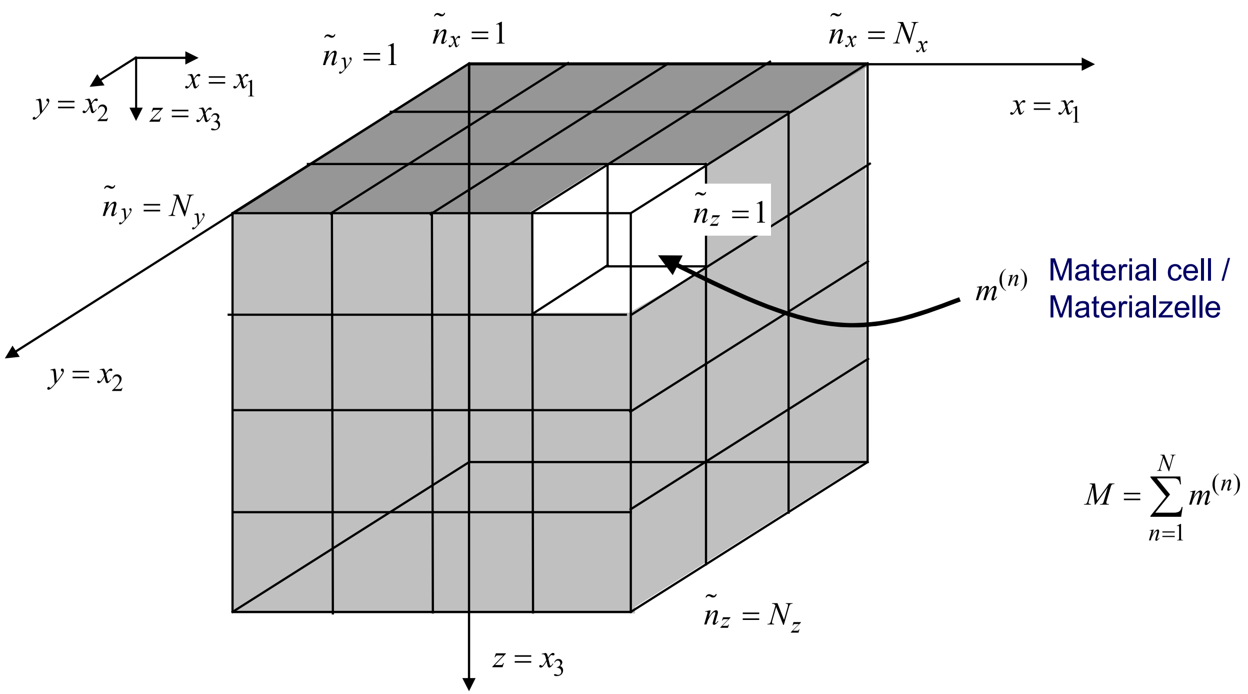
$$=\begin{bmatrix}\nu_{xx}(\underline{\mathbf{R}}) & & \\ & \nu_{yy}(\underline{\mathbf{R}}) & \\ \text{sym} & & \nu_{zz}(\underline{\mathbf{R}})\end{bmatrix}\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)$$

$$==\left[\nu_{xx}(\underline{\mathbf{R}})\underline{\mathbf{e}}_x\underline{\mathbf{e}}_x+\nu_{yy}(\underline{\mathbf{R}})\underline{\mathbf{e}}_y\underline{\mathbf{e}}_y+\nu_{zz}(\underline{\mathbf{R}})\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z\right]\bullet\left[B_x(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x+B_y(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y+B_z(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z\right]$$

$$=\underbrace{\nu_{xx}(\underline{\mathbf{R}})B_x(\underline{\mathbf{R}},t)}_{H_x(\underline{\mathbf{R}},t)}\underline{\mathbf{e}}_x+\underbrace{\nu_{yy}(\underline{\mathbf{R}})B_y(\underline{\mathbf{R}},t)}_{H_y(\underline{\mathbf{R}},t)}\underline{\mathbf{e}}_y+\underbrace{\nu_{zz}(\underline{\mathbf{R}})B_z(\underline{\mathbf{R}},t)}_{H_z(\underline{\mathbf{R}},t)}\underline{\mathbf{e}}_z$$

$$=H_x(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x+H_y(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y+H_z(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z$$

Definition of Material Cells / Definition der Materialzellen



$$n = 1 + M_x(n_x - 1) + M_y(n_y - 1) + M_z(n_z - 1)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

$$M = \sum_{n=1}^{N} m^{(n)}$$

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \rightarrow \underline{\underline{\boldsymbol{\varepsilon}}}^{(n)} \in m^{(n)} \quad n \in \mathbb{R}^N$$

$$\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \rightarrow \underline{\underline{\boldsymbol{\nu}}}^{(n)} \in m^{(n)} \quad n \in \mathbb{R}^N$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}S$$

$$\underline{\mathbf{dR}} = \underline{\mathbf{s}}\,\mathrm{d}R$$

$$\iint_S \underline{\mathbf{n}}\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\,dS = \iint_S \underline{\mathbf{e}}_x\cdot\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$

$$= \iint_S B_x(\underline{\mathbf{R}},t)\,\mathrm{d}S$$

$$= B_x^{(m)}(t)\underbrace{\iint_S \mathrm{d}S}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$

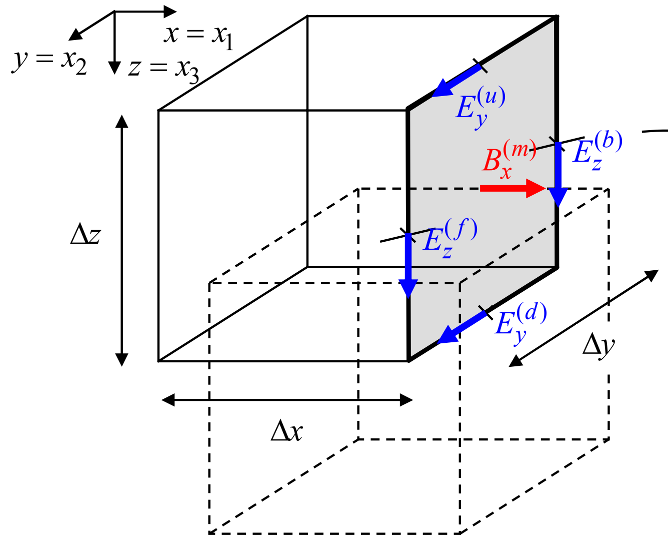
$$= B_x^{(m)}(t)\Delta y\Delta z + \underbrace{\mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]}$$

Field component in the middle / Feldkomponente in der Mitte

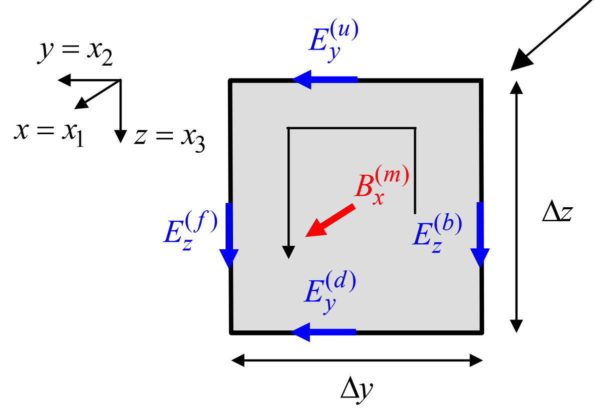
Approximation error / Approximationsfehler

$$\iint_S f(\underline{\mathbf{R}},t)\,\mathrm{d}S = f^{(m)}(t)\underbrace{\iint_S \mathrm{d}y\,\mathrm{d}z}_{=\Delta y\Delta z} + \overbrace{\mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]}$$

$$= f^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$



$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle



3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

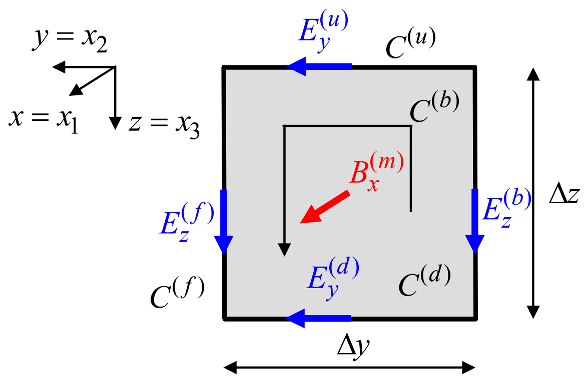
$$\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} = ?$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

$$\underline{\mathbf{dR}}_y = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_y\,\mathrm{d}y$$

$$\underline{\mathbf{dR}}_z = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_z\,\mathrm{d}z$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} = \int_{C^{(u)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(f)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(d)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(b)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}}$$

$$= \int_{C^{(u)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_y\mathrm{d}y + \int_{C^{(f)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_z\mathrm{d}z - \int_{C^{(d)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_y\mathrm{d}y - \int_{C^{(b)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_z\mathrm{d}z$$

$$= \int_{C^{(u)}}E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}}E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}}E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}}E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = \int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$\int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = E_y^{(u)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = E_y^{(u)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$

$$\int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = E_z^{(f)}(t)\underbrace{\int_{C^{(f)}} \mathrm{d}z}_{=\Delta z} + \bigcirc\left[(\Delta z)^3\right] = E_z^{(f)}(t)\Delta z + \bigcirc\left[(\Delta z)^3\right]$$

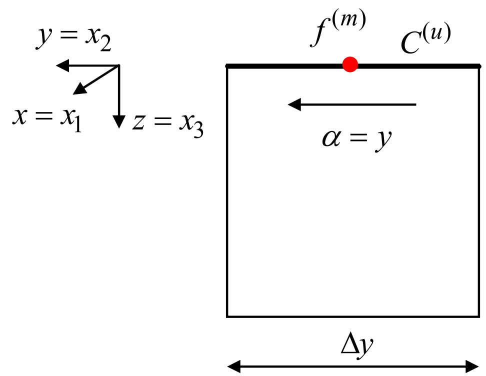
$$\int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = E_y^{(d)}(t)\underbrace{\int_{C^{(d)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = E_y^{(d)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$

$$\int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = E_z^{(b)}(t)\underbrace{\int_{C^{(b)}} \mathrm{d}z}_{=\Delta z} + \bigcirc\left[(\Delta z)^3\right] = E_z^{(b)}(t)\Delta z + \bigcirc\left[(\Delta z)^3\right]$$

Field component in the middle / Feldkomponente in der Mitte

Approximation error / Approximationsfehler

$$\int_{C^{(u)}} f(\underline{\mathbf{R}},t)\,\mathrm{d}R = f^{(m)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = f^{(m)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$



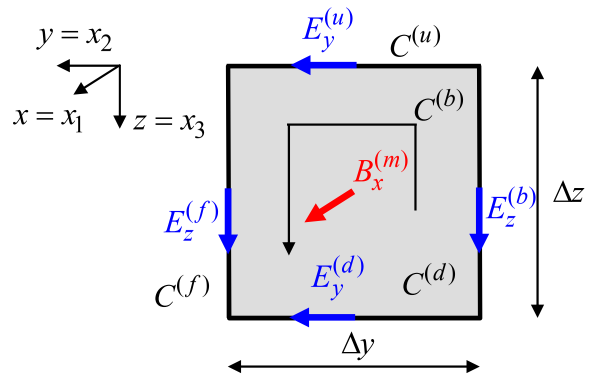
3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \mathrm{d}\underline{\mathbf{R}} = \int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$= E_y^{(u)}(t)\underbrace{\int_{C^{(u)}}\mathrm{d}y}_{=\Delta y} + E_z^{(f)}(t)\underbrace{\int_{C^{(f)}}\mathrm{d}z}_{=\Delta z} - E_y^{(d)}(t)\underbrace{\int_{C^{(d)}}\mathrm{d}y}_{=\Delta y} - E_z^{(b)}(t)\underbrace{\int_{C^{(b)}}\mathrm{d}z}_{=\Delta z}$$

$$+\bigcirc\left[(\Delta y)^3\right]+\bigcirc\left[(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \mathrm{d}\underline{\mathbf{R}} = E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z +\bigcirc\left[(\Delta y)^3\right]+\bigcirc\left[(\Delta z)^3\right]$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z$$
$$+\bigcirc\left[(\Delta y)^3\right]+\bigcirc\left[(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle

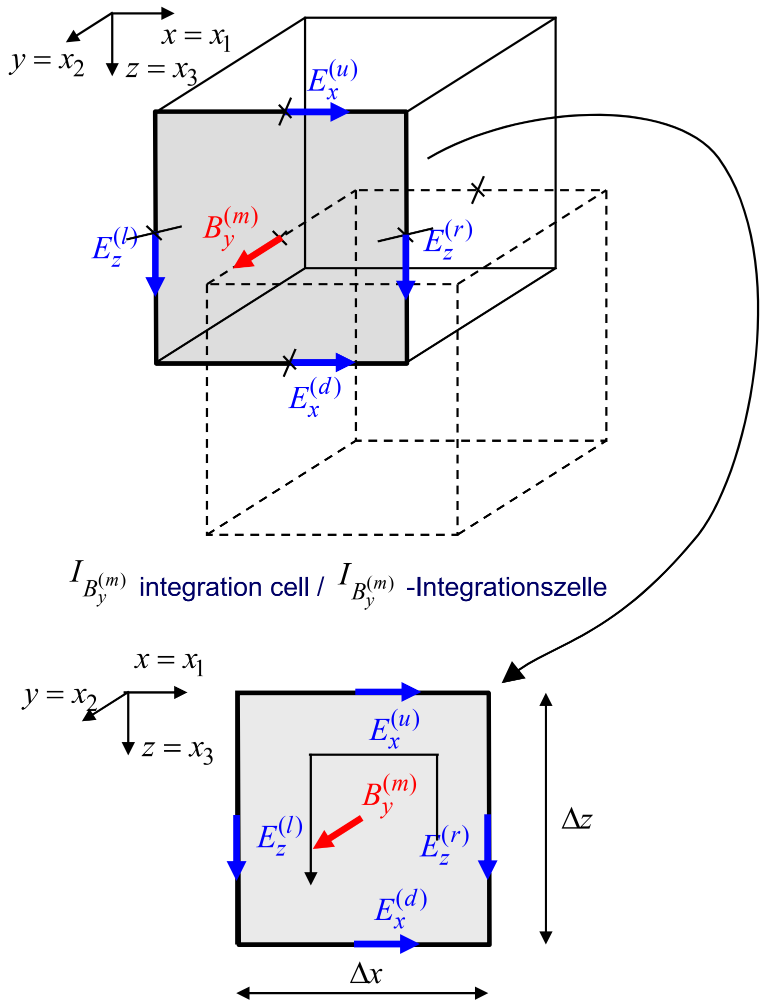
$$\iint_S \underline{\mathbf{n}}\bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\, dS = \iint_S \underline{\mathbf{e}}_x \bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$
$$= \iint_S J_{\mathrm{m}x}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$
$$= J_{\mathrm{m}x}^{(m)}(t)\underbrace{\iint_S \mathrm{d}S}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$
$$= J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle

$$\frac{d}{dt}B_x^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

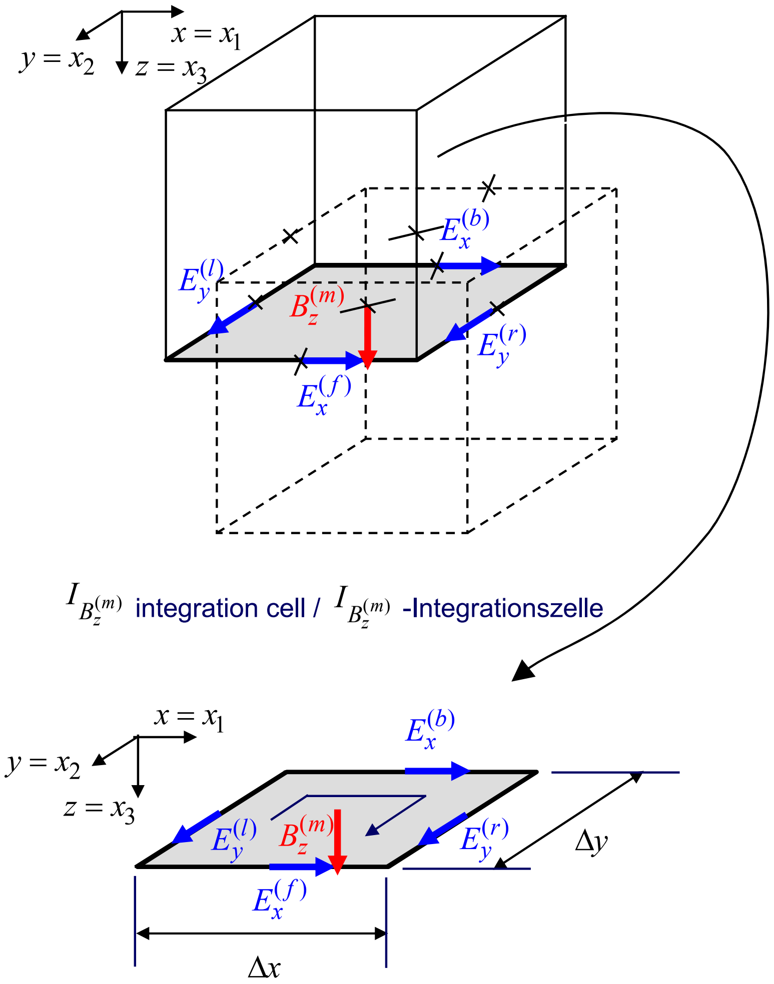


$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$I_{B_y^{(m)}}$ integration cell / $I_{B_y^{(m)}}$ -Integrationszelle

$$\frac{d}{dt} B_y^{(m)}(t)\Delta y \Delta z$$
$$= -\left[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\right]$$
$$- J_{\mathrm{my}}^{(m)}(t)\Delta y \Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen



$I_{B_z^{(m)}}$ integration cell / $I_{B_z^{(m)}}$ -Integrationszelle

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{B_z^{(m)}}$ integration cell / $I_{B_z^{(m)}}$ -Integrationszelle

$$\frac{d}{dt}B_z^{(m)}(t)\Delta x\Delta y$$
$$= -\left[E_x^{(b)}(t)\Delta x + E_y^{(r)}(t)\Delta y - E_x^{(f)}(t)\Delta x - E_y^{(l)}(t)\Delta y\right]$$
$$- J_{\mathrm{m}z}^{(m)}(t)\Delta x\Delta y$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

⇩

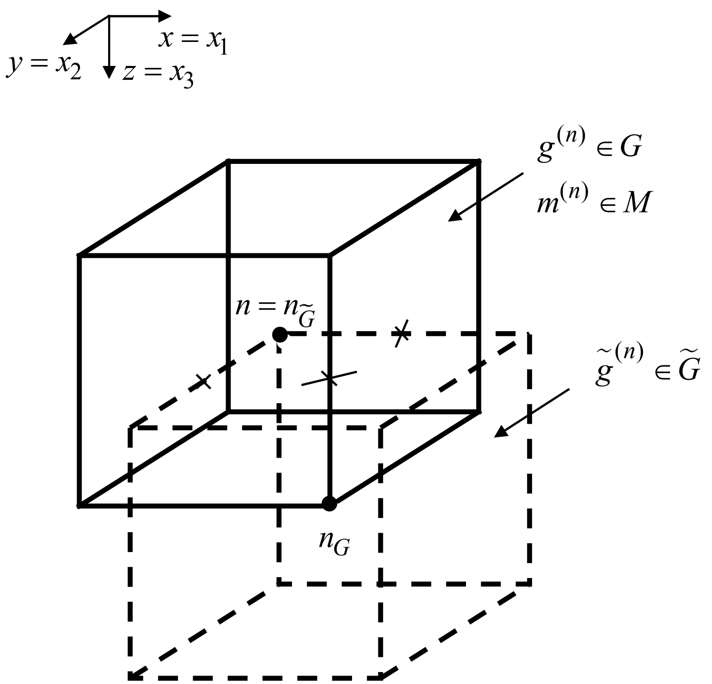
$$\frac{\mathrm{d}}{\mathrm{d}t}B_x^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z$$

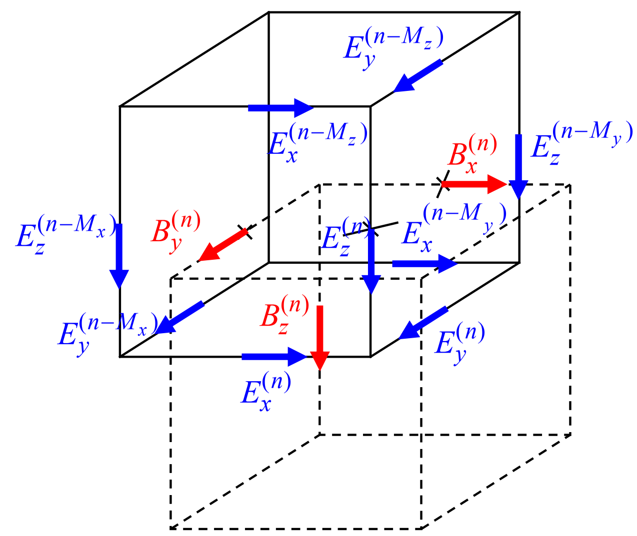
$$\frac{\mathrm{d}}{\mathrm{d}t}B_y^{(m)}(t)\Delta y\Delta z = -\left[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\right] - J_{\mathrm{m}y}^{(m)}(t)\Delta y\Delta z$$

$$\frac{\mathrm{d}}{\mathrm{d}t}B_z^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y + E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}z}^{(m)}(t)\Delta y\Delta z$$

Dual-Orthogonal Grid System in Space / Dual-orthogonales Gittersystem im Raum

3-D / 3D





Primary grid / Primäres Gitter	$G \perp \widetilde{G}$	Secondary (dual) grid Sekundäres (duales) Gitter
Primary grid / Primäres Gitter	$G = M$	Material grid Materialgitter

Global node numbering / Globale Gitternummerierung

$$n = 1 + M_x(n_x - 1) + M_y(n_y - 1) + M_z(n_z - 1)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

Local grid equations in local notation / Lokale Gittergleichungen in lokaler Notation

$$\frac{d}{dt}B_x^{(m)}(t)\Delta y\Delta z=-\left[E_y^{(u)}(t)\Delta y+E_z^{(f)}(t)\Delta z-E_y^{(d)}(t)\Delta y-E_z^{(b)}(t)\Delta z\right]-J_{\mathrm{mx}}^{(m)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(m)}(t)\Delta x\Delta z=-\left[-E_x^{(u)}(t)\Delta x+E_z^{(l)}(t)\Delta z+E_x^{(d)}(t)\Delta x-E_z^{(r)}(t)\Delta z\right]-J_{\mathrm{my}}^{(m)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(m)}(t)\Delta x\Delta y=-\left[E_x^{(b)}(t)\Delta x+E_y^{(r)}(t)\Delta y-E_x^{(f)}(t)\Delta x-E_y^{(l)}(t)\Delta y\right]-J_{\mathrm{mz}}^{(m)}(t)\Delta x\Delta y$$

Local grid equations in global grid node notation / Lokale Gittergleichungen in globaler Gitterknotennotation

$$\frac{d}{dt}B_x^{(n)}(t)\Delta y\Delta z=-\left\{\left[E_y^{(n-M_z)}(t)-E_y^{(n)}(t)\right]\Delta y+\left[E_z^{(n)}(t)-E_z^{(n-M_y)}(t)\right]\Delta z\right\}-J_{\mathrm{mx}}^{(n)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(n)}(t)\Delta x\Delta z=-\left\{\left[E_x^{(n)}(t)-E_x^{(n-M_z)}(t)\right]\Delta x+\left[E_z^{(n-M_x)}(t)-E_z^{(n)}(t)\right]\Delta z\right\}-J_{\mathrm{my}}^{(n)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(n)}(t)\Delta x\Delta y=-\left\{\left[E_x^{(n-M_y)}(t)-E_x^{(n)}(t)\right]\Delta x+\left[E_y^{(n)}(t)-E_y^{(n-M_x)}(t)\right]\Delta y\right\}-J_{\mathrm{mz}}^{(n)}(t)\Delta x\Delta y$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

Local grid equations in global grid node notation / Lokale Gittergleichungen in globaler Gitterknotennotation

$$\frac{d}{dt} B_x^{(n)}(t)\Delta y \Delta z = -\left\{ \left[E_y^{(n-M_z)}(t) - E_y^{(n)}(t) \right] \Delta y + \left[E_z^{(n)}(t) - E_z^{(n-M_y)}(t) \right] \Delta z \right\} - J_{\mathrm{mx}}^{(n)}(t)\Delta y \Delta z$$

$$\frac{d}{dt} B_y^{(n)}(t)\Delta x \Delta z = -\left\{ \left[E_x^{(n)}(t) - E_x^{(n-M_z)}(t) \right] \Delta x + \left[E_z^{(n-M_x)}(t) - E_z^{(n)}(t) \right] \Delta z \right\} - J_{\mathrm{my}}^{(n)}(t)\Delta x \Delta z$$

$$\frac{d}{dt} B_z^{(n)}(t)\Delta x \Delta y = -\left\{ \left[E_x^{(n-M_y)}(t) - E_x^{(n)}(t) \right] \Delta x + \left[E_y^{(n)}(t) - E_y^{(n-M_x)}(t) \right] \Delta y \right\} - J_{\mathrm{mz}}^{(n)}(t)\Delta x \Delta y$$

Local spatial shift operators / Lokale räumliche Schiebeoperatoren

$$S_{\pm M_i} f^{(n)} = f^{(n \pm M_i)}$$

$$S_0 f^{(n)} = f^{(n)}$$

$$S_0 = I$$

$$I\, f^{(n)} = f^{(n)}$$

Local grid equations with local spatial shift operators in global grid node notation / Lokale Gittergleichungen mit lokalen räumlichen Schiebeoperatoren in globaler Gitterknotennotation

$$\frac{d}{dt} B_x^{(n)}(t)\Delta y \Delta z = -\left\{ \left[S_{-M_z} - I \right] E_y^{(n)}(t)\Delta y + \left[I - S_{M_y} \right] E_z^{(n)}(t)\Delta z \right\} - J_{\mathrm{mx}}^{(n)}(t)\Delta y \Delta z$$

$$\frac{d}{dt} B_y^{(n)}(t)\Delta x \Delta z = -\left\{ \left[I - S_{-M_z} \right] E_x^{(n)}(t)\Delta x + \left[S_{-M_x} - I \right] E_z^{(n)}(t)\Delta z \right\} - J_{\mathrm{my}}^{(n)}(t)\Delta x \Delta z$$

$$\frac{d}{dt} B_z^{(n)}(t)\Delta x \Delta y = -\left\{ \left[S_{-M_y} - I \right] E_x^{(n)}(t)\Delta x + \left[I - S_{-M_x} \right] E_y^{(n)}(t)\Delta y \right\} - J_{\mathrm{mz}}^{(n)}(t)\Delta x \Delta y$$

3-D FIT – Local Spatial Shift Operators / 3D-FIT – Lokale räumliche Schiebeoperatoren

1. Simple spatial shift operation / Einfache räumliche Schiebeoperation

$$S_{\pm M_i} f^{(n)} = f^{(n \pm M_i)}$$

2. Identity operation / Identitätsoperation

$$I\, f^{(n)} = f^{(n)}$$

3. Multiple shift operations / Zusammengesetzte Schiebeoperationen

$$S_{\pm M_i} S_{\pm M_j} f^{(n)} = S_{\pm M_j} S_{\pm M_i} f^{(n)} = f^{(n \pm M_i \pm M_j)}$$

Special case for $M_j = -M_i$ / Speziell folgt für $M_j = -M_i$

$$S_{\pm M_i} S_{\mp M_i} = I$$

4. Local difference operator / Lokaler Differenzoperator

$$P_{\pm M_i} = \mp I \pm S_{\pm M_i}$$

5. Local averaging operator / Lokaler Mittelungsoperator

$$A_{\pm M_i} = \frac{1}{2}\left(I + S_{\pm M_i} \right)$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

Local grid equations with local spatial shift operators in global grid node notation /
Lokale Gittergleichungen mit lokalen räumlichen Schiebeoperatoren in globaler Gitterknotennotation

$$\frac{d}{dt}B_x^{(n)}(t)\Delta y\Delta z = -\left\{\left[S_{-M_z}-I\right]E_y^{(n)}(t)\Delta y+\left[I-S_{M_y}\right]E_z^{(n)}(t)\Delta z\right\}-J_{\mathrm{mx}}^{(n)}(t)\Delta y\Delta z$$

$$\frac{d}{dt}B_y^{(n)}(t)\Delta x\Delta z = -\left\{\left[I-S_{-M_z}\right]E_x^{(n)}(t)\Delta x+\left[S_{-M_x}-I\right]E_z^{(n)}(t)\Delta z\right\}-J_{\mathrm{my}}^{(n)}(t)\Delta x\Delta z$$

$$\frac{d}{dt}B_z^{(n)}(t)\Delta x\Delta y = -\left\{\left[S_{-M_y}-I\right]E_x^{(n)}(t)\Delta x+\left[I-S_{-M_x}\right]E_y^{(n)}(t)\Delta y\right\}-J_{\mathrm{mz}}^{(n)}(t)\Delta x\Delta y$$

… in local matrix form / … in lokaler Matrixform

$$\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[S]}\frac{d}{dt}\underbrace{\begin{Bmatrix}B_x^{(n)}(t)\\ B_y^{(n)}(t)\\ B_z^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{B\}^{(n)}(t)} = -\underbrace{\begin{bmatrix}0 & S_{-M_z}-I & I-S_{M_y}\\ I-S_{-M_z} & 0 & S_{-M_x}-I\\ S_{-M_y}-I & I-S_{-M_x} & 0\end{bmatrix}}_{=[\mathrm{curl}]}\underbrace{\begin{bmatrix}\Delta x & & \\ & \Delta y & \\ & & \Delta z\end{bmatrix}}_{=[R]}\underbrace{\begin{Bmatrix}E_x^{(n)}(t)\\ E_y^{(n)}(t)\\ E_z^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}(t)}$$

$$-\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[S]}\underbrace{\begin{Bmatrix}J_{\mathrm{mx}}^{(n)}(t)\\ J_{\mathrm{mx}}^{(n)}(t)\\ J_{\mathrm{mx}}^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{J_{\mathrm{m}}\}^{(n)}(t)}$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

$$\underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=[S]} \frac{d}{dt} \underbrace{\begin{Bmatrix} B_x^{(n)}(t) \\ B_y^{(n)}(t) \\ B_z^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{B\}^{(n)}(t)} = -\underbrace{\begin{bmatrix} 0 & S_{-M_z} - I & I - S_{M_y} \\ I - S_{-M_z} & 0 & S_{-M_x} - I \\ S_{-M_y} - I & I - S_{-M_x} & 0 \end{bmatrix}}_{=[\mathrm{curl}]} \underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}(t)}$$

$$-\underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=[S]} \underbrace{\begin{Bmatrix} J_{\mathrm{mx}}^{(n)}(t) \\ J_{\mathrm{mx}}^{(n)}(t) \\ J_{\mathrm{mx}}^{(n)}(t) \end{Bmatrix}^{(n)}}_{=\{J_{\mathrm{m}}\}^{(n)}(t)}$$

$$\begin{bmatrix} 0 & S_{-M_z} - I & I - S_{-M_y} \\ I - S_{-M_z} & 0 & S_{-M_x} - I \\ S_{-M_y} - I & I - S_{-M_x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -P_{-M_z} & P_{-M_y} \\ P_{-M_z} & 0 & -P_{-M_x} \\ -P_{-M_y} & P_{-M_x} & 0 \end{bmatrix} = [\mathrm{curl}]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Faraday's induction law in local matrix form / Faradaysches Induktionsgesetz in lokaler Matrixform

$$\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[S]}\frac{d}{dt}\underbrace{\begin{Bmatrix}B_x^{(n)}(t)\\ B_y^{(n)}(t)\\ B_z^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{B\}^{(n)}(t)}=-\underbrace{\begin{bmatrix}0 & -P_{-M_z} & P_{-M_y}\\ P_{-M_z} & 0 & -P_{-M_x}\\ -P_{-M_y} & P_{-M_x} & 0\end{bmatrix}}_{=[\mathrm{curl}]}\underbrace{\begin{bmatrix}\Delta x & & \\ & \Delta y & \\ & & \Delta z\end{bmatrix}}_{=[R]}\underbrace{\begin{Bmatrix}E_x^{(n)}(t)\\ E_y^{(n)}(t)\\ E_z^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}(t)}-\underbrace{\begin{bmatrix}\Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y\end{bmatrix}}_{=[S]}\underbrace{\begin{Bmatrix}J_{\mathrm{mx}}^{(n)}(t)\\ J_{\mathrm{mx}}^{(n)}(t)\\ J_{\mathrm{mx}}^{(n)}(t)\end{Bmatrix}^{(n)}}_{=\{J_{\mathrm{m}}\}^{(n)}(t)}$$

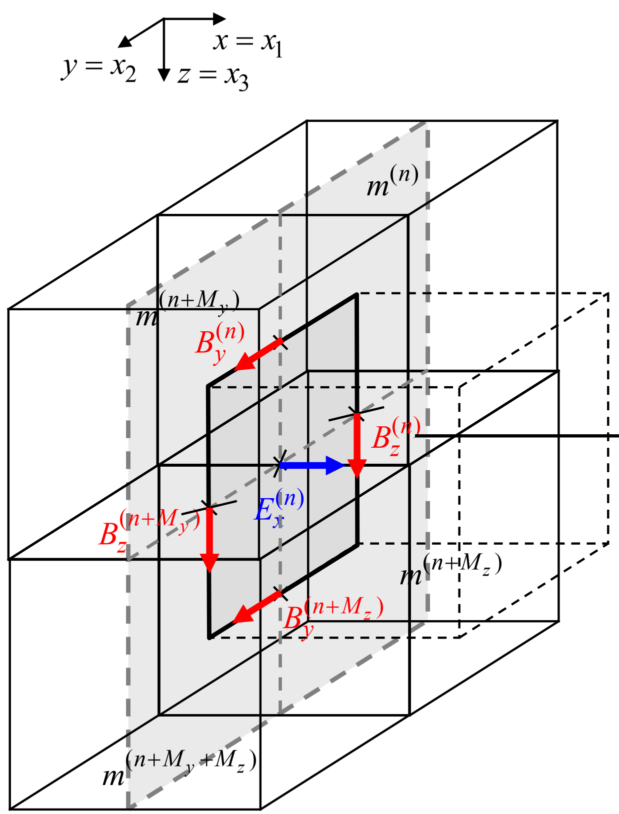
$$[S]\frac{d}{dt}\{B\}^{(n)}(t)=-[\mathrm{curl}][R]\{E\}^{(n)}(t)-[S]\{J_{\mathrm{m}}\}^{(n)}(t)$$

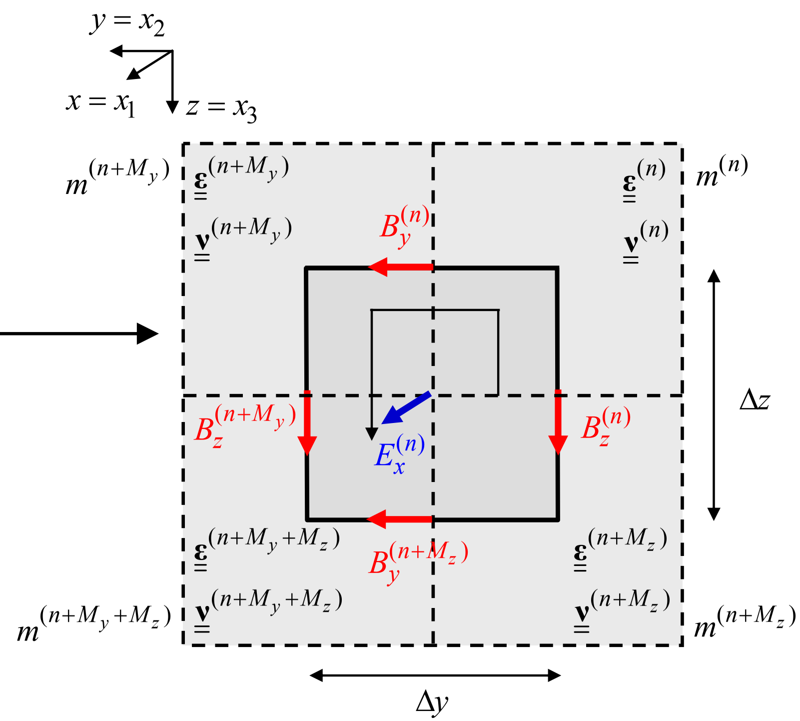
$[S]$	$\in\mathbb{R}^{3\times 3}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\{B\}^{(n)}(t)$	$\in\mathbb{R}^{3}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$[\mathrm{curl}]$	$\in\mathbb{R}^{3\times 3}$	Topological curl operator in matrix form on the grid G / Topologischer Rotationsoperator in Matrixform auf dem Gitter G
$[R]$	$\in\mathbb{R}^{3\times 3}$	Diagonal matrix of elementary lines on the grid G / Diagonalmatrix der Elementarstrecken auf dem Gitter G
$\{E\}^{(n)}(t)$	$\in\mathbb{R}^{3}$	Algebraic electric field strength vector / Algebraischer elektrische Feldstärkevektor
$\{J_{\mathrm{m}}\}^{(n)}(t)$	$\in\mathbb{R}^{3}$	Algebraic magnetic current density vector / Algebraischer magnetischer Stromdichtevektor

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle

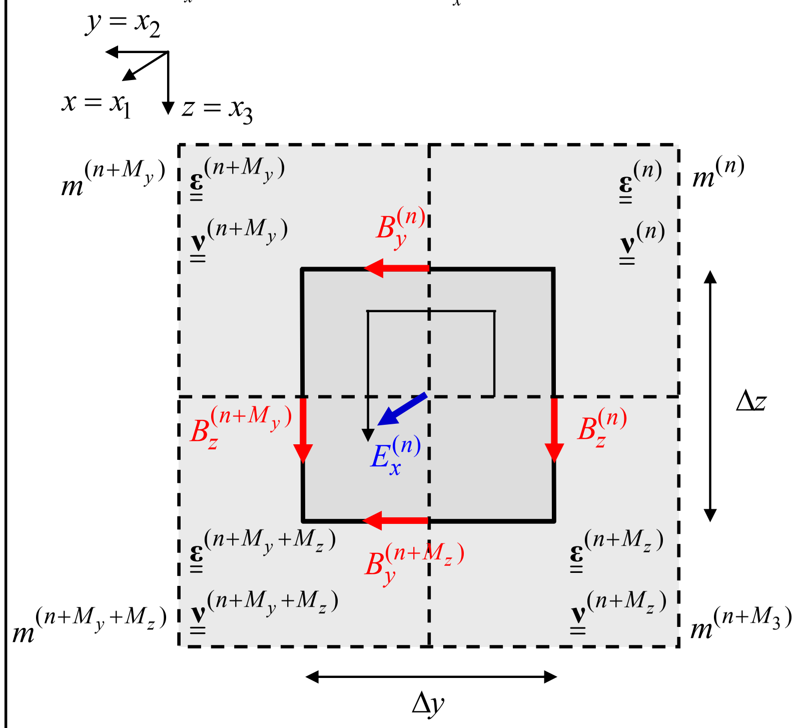




3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle



$$\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\underline{\mathbf{dS}} = \iint_S \underline{\mathbf{e}}_x\bullet\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\mathrm{d}S$$
$$= \iint_S \varepsilon_{xx}(\underline{\mathbf{R}})\,E_x(\underline{\mathbf{R}},t)\,\mathrm{d}S$$
$$= E_x^{(n)}(t)\iint_S \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S$$
$$+\mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$

$$\iint_S \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S$$
$$= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n)} + \varepsilon_{xx}^{(n+M_y)} + \varepsilon_{xx}^{(n+M_z)} + \varepsilon_{xx}^{(n+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n)}}\Delta y\Delta z$$
$$= \tilde{\varepsilon}_{xx}^{(n)}\,\Delta y\Delta z$$

$$\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\underline{\mathbf{dS}}$$
$$= E_x^{(n)}(t)\tilde{\varepsilon}_{xx}^{(n)}\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$

$$\iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$= J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]$$

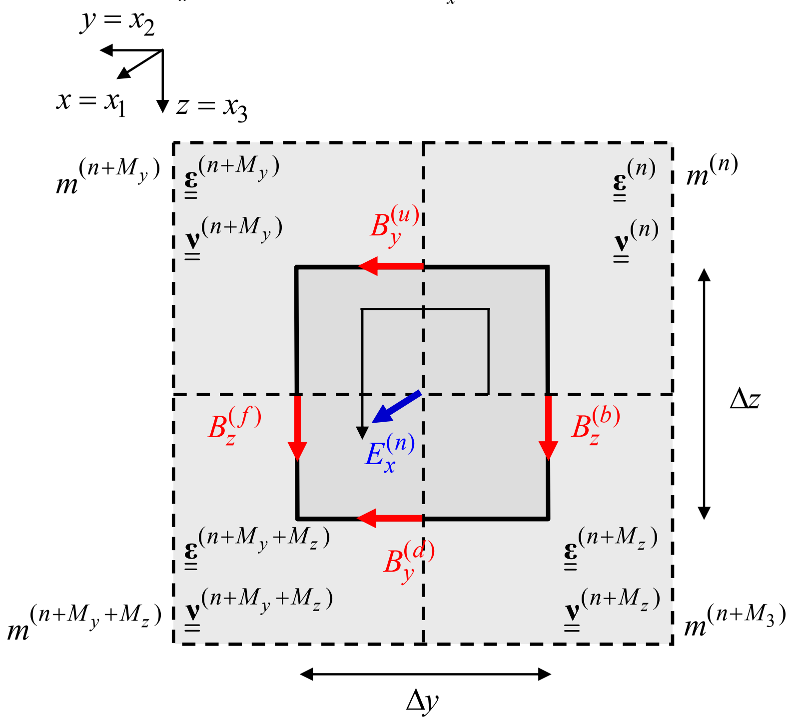
3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_e(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} = ?$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$
$$\underline{\mathbf{dR}}_y = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_y\,\mathrm{d}y$$
$$\underline{\mathbf{dR}}_z = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_z\,\mathrm{d}z$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} = \int_{C^{(u)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} + \int_{C^{(f)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \int_{C^{(d)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \int_{C^{(b)}}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}}$$

$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} = \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}})B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}})B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}})B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}})B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{B}}(\underline{\mathbf{R}},t) \right] \cdot \mathbf{d}\underline{\mathbf{R}}$$

$$= \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$\int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = B_y^{(u)}(t) \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y + \bigcirc\left[(\Delta y)^3 \right] \qquad \int_{C^{(u)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y = \frac{1}{2}\left[\nu_{yy}^{(n)} + \nu_{yy}^{(n+M_y)} \right] \Delta y$$

$$\int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = B_z^{(f)}(t) \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z + \bigcirc\left[(\Delta z)^3 \right] \qquad \int_{C^{(f)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z = \frac{1}{2}\left[\nu_{yy}^{(n+M_z)} + \nu_{yy}^{(n+M_y+M_z)} \right] \Delta z$$

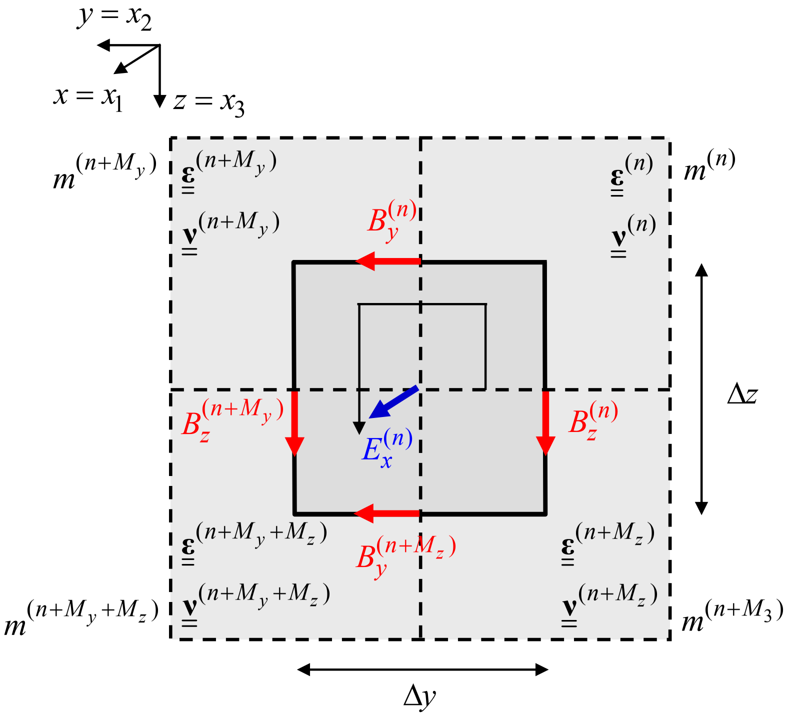
$$\int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}}) B_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = B_y^{(d)}(t) \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y + \bigcirc\left[(\Delta y)^3 \right] \qquad \int_{C^{(d)}} \nu_{yy}(\underline{\mathbf{R}})\,\mathrm{d}y = \frac{1}{2}\left[\nu_{zz}^{(n+M_y)} + \nu_{zz}^{(n+M_y+M_z)} \right] \Delta y$$

$$\int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}}) B_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = B_z^{(b)}(t) \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z + \bigcirc\left[(\Delta z)^3 \right] \qquad \int_{C^{(b)}} \nu_{zz}(\underline{\mathbf{R}})\,\mathrm{d}z = \frac{1}{2}\left[\nu_{zz}^{(n)} + \nu_{zz}^{(n+M_z)} \right] \Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle



$$\oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}}$$

$$= \underbrace{\frac{1}{2}\left[\nu_{yy}^{(n)} + \nu_{yy}^{(n+M_y)}\right]}_{=\tilde{\nu}_{yy}^{(n)}} B_y^{(n)}(t)\Delta y$$

$$- \underbrace{\frac{1}{2}\left[\nu_{yy}^{(n+M_z)} + \nu_{yy}^{(n+M_y+M_z)}\right]}_{=\tilde{\nu}_{yy}^{(n+M_z)}} B_y^{(n+M_z)}(t)\Delta y$$

$$+ \underbrace{\frac{1}{2}\left[\nu_{zz}^{(n+M_y)} + \nu_{zz}^{(n+M_y+M_z)}\right]}_{=\tilde{\nu}_{zz}^{(n+M_y)}} B_z^{(n+M_y)}(t)\Delta z$$

$$- \underbrace{\frac{1}{2}\left[\nu_{zz}^{(n)} + \nu_{zz}^{(n+M_z)}\right]}_{=\tilde{\nu}_{zz}^{(n)}} B_z^{(n)}(t)\Delta z$$

$$= \tilde{\nu}_{yy}^{(n)} B_y^{(n)}(t)\Delta y - \tilde{\nu}_{yy}^{(n+M_z)} B_y^{(n+M_z)}(t)\Delta y$$
$$+ \tilde{\nu}_{zz}^{(n+M_y)} B_z^{(n+M_y)}(t)\Delta z - \tilde{\nu}_{zz}^{(n)} B_z^{(n)}(t)\Delta z$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

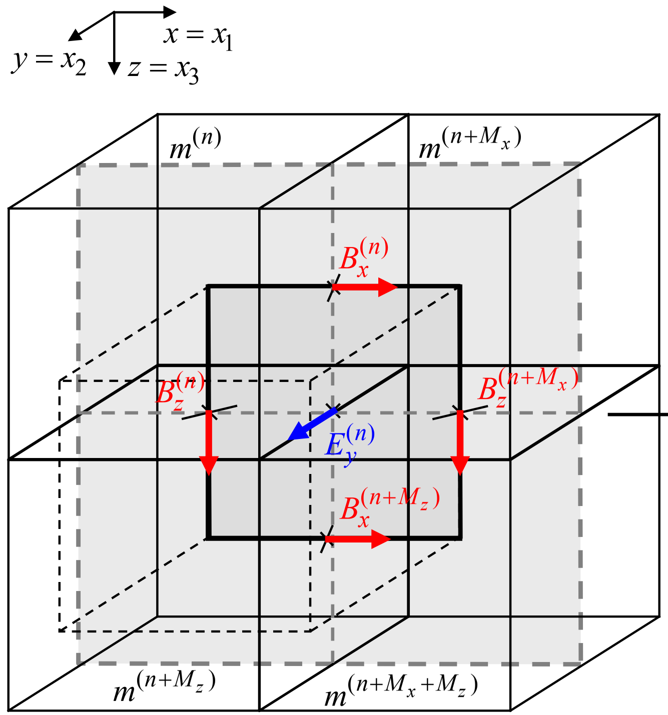
$I_{E_x^{(m)}}$ integration cell / $I_{E_x^{(m)}}$ -Integrationszelle

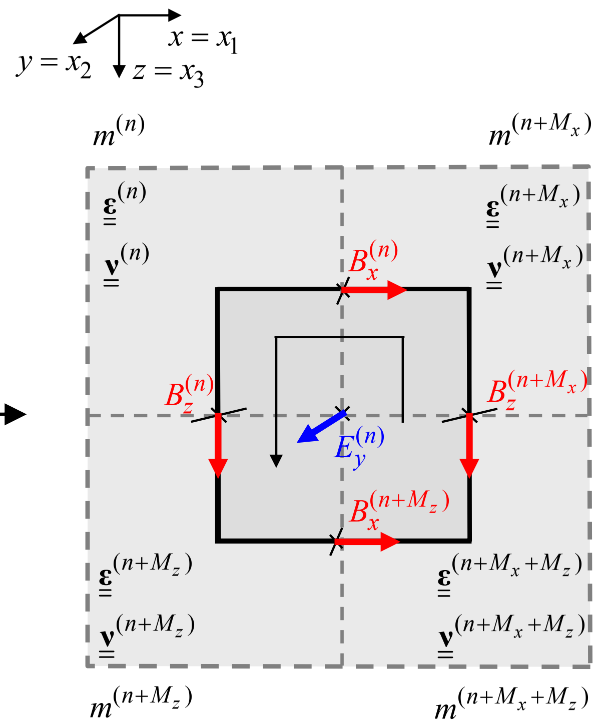
$$\begin{aligned}\tilde{\varepsilon}_{xx}^{(n)}\frac{d}{dt}E_x^{(n)}(t)\Delta y\Delta z &= \tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - \tilde{\nu}_{yy}^{(n+M_z)}B_y^{(n+M_z)}(t)\Delta y\\ &\quad+\tilde{\nu}_{zz}^{(n+M_y)}B_z^{(n+M_y)}(t)\Delta z - \tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z\\ &=\left(I-S_{M_z}\right)\tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y+\left(S_{M_y}-I\right)\tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z\end{aligned}$$

3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_y^{(m)}}$ integration cell / $I_{E_y^{(m)}}$ -Integrationszelle

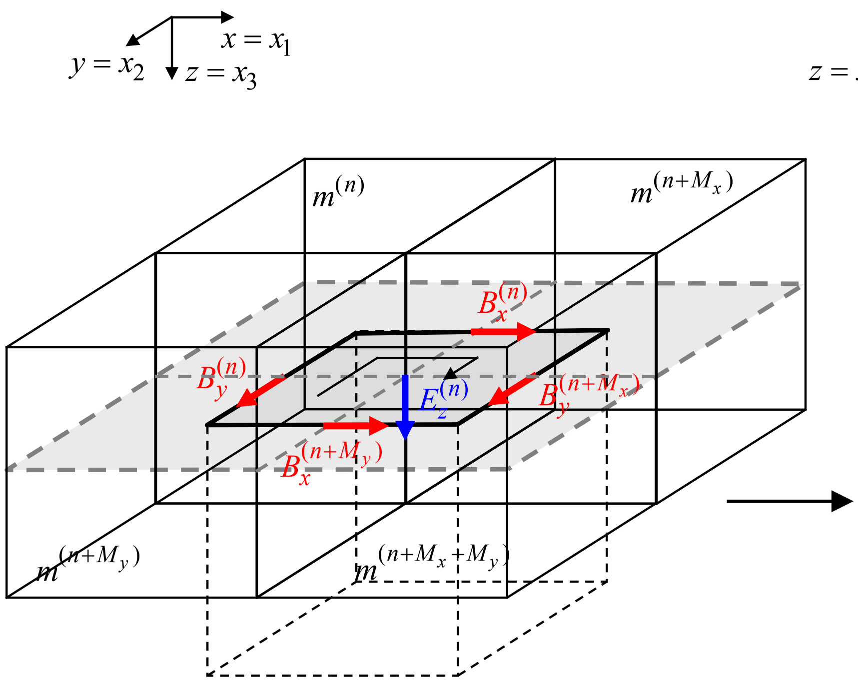


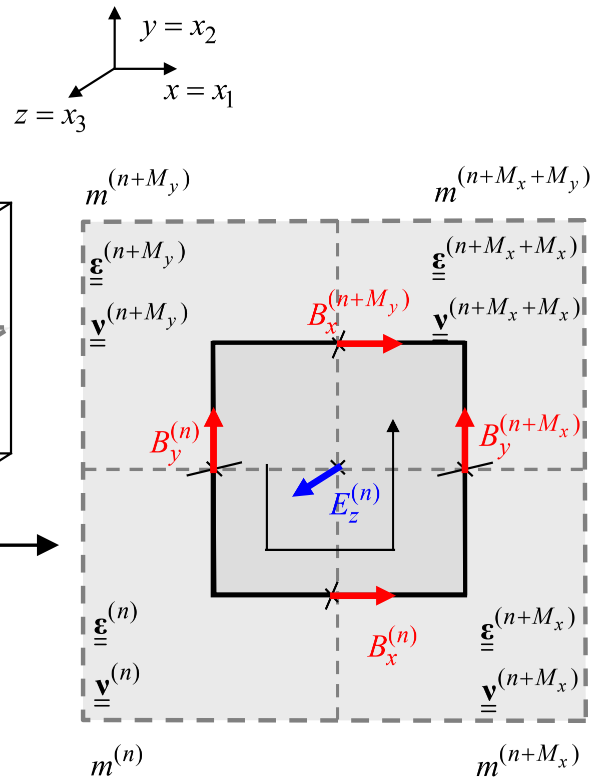


3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{E_z^{(m)}}$ integration cell / $I_{E_z^{(m)}}$ -Integrationszelle





3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{d}{dt}\iint_S \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\left[\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\Downarrow$$

$$\begin{aligned}
\tilde{\varepsilon}_{xx}^{(n)}\frac{d}{dt}E_x^{(n)}(t)\Delta y\Delta z &= \tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - \tilde{\nu}_{yy}^{(n+M_z)}B_y^{(n+M_z)}(t)\Delta y \\
&\quad + \tilde{\nu}_{zz}^{(n+M_y)}B_z^{(n+M_y)}(t)\Delta z - \tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z \\
&= \left(I - S_{M_z}\right)\tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y + \left(S_{M_y} - I\right)\tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ex}}^{(n)}(t)\Delta y\Delta z
\end{aligned}$$

$$\begin{aligned}
\tilde{\varepsilon}_{yy}^{(n)}\frac{d}{dt}E_y^{(n)}(t)\Delta x\Delta z &= \tilde{\nu}_{xx}^{(n+M_3)}B_x^{(n+M_3)}(t)\Delta x - \tilde{\nu}_{xx}^{(n_z)}B_x^{(n)}(t)\Delta x \\
&\quad + \tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - \tilde{\nu}_{zz}^{(n+M_x)}B_z^{(n+M_x)}(t)\Delta z - J_{\mathrm{ey}}^{(n)}(t)\Delta x\Delta z \\
&= \left(S_{M_z} - I\right)\tilde{\nu}_{xx}^{(n)}B_x^{(n)}(t)\Delta x + \left(I - S_{M_x}\right)\tilde{\nu}_{zz}^{(n)}B_z^{(n)}(t)\Delta z - J_{\mathrm{ey}}^{(n)}(t)\Delta y\Delta z
\end{aligned}$$

$$\begin{aligned}
\tilde{\varepsilon}_{zz}^{(n)}\frac{d}{dt}E_z^{(n)}(t)\Delta x\Delta y &= \tilde{\nu}_{xx}^{(n)}B_x^{(n)}(t)\Delta x - \tilde{\nu}_{xx}^{(n+M_y)}B_x^{(n+M_y)}(t)\Delta x \\
&\quad + \tilde{\nu}_{yy}^{(n+M_x)}B_y^{(n+M_x)}(t)\Delta y - \tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - J_{\mathrm{ez}}^{(n)}(t)\Delta x\Delta y \\
&= \left(I - S_{M_y}\right)\tilde{\nu}_{xx}^{(n)}B_x^{(n)}(t)\Delta x + \left(S_{M_x} - I\right)\tilde{\nu}_{yy}^{(n)}B_y^{(n)}(t)\Delta y - J_{\mathrm{ex}}^{(n)}(t)\Delta x\Delta y
\end{aligned}$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

$$\underbrace{\begin{bmatrix} \tilde{\varepsilon}_{xx}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)} \end{bmatrix}}_{=\left[\widetilde{\varepsilon}\right]^{(n)}} \underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=\left[\widetilde{S}\right]} \frac{d}{dt} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}}_{=\{E\}^{(n)}(t)}$$

$$= \underbrace{\begin{bmatrix} 0 & I - S_{M_z} & S_{M_y} - I \\ S_{M_z} - I & 0 & i - S_{M_x} \\ I - S_{M_y} & S_{M_x} - I & 0 \end{bmatrix}}_{=\left[\widetilde{\text{curl}}\right]} \underbrace{\begin{bmatrix} \tilde{\nu}_{xx}^{(n)} & & \\ & \tilde{\nu}_{yy}^{(n)} & \\ & & \tilde{\nu}_{zz}^{(n)} \end{bmatrix}}_{=\left[\widetilde{\nu}\right]^{(n)}} \underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=\left[\widetilde{R}\right]} \underbrace{\begin{Bmatrix} B_x^{(n)}(t) \\ B_y^{(n)}(t) \\ B_z^{(n)}(t) \end{Bmatrix}}_{=\{B\}^{(n)}(t)} - \underbrace{\begin{bmatrix} \Delta y \Delta z & & \\ & \Delta x \Delta z & \\ & & \Delta x \Delta y \end{bmatrix}}_{=\left[\widetilde{S}\right]} \underbrace{\begin{Bmatrix} J_{\text{ex}}^{(n)}(t) \\ J_{\text{ex}}^{(n)}(t) \\ J_{\text{ex}}^{(n)}(t) \end{Bmatrix}}_{=\{J_{\text{e}}\}^{(n)}(t)}$$

$$\begin{bmatrix} 0 & I - S_{M_z} & S_{M_y} - I \\ S_{M_z} - I & 0 & i - S_{M_x} \\ I - S_{M_y} & S_{M_x} - I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -P_{M_z} & P_{M_y} \\ P_{M_z} & 0 & -P_{M_x} \\ -P_{M_y} & P_{M_x} & 0 \end{bmatrix} = \left[\widetilde{\text{curl}}\right]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

$$\underbrace{\begin{bmatrix} \tilde{\varepsilon}_{xx}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)} \end{bmatrix}}_{=[\tilde{\varepsilon}]^{(n)}} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y \end{bmatrix}}_{=[\tilde{S}]} \frac{d}{dt} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}}_{=\{E\}^{(n)}(t)} = \underbrace{\begin{bmatrix} 0 & -P_{M_z} & P_{M_y} \\ P_{M_z} & 0 & -P_{M_x} \\ -P_{M_y} & P_{M_x} & 0 \end{bmatrix}}_{=[\widetilde{\mathrm{curl}}]} \underbrace{\begin{bmatrix} \tilde{\nu}_{xx}^{(n)} & & \\ & \tilde{\nu}_{yy}^{(n)} & \\ & & \tilde{\nu}_{zz}^{(n)} \end{bmatrix}}_{=[\tilde{\nu}]^{(n)}} \underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[\tilde{R}]} \underbrace{\begin{Bmatrix} B_x^{(n)}(t) \\ B_y^{(n)}(t) \\ B_z^{(n)}(t) \end{Bmatrix}}_{=\{B\}^{(n)}(t)} - \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta y \end{bmatrix}}_{=[\tilde{S}]} \underbrace{\begin{Bmatrix} J_{\mathrm{ex}}^{(n)}(t) \\ J_{\mathrm{ex}}^{(n)}(t) \\ J_{\mathrm{ex}}^{(n)}(t) \end{Bmatrix}}_{=\{J_\mathrm{e}\}^{(n)}(t)}$$

$$[\tilde{\varepsilon}]^{(n)}[\tilde{S}]\frac{d}{dt}\{E\}^{(n)}(t) = [\widetilde{\mathrm{curl}}][\tilde{\nu}]^{(n)}[\tilde{R}]\{B\}^{(n)}(t) - [\tilde{S}]\{J_\mathrm{e}\}^{(n)}(t)$$

$[\tilde{\varepsilon}]^{(n)}$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of permittivities on the grid $\tilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\tilde{G}$
$[\tilde{S}]$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of elementary surfaces on the grid $\tilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\tilde{G}$
$\{E\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$[\widetilde{\mathrm{curl}}]$	$\in \mathbb{R}^{3\times 3}$	Topological curl operator in matrix form on the grid $\tilde{G}$ / Topologischer Rotationsoperator in Matrixform auf dem Gitter $\tilde{G}$
$[\tilde{\nu}]^{(n)}$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of impermeabilities on the grid $\tilde{G}$ / Diagonalmatrix der Impermeabilitäten auf dem Gitter $\tilde{G}$
$[\tilde{R}]$	$\in \mathbb{R}^{3\times 3}$	Diagonal matrix of elementary lines on the grid $\tilde{G}$ / Diagonalmatrix der Elementarstrecken auf dem Gitter $\tilde{G}$
$\{B\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$\{J_\mathrm{e}\}^{(n)}(t)$	$\in \mathbb{R}^{3}$	Algebraic electric current density vector / Algebraischer elektrischer Stromdichtevektor

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler und globaler Matrixform

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{R}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}}$$

$$\frac{d}{dt}\iint_S \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}} = \oint_{C=\partial S} \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{R}} - \iint_S \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}}$$

Discrete grid equations in local matrix form / Diskrete Gittergleichungen in lokaler Matrixform

$$[S]\frac{d}{dt}\{B\}^{(n)}(t) = -[\mathrm{curl}][R]\{E\}^{(n)}(t) \quad -[S]\{J_\mathrm{m}\}^{(n)}(t)$$

$$\widetilde{[\varepsilon]}^{(n)}\widetilde{[S]}\frac{d}{dt}\{E\}^{(n)}(t) = \widetilde{[\mathrm{curl}]}\widetilde{[\nu]}^{(n)}\widetilde{[R]}\{B\}^{(n)}(t) - \widetilde{[S]}\{J_\mathrm{e}\}^{(n)}(t)$$

Discrete grid equations in global matrix form / Diskrete Gittergleichungen in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

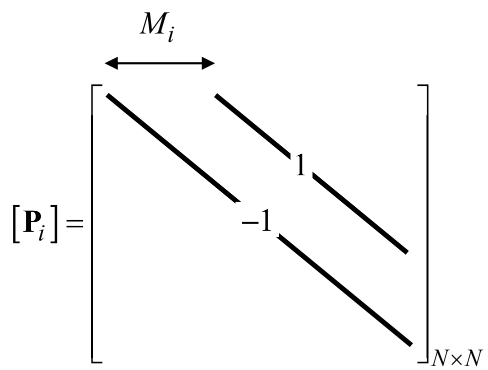
$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\boldsymbol{\nu}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_\mathrm{e}\}(t)$$

Elementary Difference Matrix $[\mathbf{P}_i]$ (P Matrix) / Elementare Differenzmatrix $[\mathbf{P}_i]$ (P-Matrix)

Elementary difference operator in global matrix form (P matrix) /
Elementarer Differenzoperator in globaler Matrixform (P-Matrix)

$$[\mathbf{P}_{\pm i}] := \left([\mathbf{P}_{\pm i}]\right)_{jk}, \quad j,k \in \{1,2,\ldots,N\}$$

$$\left([\mathbf{P}_{\pm i}]\right)_{jk} = \begin{cases} \mp 1 & j = k \\ \pm 1 & j = k \mp M_i \text{ or / bzw. } k = j \pm M_i; \quad i = x,y,z; \quad j,k \in \{1,2,\ldots,N\} \\ 0 & \text{else / sonst} \end{cases}$$



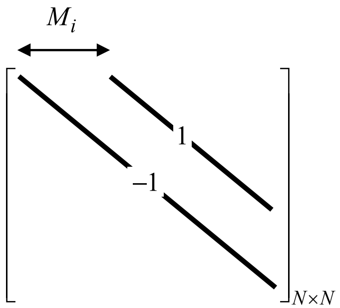
The P matrix has only two bands /
Die P-Matrix hat nur zwei Bänder

Elementary Difference Matrix $[\mathbf{P}_i]$ (P Matrix) (…) / Elementare Differenzmatrix $[\mathbf{P}_i]$ (P-Matrix) (…)

The P matrix can be represented by a sum of an identity matrix $[\mathbf{I}]$ and a band matrix $[\mathbf{B}]$ /
Die P-Matrix kann als Summe aus einer Einheitsmatrix (Identitätsmatrix) $[\mathbf{I}]$ und Bandmatrix $[\mathbf{B}]$ dargestellt werden

$$[\mathbf{P}_{\pm i}] := \mp[\mathbf{I}] \pm [\mathbf{B}_{\pm i}], \qquad i = \{x, y, z\}$$

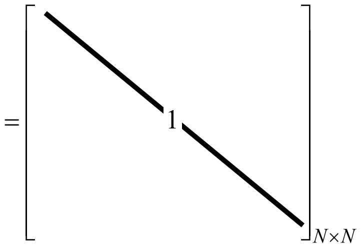
$[\mathbf{P}_i] =$



Identity matrix / Einheitsmatrix (Identitätsmatrix)

$$([\mathbf{I}])_{ij} = \delta_{ij} \qquad i, j \in \{1, 2, \ldots, N\}$$

$[\mathbf{I}] =$

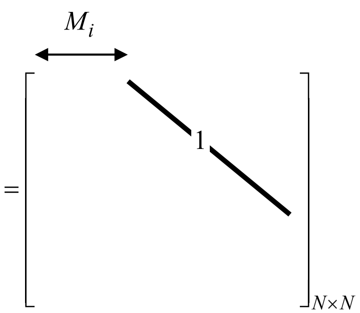


Band matrix / Bandmatrix

$$([\mathbf{B}]_{\pm i})_{jk} = \begin{cases} 1 & j = k \mp M_i \text{ or / bzw. } k = j \pm M_i \\ 0 & \text{else / sonst} \end{cases}$$

$$i = x, y, z; \; j, k \in \{1, 2, \ldots, N\}$$

$[\mathbf{B}_i] =$



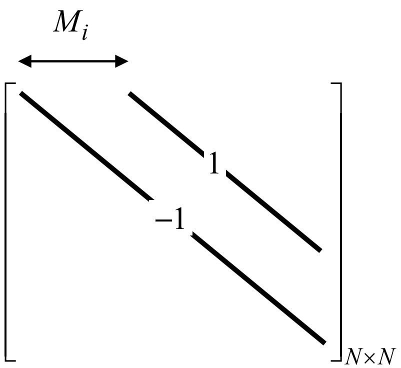
Properties of the Difference Matrix $[\mathrm{P}_i]$ (P Matrix) / Eigenschaften der Differenzmatrix $[\mathrm{P}_i]$ (P-Matrix)

$$[\mathbf{P}_{\pm i}] := \mp[\mathbf{I}] \pm [\mathbf{B}_{\pm i}], \quad i = \{x, y, z\}$$

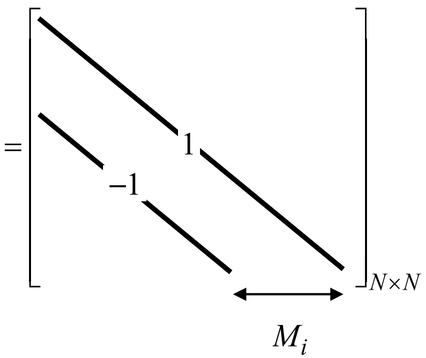
$$[\mathbf{P}_i] := -[\mathbf{I}] + [\mathbf{B}_i], \quad i = \{x, y, z\}$$

$$[\mathbf{P}_{-i}] := [\mathbf{I}] - [\mathbf{B}_{-i}], \quad i = \{x, y, z\}$$

$[\mathbf{P}_i] =$

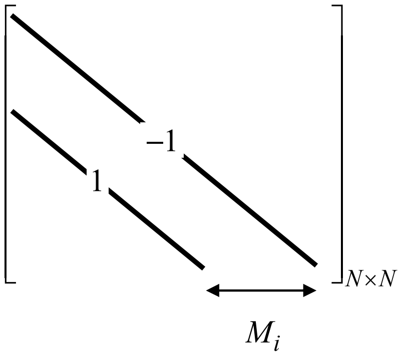


$[\mathbf{P}_{-i}] =$

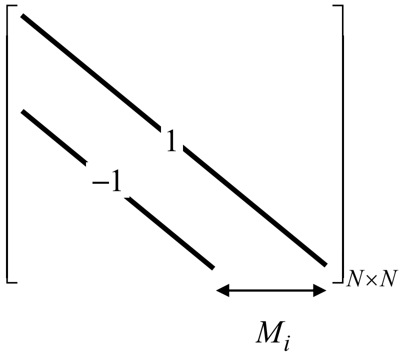


Property / Eigenschaft $\quad -[\mathbf{P}_i]^{\mathrm{T}} = [\mathbf{P}_{-i}]$

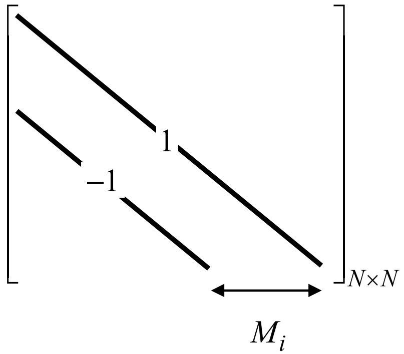
$[\mathbf{P}_i]^{\mathrm{T}} =$



$-[\mathbf{P}_i]^{\mathrm{T}} =$



$[\mathbf{P}_{-i}] =$



Discrete Global Gradient, Divergence, and Curl Operator / Diskreter globaler Gradienten-, Divergenz- und Rotationsoperator

Discrete gradient operator /
Diskreter Gradientenoperator

$$[\mathbf{grad}] = \begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}_{3N\times N}$$

$$[\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix}_{3N\times N}$$

Discrete curl operator /
Diskreter Rotationsoperator

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^{\mathrm{T}} & -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} & [\mathbf{0}] & [\mathbf{P}_x]^{\mathrm{T}} \\ [\mathbf{P}_y]^{\mathrm{T}} & -[\mathbf{P}_x]^{\mathrm{T}} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

Discrete divergence operator /
Diskreter Divergenzoperator

$$[\mathbf{div}] := \left[-[\mathbf{P}_x]^{\mathrm{T}}, -[\mathbf{P}_y]^{\mathrm{T}}, -[\mathbf{P}_z]^{\mathrm{T}} \right]_{N\times 3N}$$

$$[\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x], [\mathbf{P}_y], [\mathbf{P}_z] \right]_{N\times 3N}$$

The matrix operators /
Die Matrixoperatoren

$$[\mathbf{grad}] \quad [\widetilde{\mathbf{grad}}]$$
$$[\mathbf{div}] \quad [\widetilde{\mathbf{div}}]$$
$$[\mathbf{curl}] \quad [\widetilde{\mathbf{curl}}]$$

are **global** matrix operators /
sind **globale** Matrixoperatoren

Properties of the Global Matrix Operators / Eigenschaften der globalen Matrixoperatoren

Some properties of the global matrix operators of the dual grid system /
Einige Eigenschaften der globalen Matrixoperatoren des dualen Gittersystems

$$-\left[\widetilde{\mathbf{div}}\right] = \left[\mathbf{grad}\right]^{\mathrm{T}}$$

$$\left[\widetilde{\mathbf{grad}}\right]^{\mathrm{T}} = \left[\mathbf{div}\right]$$

$$\left[\mathbf{curl}\right] = \left[\widetilde{\mathbf{curl}}\right]^{\mathrm{T}}$$

Conservation of important vector identities /
Erhaltung von wichtigen Vektoridentitäten

Vector identities /
Vektoridentitäten

$$\begin{aligned} \mathrm{curl\ grad} &= \nabla \times \nabla = \underline{\mathbf{0}} \\ \mathrm{div\ curl} &= \nabla \bullet \nabla = 0 \end{aligned}$$

$$\Updownarrow$$

are conserved in the dual grid system /
bleiben im dualen Gittersystem erhalten

$$\left[\mathbf{curl}\right]\left[\mathbf{grad}\right] = \left[\mathbf{0}\right]$$

$$\left[\widetilde{\mathbf{curl}}\right]\left[\widetilde{\mathbf{grad}}\right] = \left[\mathbf{0}\right]$$

$$\left[\mathbf{div}\right]\left[\mathbf{curl}\right] = \left[\mathbf{0}\right]$$

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\mathbf{curl}}\right] = \left[\mathbf{0}\right]$$

Properties of the Global Matrix Operators / Eigenschaften der globalen Matrixoperatoren

Consistency test / Konsistenztest

$$[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix}$$

$$= \begin{bmatrix} [\mathbf{P}_y][\mathbf{P}_z] - [\mathbf{P}_z][\mathbf{P}_y] \\ [\mathbf{P}_z][\mathbf{P}_x] - [\mathbf{P}_x][\mathbf{P}_z] \\ [\mathbf{P}_x][\mathbf{P}_y] - [\mathbf{P}_y][\mathbf{P}_x] \end{bmatrix}$$

$$[\mathbf{P}_i][\mathbf{P}_j] - [\mathbf{P}_j][\mathbf{P}_i] = \left(-[\mathbf{I}] + [\mathbf{B}_i]\right)\left(-[\mathbf{I}] + [\mathbf{B}_j]\right) - \left(-[\mathbf{I}] + [\mathbf{B}_j]\right)\left(-[\mathbf{I}] + [\mathbf{B}_i]\right)$$

$$= \left(-[\mathbf{I}][\mathbf{I}] - [\mathbf{I}][\mathbf{B}_j] - [\mathbf{B}_i][\mathbf{I}] + [\mathbf{B}_i][\mathbf{B}_j]\right)$$

$$-\left(-[\mathbf{I}][\mathbf{I}] - [\mathbf{I}][\mathbf{B}_i] - [\mathbf{B}_j][\mathbf{I}] + [\mathbf{B}_j][\mathbf{B}_i]\right)$$

$$= \left(-[\mathbf{I}] - [\mathbf{B}_j] - [\mathbf{B}_i] + [\mathbf{B}_i][\mathbf{B}_j]\right)$$

$$-\left(-[\mathbf{I}] - [\mathbf{B}_i] - [\mathbf{B}_j] + [\mathbf{B}_j][\mathbf{B}_i]\right)$$

$$= -[\mathbf{I}] - [\mathbf{B}_j] - [\mathbf{B}_i] + [\mathbf{B}_i][\mathbf{B}_j] + [\mathbf{I}] + [\mathbf{B}_i] + [\mathbf{B}_j] - [\mathbf{B}_j][\mathbf{B}_i]$$

$$= [\mathbf{B}_i][\mathbf{B}_j] - [\mathbf{B}_j][\mathbf{B}_i]$$

Properties of the Global Matrix Operators / Eigenschaften der globalen Matrixoperatoren

With the property /
Mit der Eigenschaft

$$\left(\left[\mathbf{B}_{\pm i}\right]\left[\mathbf{B}_{\pm j}\right]\right)_{kl} = \begin{cases} 1 & k = l \mp M_i \mp M_j \\ 0 & \text{else / sonst} \end{cases}$$

⇒ *i* and *j* can be arbitrarily interchanged /
i und *j* können beliebig vertauscht werden

⇒ This means that the matrices $\left[\mathbf{B}_{\pm i}\right]$ and $\left[\mathbf{B}_{\pm j}\right]$ as well as $\left[\mathbf{P}_{\pm i}\right]$ and $\left[\mathbf{P}_{\pm j}\right]$ are commutative!
Das bedeutet, dass die Matrizen $\left[\mathbf{B}_{\pm i}\right]$ und $\left[\mathbf{B}_{\pm j}\right]$ als auch $\left[\mathbf{P}_{\pm i}\right]$ und $\left[\mathbf{P}_{\pm j}\right]$ kommutativ sind!

⇒

$$\left[\mathbf{B}_{\pm i}\right]\left[\mathbf{B}_{\pm j}\right] = \left[\mathbf{B}_{\pm j}\right]\left[\mathbf{B}_{\pm i}\right]$$

$$\left[\mathbf{P}_{\pm i}\right]\left[\mathbf{P}_{\pm j}\right] = \left[\mathbf{P}_{\pm j}\right]\left[\mathbf{P}_{\pm i}\right]$$

⇒

$$\begin{aligned} \left[\widetilde{\mathbf{curl}}\right]\left[\widetilde{\mathbf{grad}}\right] &= \left[\mathbf{P}_i\right]\left[\mathbf{P}_j\right] - \left[\mathbf{P}_j\right]\left[\mathbf{P}_i\right] \\ &= \left[\mathbf{B}_i\right]\left[\mathbf{B}_j\right] - \underbrace{\left[\mathbf{B}_j\right]\left[\mathbf{B}_i\right]}_{=\left[\mathbf{B}_i\right]\left[\mathbf{B}_j\right]} \\ &= \left[\mathbf{B}_i\right]\left[\mathbf{B}_j\right] - \left[\mathbf{B}_i\right]\left[\mathbf{B}_j\right] \\ &= \left[\mathbf{0}\right] \end{aligned}$$

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in lokaler und globaler Matrixform

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{d}{dt}\iint_S \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S} \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

Discrete grid equations in local matrix form /
Diskrete Gittergleichungen in lokaler Matrixform

$$[S]\frac{d}{dt}\{B\}^{(n)}(t) = -[\mathrm{curl}][R]\{E\}^{(n)}(t) - [S]\{J_\mathrm{m}\}^{(n)}(t)$$

$$\widetilde{[\varepsilon]}^{(n)}\widetilde{[S]}\frac{d}{dt}\{E\}^{(n)}(t) = \widetilde{[\mathrm{curl}]}\widetilde{[\nu]}^{(n)}\widetilde{[R]}\{B\}^{(n)}(t) - \widetilde{[S]}\{J_\mathrm{e}\}^{(n)}(t)$$

$$n = 1, 2, \ldots, N$$

Discrete grid equations in global matrix form /
Diskrete Gittergleichungen in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) - [\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\boldsymbol{\nu}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_\mathrm{e}\}(t)$$

3-D FIT – … Discrete Grid Equations in Gobal Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Faraday's induction law in global matrix form /
Faradaysches Induktionsgesetz in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$[\mathbf{S}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\{\mathbf{B}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$[\mathbf{curl}]$	$\in \mathbb{R}^{3N\times 3N}$	Topological curl operator in matrix form on the grid G / Topologischer Rotationsoperator in Matrixform auf dem Gitter G
$[\mathbf{R}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary lines on the grid G / Diagonalmatrix der Elementarstrecken auf dem Gitter G
$\{\mathbf{E}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrische Feldstärkevektor
$\{\mathbf{J}_\mathrm{m}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic current density vector / Algebraischer magnetischer Stromdichtevektor

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Faraday's induction law in global matrix form /
Faradaysches Induktionsgesetz in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$[\mathbf{S}] = \begin{bmatrix} \Delta y\Delta z & & & & & & & \\ & \ddots & & & & & & \\ & & \Delta y\Delta z & & & & & \\ & & & \Delta x\Delta y & & & & \\ & & & & \ddots & & & \\ & & & & & \Delta x\Delta y & & \\ & & & & & & \Delta x\Delta y & \\ & & & & & & \ddots & \\ & & & & & & & \Delta x\Delta y \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} [\mathrm{diag}\{\Delta y\Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x\Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x\Delta y\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^\mathrm{T} & -[\mathbf{P}_y]^\mathrm{T} \\ -[\mathbf{P}_z]^\mathrm{T} & [\mathbf{0}] & [\mathbf{P}_x]^\mathrm{T} \\ [\mathbf{P}_y]^\mathrm{T} & -[\mathbf{P}_x]^\mathrm{T} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{R}] = \begin{bmatrix} \Delta x & & & & & & & & \\ & \ddots & & & & & & & \\ & & \Delta x & & & & & & \\ & & & \Delta y & & & & & \\ & & & & \ddots & & & & \\ & & & & & \Delta y & & & \\ & & & & & & \Delta z & & \\ & & & & & & & \ddots & \\ & & & & & & & & \Delta z \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} [\mathrm{diag}\{\Delta x\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta y\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta z\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$\{\mathbf{B}\}(t) = \begin{Bmatrix} \{B_x\}(t) \\ \{B_y\}(t) \\ \{B_z\}(t) \end{Bmatrix}_{3N} \qquad \{B_i\}(t) = \begin{Bmatrix} B_i^{(1)}(t) \\ B_i^{(2)}(t) \\ \vdots \\ B_i^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Faraday's induction law in global matrix form /
Faradaysches Induktionsgesetz in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$\{\mathbf{E}\}(t) = \begin{Bmatrix} \{E_x\}(t) \\ \{E_y\}(t) \\ \{E_z\}(t) \end{Bmatrix}_{3N} \qquad \{E_i\}(t) = \begin{Bmatrix} E_i^{(1)}(t) \\ E_i^{(2)}(t) \\ \vdots \\ E_i^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

$$\{\mathbf{J}_\mathrm{m}\}(t) = \begin{Bmatrix} \{J_{\mathrm{m}x}\}(t) \\ \{J_{\mathrm{m}y}\}(t) \\ \{J_{\mathrm{m}z}\}(t) \end{Bmatrix}_{3N} \qquad \{J_{\mathrm{m}i}\}(t) = \begin{Bmatrix} J_{\mathrm{m}i}^{(1)}(t) \\ J_{\mathrm{m}i}^{(2)}(t) \\ \vdots \\ J_{\mathrm{m}i}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$[\widetilde{\boldsymbol{\varepsilon}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of permittivities on the grid $\widetilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\widetilde{G}$
$[\widetilde{\mathbf{S}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\widetilde{G}$
$\{\mathbf{E}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$[\widetilde{\mathrm{curl}}]$	$\in \mathbb{R}^{3N\times 3N}$	Topological curl operator in matrix form on the grid $\widetilde{G}$ / Topologischer Rotationsoperator in Matrixform auf dem Gitter $\widetilde{G}$
$[\widetilde{\boldsymbol{\nu}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of impermeabilities on the grid $\widetilde{G}$ / Diagonalmatrix der Impermeabilitäten auf dem Gitter $\widetilde{G}$
$[\widetilde{\mathbf{R}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary lines on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarstrecken auf dem Gitter $\widetilde{G}$
$\{\mathbf{B}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$\{\mathbf{J}_\mathrm{e}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric current density vector / Algebraischer elektrischer Stromdichtevektor

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$[\widetilde{\varepsilon}] = \begin{bmatrix} \tilde{\varepsilon}_{xx}^{(1)} & & & & & & & & \\ & \ddots & & & & & & & \\ & & \tilde{\varepsilon}_{xx}^{(N)} & & & & & & \\ & & & \tilde{\varepsilon}_{yy}^{(1)} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{\varepsilon}_{yy}^{(N)} & & & \\ & & & & & & \tilde{\varepsilon}_{zz}^{(1)} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \tilde{\varepsilon}_{zz}^{(N)} \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\{\tilde{\varepsilon}_{xx}^{(1)}, \tilde{\varepsilon}_{xx}^{(2)}, \ldots, \tilde{\varepsilon}_{xx}^{(N)}\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\{\tilde{\varepsilon}_{yy}^{(1)}, \tilde{\varepsilon}_{yy}^{(2)}, \ldots, \tilde{\varepsilon}_{yy}^{(N)}\}_N\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\{\tilde{\varepsilon}_{zz}^{(1)}, \tilde{\varepsilon}_{zz}^{(2)}, \ldots, \tilde{\varepsilon}_{zz}^{(N)}\}_N\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$$[\widetilde{\mathbf{S}}] = \begin{bmatrix} \Delta y\Delta z & & & & & & & & \\ & \ddots & & & & & & & \\ & & \Delta y\Delta z & & & & & & \\ & & & \Delta x\Delta z & & & & & \\ & & & & \ddots & & & & \\ & & & & & \Delta x\Delta z & & & \\ & & & & & & \Delta x\Delta y & & \\ & & & & & & & \ddots & \\ & & & & & & & & \Delta y\Delta z \end{bmatrix}_{3N\times 3N} = \begin{bmatrix} [\mathrm{diag}\{\Delta y\Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x\Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x\Delta y\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{R}}] = \begin{bmatrix} \Delta x & & & & & & & & \\ & \ddots & & & & & & & \\ & & \Delta x & & & & & & \\ & & & \Delta y & & & & & \\ & & & & \ddots & & & & \\ & & & & & \Delta y & & & \\ & & & & & & \Delta z & & \\ & & & & & & & \ddots & \\ & & & & & & & & \Delta z \end{bmatrix}_{3N\times 3N} = \begin{bmatrix} [\mathrm{diag}\{\Delta x\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta y\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta z\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$\{\mathbf{J}_\mathrm{e}\}(t) = \begin{Bmatrix} \{J_\mathrm{ex}\}(t) \\ \{J_\mathrm{ey}\}(t) \\ \{J_\mathrm{ez}\}(t) \end{Bmatrix}_{3N} \qquad \{J_{\mathrm{e}i}\}(t) = \begin{Bmatrix} J_{\mathrm{e}i}^{(1)}(t) \\ J_{\mathrm{e}i}^{(2)}(t) \\ \vdots \\ J_{\mathrm{e}i}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\mathbf{v}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_\mathrm{e}\}(t)$$

$$\widetilde{[\nu]} = \begin{bmatrix} \tilde{\nu}_{xx}^{(1)} & & & & & & & & \\ & \ddots & & & & & & & \\ & & \tilde{\nu}_{xx}^{(N)} & & & & & & \\ & & & \tilde{\nu}_{yy}^{(1)} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{\nu}_{yy}^{(N)} & & & \\ & & & & & & \tilde{\nu}_{zz}^{(1)} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \tilde{\nu}_{zz}^{(N)} \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\{\tilde{\nu}_{xx}^{(1)}, \tilde{\nu}_{xx}^{(2)}, \ldots, \tilde{\nu}_{xx}^{(N)}\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\{\tilde{\nu}_{yy}^{(1)}, \tilde{\nu}_{yy}^{(2)}, \ldots, \tilde{\nu}_{yy}^{(N)}\}_N\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\{\tilde{\nu}_{zz}^{(1)}, \tilde{\nu}_{zz}^{(2)}, \ldots, \tilde{\nu}_{zz}^{(N)}\}_N\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in lokaler und globaler Matrixform

The two discrete grid equations in global matrix form read /
Die beiden diskreten Gittergleichungen in globaler Matrixform lauten

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = \quad [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

We arrange the last equations in the form /
Wir bringen die letzten beiden Gleichungen in die Form

$$\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{S}]^{-1} \quad [\mathbf{curl}] \quad [\mathbf{R}]\{\mathbf{E}\}(t) - [\mathbf{S}]^{-1} \quad [\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$\frac{d}{dt}\{\mathbf{E}\}(t) = \quad [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} \, [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$$[\mathbf{S}]^{-1}[\mathbf{S}] = [\mathbf{I}]$$

$$[\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{S}}] = \underbrace{[\widetilde{\mathbf{S}}]^{-1}[\widetilde{\mathbf{S}}]}_{=[\mathbf{I}]}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} = [\widetilde{\boldsymbol{\varepsilon}}]^{-1}$$

$$\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{S}]^{-1} \quad [\mathbf{curl}] \quad [\mathbf{R}]\{\mathbf{E}\}(t) - \quad \{\mathbf{J}_\mathrm{m}\}(t)$$

$$\frac{d}{dt}\{\mathbf{E}\}(t) = \quad [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} \, [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\boldsymbol{\varepsilon}}]^{-1} \, \{\mathbf{J}_\mathrm{e}\}(t)$$

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in lokaler und globaler Matrixform

The two discrete grid equations in global matrix form read /
Die beiden diskreten Gittergleichungen in globaler Matrixform lauten

$$\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{S}]^{-1} \quad [\mathbf{curl}] \quad [\mathbf{R}]\{\mathbf{E}\}(t) - \quad \{\mathbf{J}_\mathrm{m}\}(t)$$

$$\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}(t)$$

Now we write these two matrix equations in matrix form and find a first-order system of differential equations /
Nun schreiben wir die beiden Matrixgleichungen in Matrixform und finden das folgende System von Differentialgleichungen erster Ordnung

$$\frac{d}{dt}\{\mathbf{y}\}(t) = [\mathbf{A}]\{\mathbf{y}\}(t) + \{\mathbf{q}\}(t)$$

with / mit

Solution vector / Lösungsvektor

$$\{\mathbf{y}\}(t) = \begin{Bmatrix} \{\mathbf{B}\}(t) \\ \{\mathbf{E}\}(t) \end{Bmatrix}$$

System matrix / Systemmatrix

$$[\mathbf{A}] = \begin{bmatrix} [0] & [\mathbf{S}]^{-1}[\mathbf{curl}][\mathbf{R}] \\ [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}] & [0] \end{bmatrix}$$

Source vector / Quellvektor

$$\{\mathbf{q}\}(t) = \begin{Bmatrix} - \quad \{\mathbf{J}_\mathrm{m}\}(t) \\ -[\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}(t) \end{Bmatrix}$$

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

A general solution of the initial value problem (IVP) with the initial value $\{\mathbf{y}\}(t_0)$ is /
Eine allgemeine Lösung des Anfangswertproblems (AWP) mit dem Anfangswert $\{\mathbf{y}\}(t_0)$ ist

$$\{\mathbf{y}\}(t) = \{\mathbf{y}\}(t_0) + \underbrace{\int_{t'=t_0}^{t} \underbrace{\{[\mathbf{A}]\{\mathbf{y}\}(t) + \{\mathbf{q}\}(t)\}}_{=\{\dot{\mathbf{y}}\}(t)} \, \mathrm{d}t}_{\text{time integration / zeitliche Integration}}$$

- implicit time integration / implizierte Zeitintegration
- explicit time integration / explizite Zeitintegration

Explicit time integration / Explizite Zeitintegration

$$\{\mathbf{B}\}(t) = \boxed{\{\mathbf{B}\}(t_0)} + \int_{t'=t_0}^{t} \{\dot{\mathbf{B}}\}(t') \, \mathrm{d}t'$$

$$\{\mathbf{E}\}(t) = \boxed{\{\mathbf{E}\}(t_0)} + \int_{t'=t_0}^{t} \{\dot{\mathbf{E}}\}(t') \, \mathrm{d}t'$$

Initial value / Anfangswert

$t = [0, T]$; T: time interval to be simulated / zu simulierendes Zeitinterval

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

Discretization in time on a staggered grid in time /
Diskretisierung in der Zeit auf einem versetzten Gitter in der Zeit

$$\{\mathbf{B}\}(t) \rightarrow \{\mathbf{B}\}(n_t \Delta t) \rightarrow \{\mathbf{B}\}^{(n_t)}$$

$$\{\mathbf{E}\}(t) \rightarrow \{\mathbf{E}\}\left[\left(n_t + \frac{1}{2}\right)\Delta t\right] \rightarrow \{\mathbf{E}\}^{(n_t + 1/2)}$$

$$\{\mathbf{B}\}(t) = \{\mathbf{B}\}(t_0) + \int_{t'=t_0}^{t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{E}\}(t) = \{\mathbf{E}\}(t_0) + \int_{t'=t_0}^{t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t - 1)} + \int_{t'=(n_t-1)\Delta t}^{n_t \Delta t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{E}\}^{(n_t + 1/2)} = \{\mathbf{E}\}^{(n_t - 1/2)} + \int_{t'=(n_t-1/2)\Delta t}^{(n_t+1/2)\Delta t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t'$$

Mid point rule /
Mittelpunktsregel

$$\int_{t'=(n_t-1)\Delta t}^{n_t \Delta t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t' = \{\dot{\mathbf{B}}\}\left[(n_t - 1/2)\Delta t\right]\Delta t = \{\dot{\mathbf{B}}\}^{(n_t - 1/2)}\Delta t$$

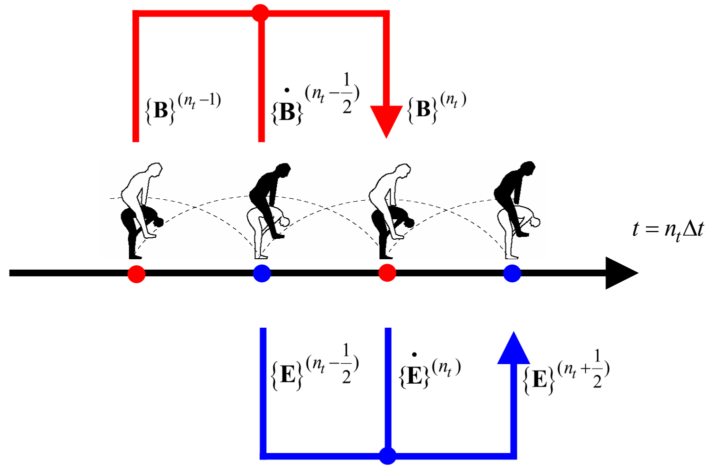
$$\int_{t'=(n_t-1/2)\Delta t}^{(n_t+1/2)\Delta t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t' = \{\dot{\mathbf{E}}\}(n_t \Delta t)\Delta t = \{\dot{\mathbf{E}}\}^{(n_t)}\Delta t$$

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

The leapfrog structure of the algorithm in time /
Die Bocksprung-Struktur des Algorithmus in der Zeit

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \Delta t \{\dot{\mathbf{B}}\}^{(n_t-1/2)}$$

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \Delta t \{\dot{\mathbf{E}}\}^{(n_t)}$$



3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

Electromagnetic grid equations (EMGE) of the so-called
Electromagnetic Finite Integration Technique (EMFIT) algorithm /
Elektromagnetische Gittergleichungen (EMGG) des so genannten
Elektromagnetischen Finite Integrationstechnik (EMFIT) Algorithmus

Faraday's induction grid equation / Faradaysche Induktionsgittergleichung

$$\{\dot{\mathbf{B}}\}^{(n_t-1/2)} = -[\mathbf{S}]^{-1}[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}^{(n_t-1/2)} - \{\mathbf{J}_\mathrm{m}\}^{(n_t-1/2)}$$

Time integration / Zeitintegration

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \Delta t\{\dot{\mathbf{B}}\}^{(n_t-1/2)}$$

Ampère-Maxwell's circuital grid equation / Ampère-Maxwellsche Durchflutungsgittergleichung

$$\{\dot{\mathbf{E}}\}^{(n_t} = [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}\widetilde{[\mathbf{curl}]}[\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}^{(n_t)} - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}^{(n_t)}$$

Time integration / Zeitintegration

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \Delta t\{\dot{\mathbf{E}}\}^{(n_t)}$$

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

Electromagnetic grid equations (EMGE) of the so-called EMFIT algorithm /
Elektromagnetische Gittergleichungen (EMGG) des so genannten EMFIT-Algorithmus

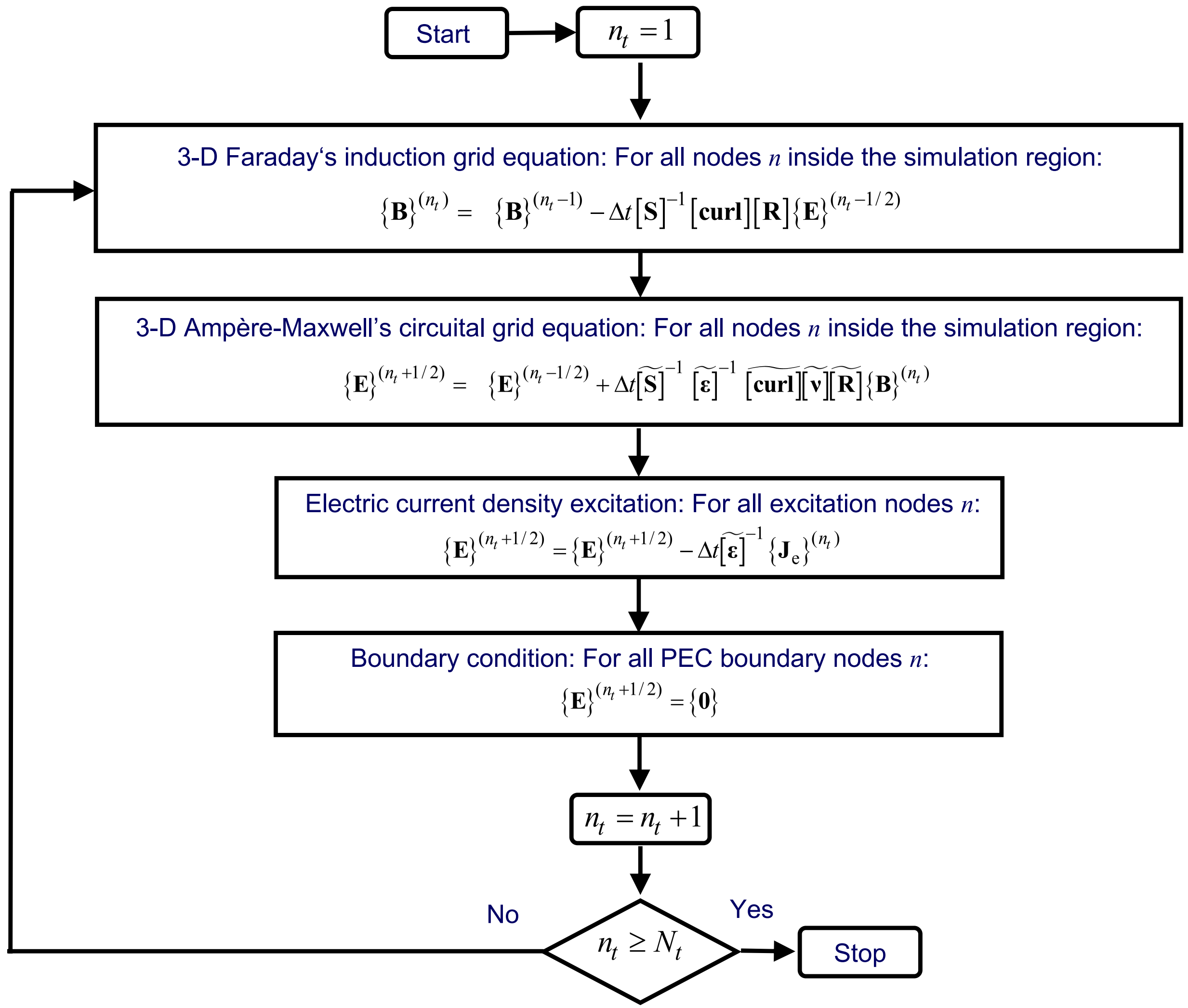
Time-integrated Faraday's induction grid equation /
Zeitlich integrierte Faradaysche Induktionsgittergleichung

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \Delta t\left[-[\mathbf{S}]^{-1}[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}^{(n_t-1/2)} - \{\mathbf{J}_\mathrm{m}\}^{(n_t-1/2)}\right]$$

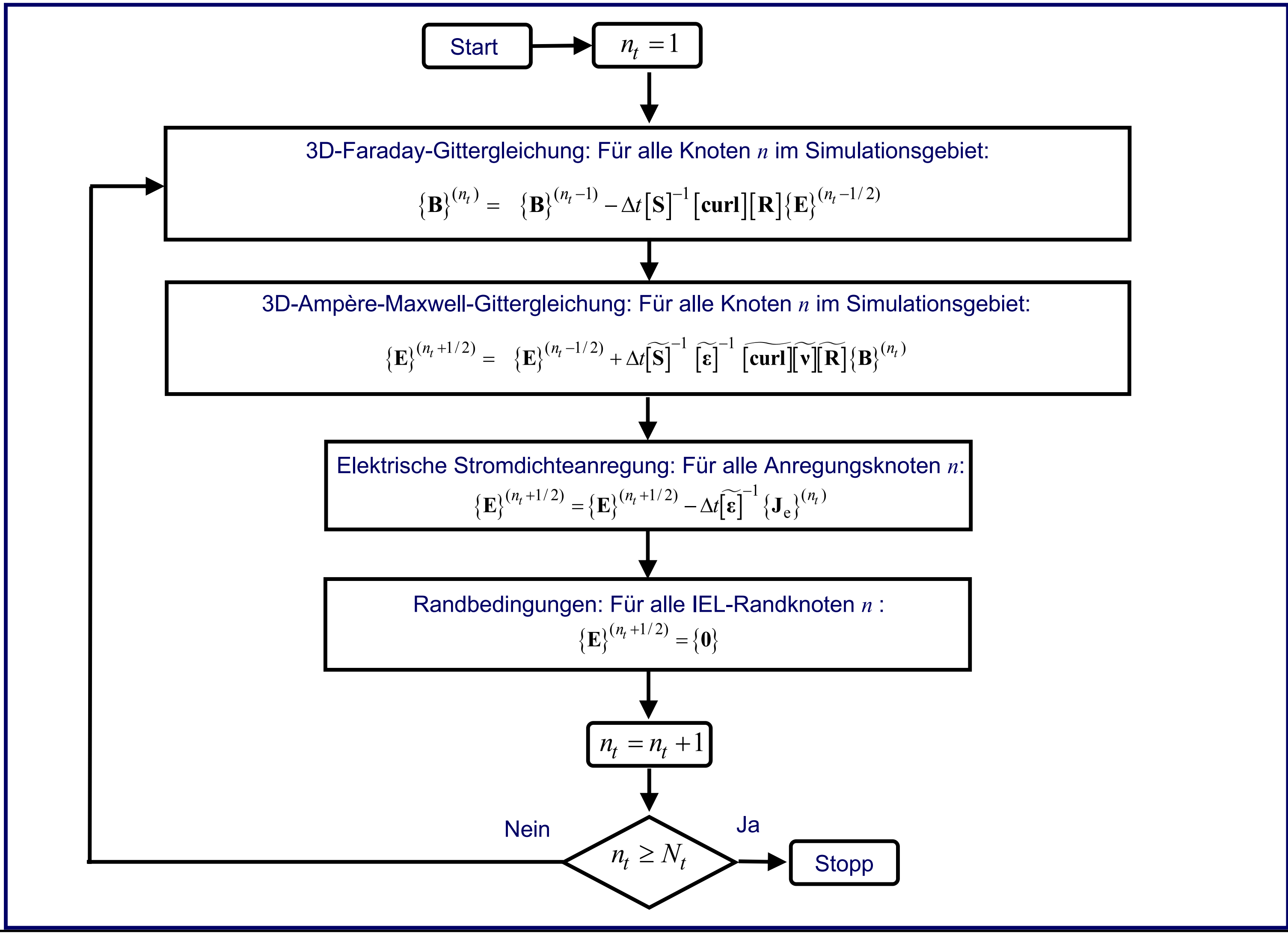
Time-integrated Ampère-Maxwell's circuital grid equation /
Zeitlich integrierte Ampère-Maxwellsche Durchflutungsgittergleichung

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \Delta t\left[[\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}^{(n_t)} - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}^{(n_t)}\right]$$

3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



3-D FIT – … Normalized … Grid Equations / 3D-FIT – ... normierte ... Gittergleichungen

Normalized electromagnetic grid equations (EMGE) of the so-called EMFIT algorithm /
Normierte elektromagnetische Gittergleichungen (EMGG) des so genannten EMFIT-Algorithmus

Normalized time-integrated Faraday's induction grid equation /
Normierte zeitlich integrierte Faradaysche Induktionsgittergleichung

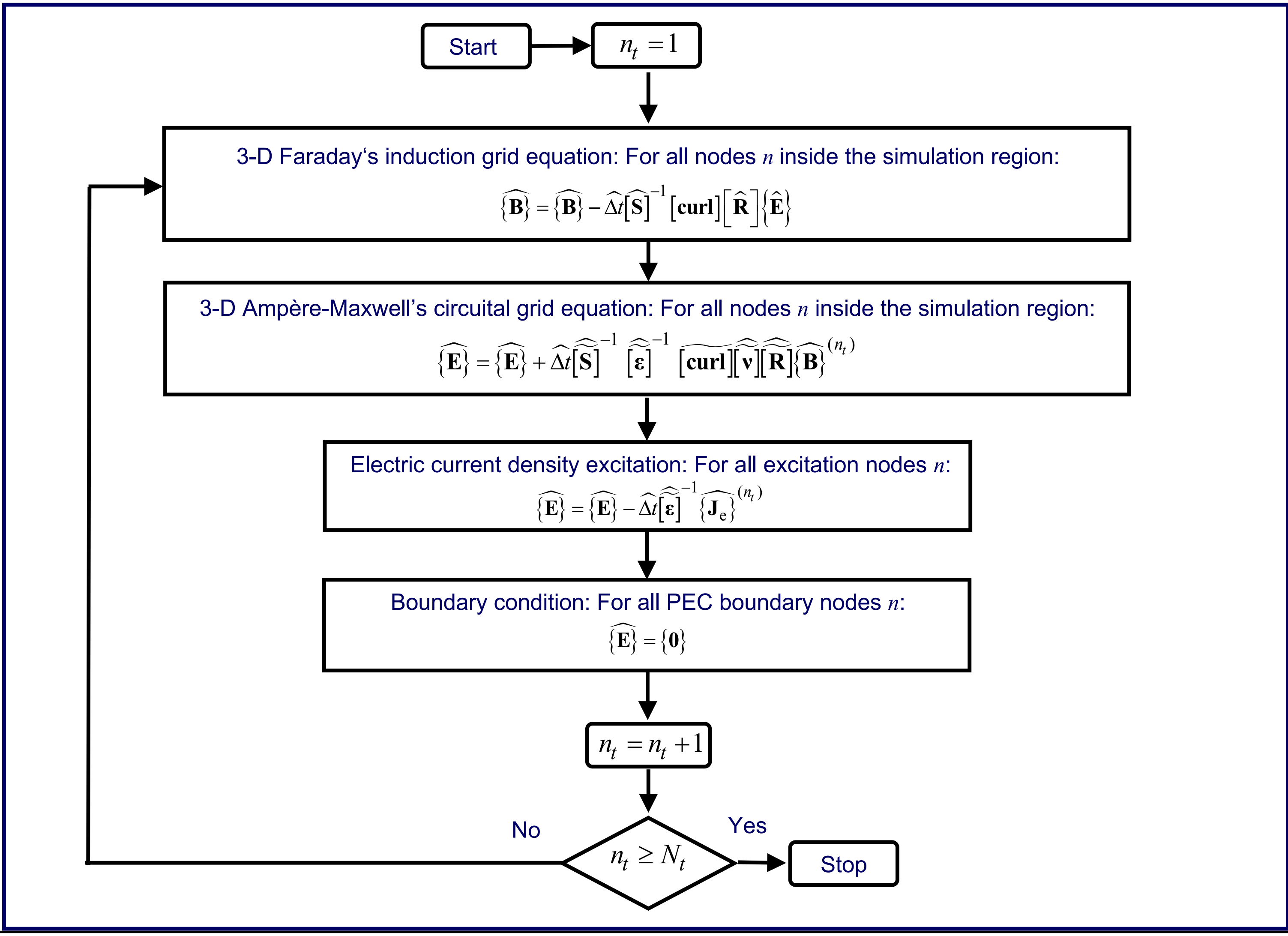
$$\widehat{\{\mathbf{B}\}}^{(n_t)} = \widehat{\{\mathbf{B}\}}^{(n_t-1)} + \widehat{\Delta t}\left[-\left[\widehat{\mathbf{S}}\right]^{-1} \left[\mathbf{curl}\right]\left[\widehat{\mathbf{R}}\right]\left\{\widehat{\mathbf{E}}\right\}^{(n_t-1/2)} - \widehat{\{\mathbf{J}_\mathrm{m}\}}^{(n_t-1/2)} \right]$$

Normalized time-integrated Ampère-Maxwell's circuital grid equation /
Normierte zeitlich integrierte Ampère-Maxwellsche Durchflutungsgittergleichung

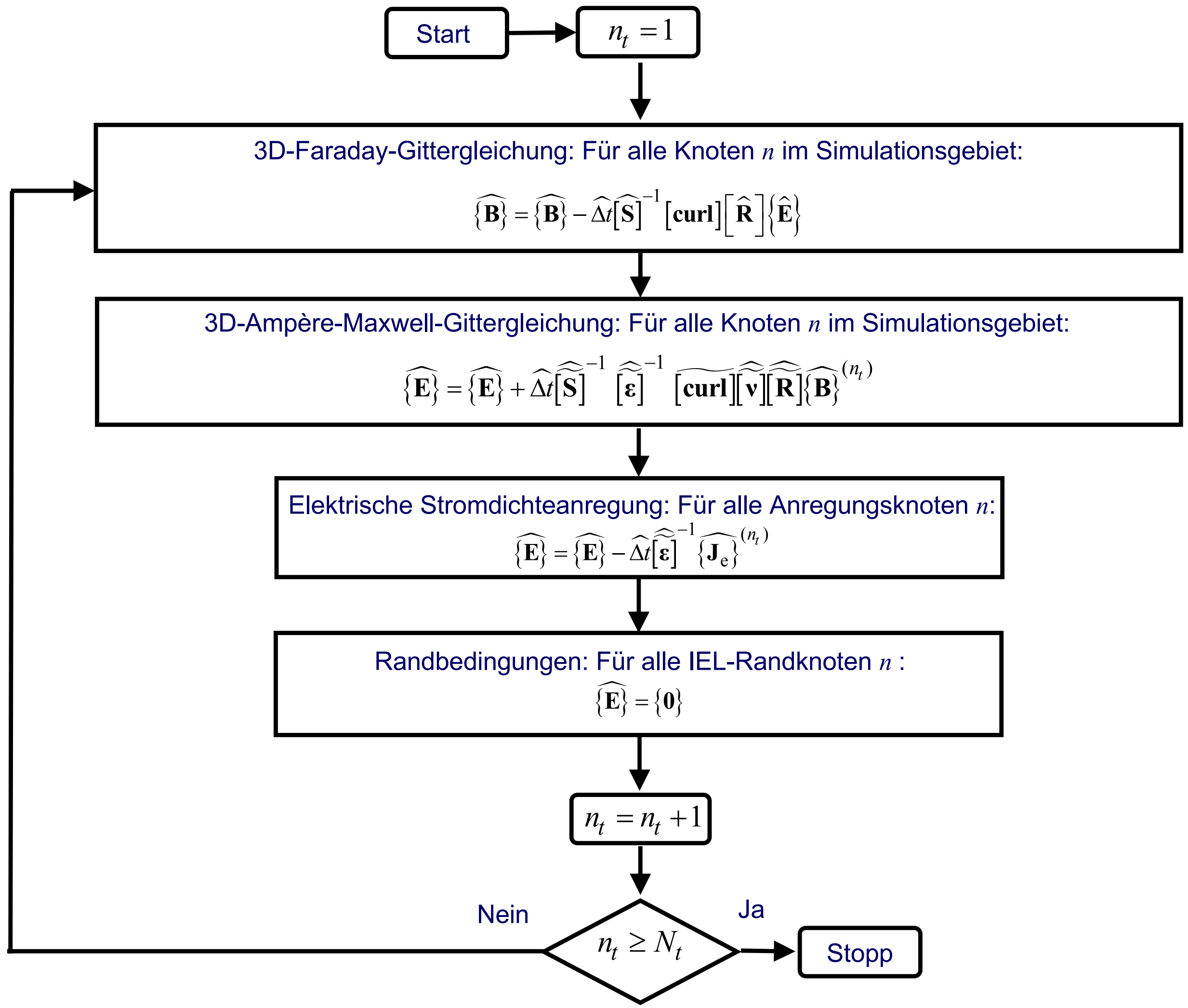
$$\widehat{\{\mathbf{E}\}}^{(n_t+1/2)} = \widehat{\{\mathbf{E}\}}^{(n_t-1/2)} + \widehat{\Delta t}\left[\widehat{\widehat{[\mathbf{S}]}}^{-1} \widehat{\widehat{[\boldsymbol{\varepsilon}]}}^{-1} \widetilde{[\mathbf{curl}]}\widehat{\widehat{[\boldsymbol{\nu}]}}\widehat{\widehat{[\mathbf{R}]}}\widehat{\{\mathbf{B}\}}^{(n_t)} - \widehat{\widehat{[\boldsymbol{\varepsilon}]}}^{-1} \widehat{\{\mathbf{J}_\mathrm{e}\}}^{(n_t)} \right]$$

In a computer implementation we can neglect the integer time step counter n_t. /
In der Rechnerimplementierung kann der ganzzahlige Zeitschrittzähler n_t unterdrückt werden.

3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Maxwell's equations in integral form / Maxwellsche Gleichungen in Integralform	**FIT** Maxwell's grid equations / Maxwellsche Gittergleichungen
$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$	$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$
$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$	$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\mathbf{R}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$

$$\left.\begin{aligned}
\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} &= \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV \\
\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} &= \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV
\end{aligned}\right\} \; ?$$

FIT Discretization of the 3rd Maxwell Equation / FIT-Diskretisierung der 3. Maxwellschen Gleichung

Integral form / Integralform

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

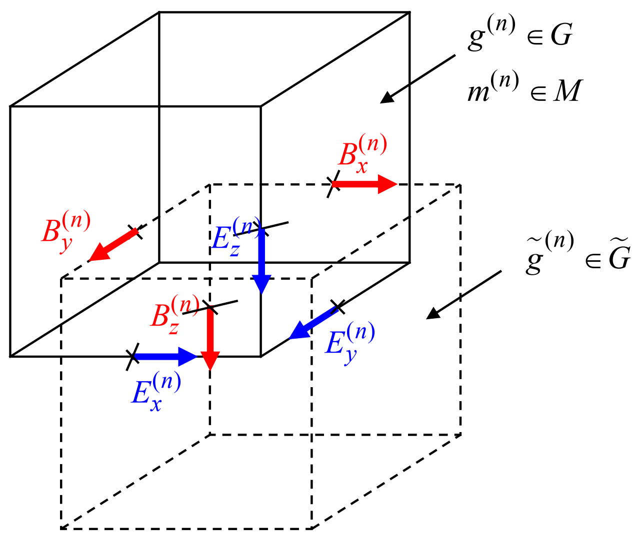
$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV$$

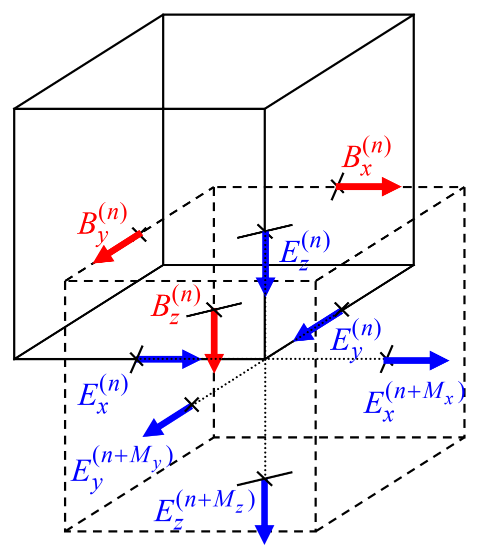
Differential form / Differentialform

$$\nabla\bullet \underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_e(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] = \rho_e(\underline{\mathbf{R}},t)$$

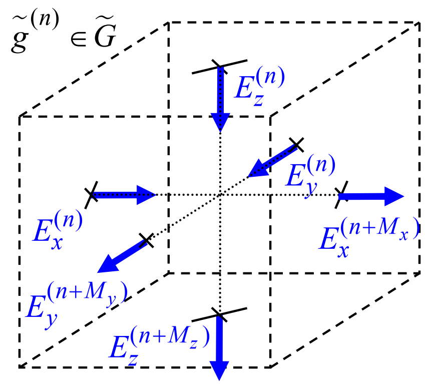


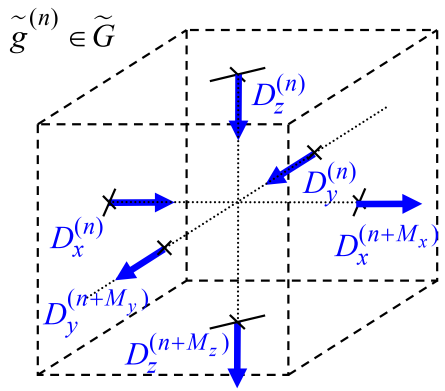


FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$





$$D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

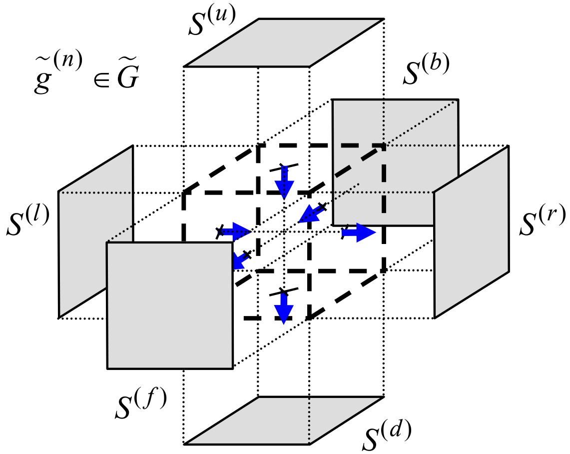
$$D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$D_y^{(n)} = \tilde{\varepsilon}_{yy}^{(n)} E_x^{(n)}$$

$$D_y^{(n+M_y)} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}$$

$$D_z^{(n)} = \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}$$

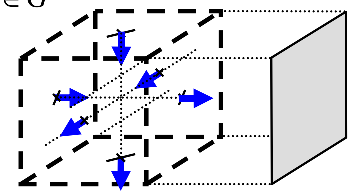
$$D_z^{(n+M_z)} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}$$



$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} + \iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$+\iint_{S^{(f)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} + \iint_{S^{(b)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$+\iint_{S^{(d)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} + \iint_{S^{(u)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$\tilde{g}^{(n)} \in \tilde{G}$



$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z$

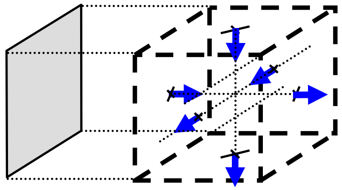
$S^{(r)}: \quad D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\begin{aligned}
\iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} &= \iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathrm{e}}_x\,\mathrm{d}y\mathrm{d}z \\
&= \iint_{S^{(r)}} D_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= \iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}}) E_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= E_x^{(n+M_x)}(t) \underbrace{\iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}y\mathrm{d}z}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right] \\
&= \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right] \\
&= D_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]
\end{aligned}$$

$$\begin{aligned}
&\iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S \\
&= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n+M_x)} + \varepsilon_{xx}^{(n+M_x+M_y)} + \varepsilon_{xx}^{(n+M_x+M_z)} + \varepsilon_{xx}^{(n+M_x+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}}\Delta y\Delta z \\
&= \tilde{\varepsilon}_{xx}^{(n+M_x)}\Delta y\Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$\tilde{g}^{(n)} \in \tilde{G}$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$

$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = -\underline{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z$

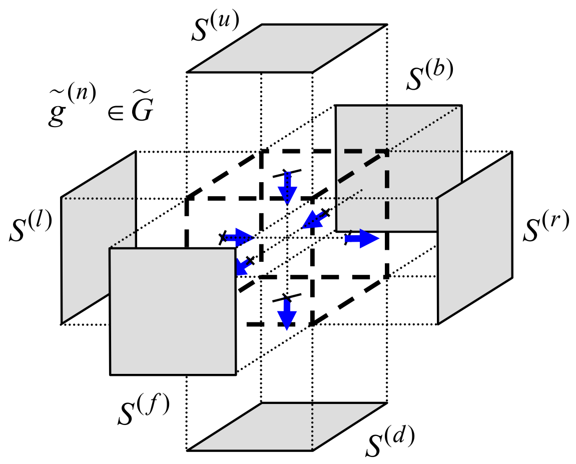
$S^{(l)}: \quad D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$

$$\begin{aligned}
\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} &= -\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathrm{e}}_x\,\mathrm{d}y\mathrm{d}z \\
&= -\iint_{S^{(l)}} D_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= -\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}}) E_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= -E_x^{(n)}(t) \underbrace{\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}y\mathrm{d}z}_{=\tilde{\varepsilon}_{xx}^{(n)}\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right] \\
&= -\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right] \\
&= -D_x^{(n)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]
\end{aligned}$$

$$\begin{aligned}
&\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S \\
&= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n)} + \varepsilon_{xx}^{(n+M_y)} + \varepsilon_{xx}^{(n+M_z)} + \varepsilon_{xx}^{(n+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n)}} \Delta y \Delta z \\
&= \tilde{\varepsilon}_{xx}^{(n)} \Delta y \Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$\tilde{g}^{(n)} \in \tilde{G}$



$$S^{(l)}: \quad D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

$$S^{(r)}: \quad D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$S^{(b)}: \quad D_y^{(n)} = \tilde{\varepsilon}_{yy}^{(n)} E_x^{(n)}$$

$$S^{(f)}: \quad D_y^{(n+M_y)} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}$$

$$S^{(u)}: \quad D_z^{(n)} = \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}$$

$$S^{(d)}: \quad D_z^{(n+M_z)} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$

$$\iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$
$$= D_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$

$$\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$
$$= -D_x^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$

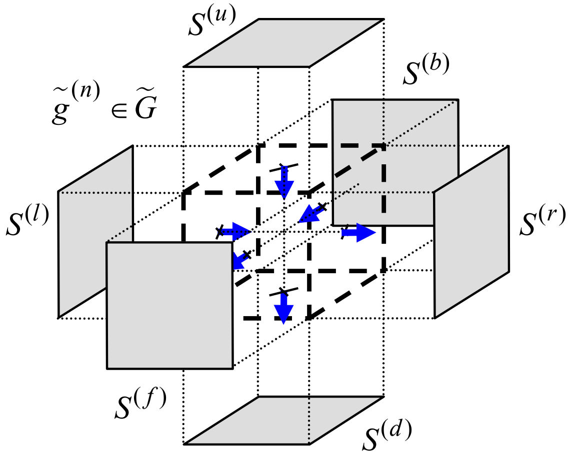
$$\iint_{S^{(f)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$
$$= D_y^{(n+M_y)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$

$$\iint_{S^{(b)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$
$$= -D_y^{(n)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$

$$\iint_{S^{(d)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$
$$= D_z^{(n+M_z)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$

$$\iint_{S^{(u)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$
$$= -D_z^{(n)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$= \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z - \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z$$
$$+\tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t)\Delta x\Delta z - \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z$$
$$+\tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t)\Delta x\Delta y - \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$
$$= D_x^{(n+M_x)}(t)\Delta y\Delta z - D_x^{(n)}(t)\Delta y\Delta z$$
$$+D_y^{(n+M_y)}(t)\Delta x\Delta z - D_y^{(n)}(t)\Delta x\Delta z$$
$$+D_z^{(n+M_z)}(t)\Delta x\Delta y - D_z^{(n)}(t)\Delta x\Delta y$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$= \left[\tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t) - \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\right]\Delta y\Delta z$$
$$+\left[\tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t) - \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\right]\Delta x\Delta z$$
$$+\left[\tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t) - \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\right]\Delta x\Delta y$$
$$= \left[D_x^{(n+M_x)}(t) - D_x^{(n)}(t)\right]\Delta y\Delta z$$
$$+\left[D_y^{(n+M_y)}(t) - D_y^{(n)}(t)\right]\Delta x\Delta z$$
$$+\left[D_z^{(n+M_z)}(t) - D_z^{(n)}(t)\right]\Delta x\Delta y$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$= \left(S_{M_x} - I\right)\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z$$
$$+\left(S_{M_y} - I\right)\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z$$
$$+\left(S_{M_z} - I\right)\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$
$$= \left(S_{M_x} - I\right) D_x^{(n)}(t)\Delta y\Delta z$$
$$+\left(S_{M_y} - I\right) D_y^{(n)}(t)\Delta x\Delta z$$
$$+\left(S_{M_z} - I\right) D_z^{(n)}(t)\Delta x\Delta y$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$

$$= \underbrace{\begin{bmatrix} S_{M_x}-I & S_{M_y}-I & S_{M_z}-I \end{bmatrix}}_{=\widetilde{[\mathrm{div}]}} \underbrace{\begin{bmatrix} \tilde{\varepsilon}_{zz}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)} \end{bmatrix}}_{=\widetilde{[\varepsilon]}^{(n)}} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta z \end{bmatrix}}_{=\widetilde{[S]}} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}}_{=\{E\}^{(n)}(t)}$$

$$= \widetilde{[\mathrm{div}]}\widetilde{[\varepsilon]}^{(n)}\widetilde{[S]}\{E\}^{(n)}(t)$$

$$= \left(S_{M_x}-I\right) D_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right) D_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right) D_z^{(n)}(t)\Delta x\Delta y$$

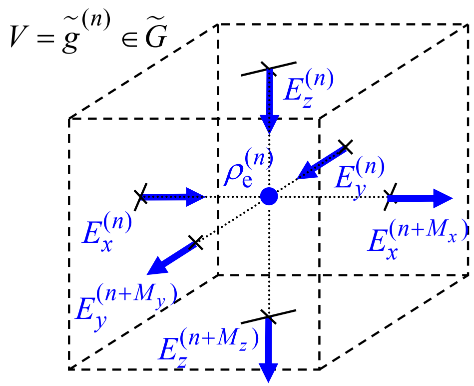
$$= \underbrace{\begin{bmatrix} S_{M_x}-I & S_{M_y}-I & S_{M_z}-I \end{bmatrix}}_{=\widetilde{[\mathrm{div}]}} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta z \end{bmatrix}}_{=\widetilde{[S]}} \underbrace{\begin{Bmatrix} D_x^{(n)}(t) \\ D_y^{(n)}(t) \\ D_z^{(n)}(t) \end{Bmatrix}}_{=\{D\}^{(n)}(t)}$$

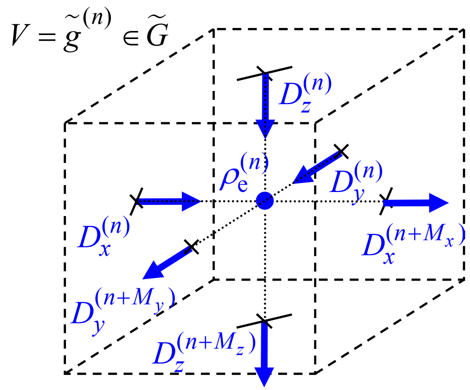
$$= \widetilde{[\mathrm{div}]}\widetilde{[S]}\{D\}^{(n)}(t)$$

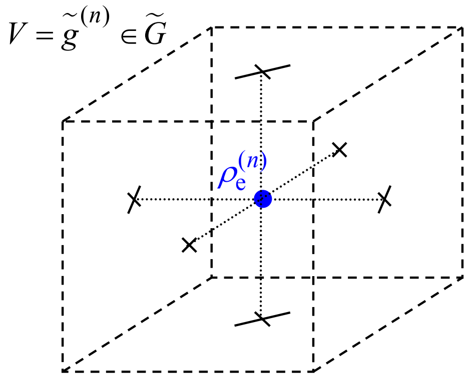
FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\cdot \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$







$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z + \bigcirc\left[(\Delta x)^3\,\Delta y\Delta z + \Delta x(\Delta y)^3\,\Delta z + \Delta x\Delta y(\Delta z)^3\right]$$

$$= Q_\mathrm{e}^{(n)}(t) + \bigcirc\left[(\Delta x)^3\,\Delta y\Delta z + \Delta x(\Delta y)^3\,\Delta z + \Delta x\Delta y(\Delta z)^3\right]$$

$$= \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z + \bigcirc\left[(\Delta x)^5\right] \quad \text{if} \quad \Delta x \approx \Delta y \approx \Delta z$$

$$= Q_\mathrm{e}^{(n)}(t) + \bigcirc\left[(\Delta x)^5\right] \quad \text{if} \quad \Delta x \approx \Delta y \approx \Delta z$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\oiint_{S=\partial V} \mathbf{D}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = Q_\mathrm{e}(t)$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = Q_\mathrm{e}(t)$$

Discrete grid equations in local matrix form /
Diskrete Gittergleichungen in lokaler Matrixform

$$[\widetilde{\mathrm{div}}][\widetilde{\varepsilon}]^{(n)}[\widetilde{S}]\{E\}^{(n)}(t) = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z = Q_\mathrm{e}^{(n)}(t)$$

$$[\widetilde{\mathrm{div}}][\widetilde{S}]\{D\}^{(n)}(t) = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z = Q_\mathrm{e}^{(n)}(t)$$

Discrete grid equations in global matrix form /
Diskrete Gittergleichungen in globaler Matrixform

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}(t) = \{\mathbf{Q}_\mathrm{e}\}(t)$$

$$[\widetilde{\mathbf{div}}][\widetilde{\mathbf{S}}]\{\mathbf{D}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}(t) = \{\mathbf{Q}_\mathrm{e}\}(t)$$

with / mit $\quad [\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x],[\mathbf{P}_y],[\mathbf{P}_z]\right]_{N\times 3N}$

Discrete Local and Global Gradient, Divergence, and Curl Operators / Diskrete lokale und globale Gradienten-, Divergenz- und Rotationsoperatoren

Discrete gradient operator / Diskreter Gradientenoperator

$$[\mathbf{grad}] = \begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}_{3N\times N}$$

$$[\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix}_{3N\times N}$$

Discrete curl operator / Diskreter Rotationsoperator

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^{\mathrm{T}} & -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} & [\mathbf{0}] & [\mathbf{P}_x]^{\mathrm{T}} \\ [\mathbf{P}_y]^{\mathrm{T}} & -[\mathbf{P}_x]^{\mathrm{T}} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

Discrete divergence operator / Diskreter Divergenzoperator

$$[\mathbf{div}] := \left[-[\mathbf{P}_x]^{\mathrm{T}}, -[\mathbf{P}_y]^{\mathrm{T}}, -[\mathbf{P}_z]^{\mathrm{T}} \right]_{N\times 3N}$$

$$[\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x], [\mathbf{P}_y], [\mathbf{P}_z] \right]_{N\times 3N}$$

$$\text{curl grad} = \nabla\times\nabla = \underline{\mathbf{0}}$$

$$\text{div curl} = \nabla\bullet\nabla = 0$$

$$-[\widetilde{\mathbf{div}}] = [\mathbf{grad}]^{\mathrm{T}}$$

$$[\widetilde{\mathbf{grad}}]^{\mathrm{T}} = [\mathbf{div}]$$

$$[\mathbf{curl}] = [\widetilde{\mathbf{curl}}]^{\mathrm{T}}$$

$$[\mathbf{curl}][\mathbf{grad}] = [\mathbf{0}]$$

$$[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{grad}}] = [\mathbf{0}]$$

$$[\mathbf{div}][\mathbf{curl}] = [\mathbf{0}]$$

$$[\widetilde{\mathbf{div}}][\widetilde{\mathbf{curl}}] = [\mathbf{0}]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form /
Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{E}\right\}(t)=\left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}(t)=\left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$$

$\left[\widetilde{\mathbf{div}}\right]$	$\in \mathbb{R}^{N\times 3N}$	Topological divergence operator in matrix form on the grid $\widetilde{G}$ / Topologischer Divergenzoperator in Matrixform auf dem Gitter $\widetilde{G}$
$\left[\widetilde{\boldsymbol{\varepsilon}}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of permittivities on the grid $\widetilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\widetilde{G}$
$\left[\widetilde{\mathbf{S}}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\widetilde{G}$
$\left\{\mathbf{E}\right\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$\left[\widetilde{\mathbf{V}}\right]$	$\in \mathbb{R}^{N\times N}$	Diagonal matrix of elementary volumes on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarvolumina auf dem Gitter $\widetilde{G}$
$\left\{\rho_{\mathrm{e}}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic electric charge density vector / Algebraischer elektrischer Ladungsdichtevektor
$\left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic electric charge vector / Algebraischer elektrischer Ladungsvektor

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Magnetic Gauss' grid equation – 4th Maxwell's grid equation in global matrix form /
Magnetische Gaußsche Gittergleichung – 4. Maxwellsche Gittergleichung in globaler Matrixform

$$\left[\mathbf{div}\right]\left[\mathbf{S}\right]\left\{\mathbf{B}\right\}(t)=\left[\mathbf{V}\right]\left\{\boldsymbol{\rho}_{\mathrm{m}}\right\}(t)=\left\{\mathbf{Q}_{\mathrm{m}}\right\}(t)$$

$\left[\mathbf{div}\right]$	$\in \mathbb{R}^{N\times 3N}$	Topological divergence operator in matrix form on the grid G / Topologischer Divergenzoperator in Matrixform auf dem Gitter G
$\left[\mathbf{S}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\left\{\mathbf{B}\right\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$\left[\mathbf{V}\right]$	$\in \mathbb{R}^{N\times N}$	Diagonal matrix of elementary volumes on the grid G / Diagonalmatrix der Elementarvolumina auf dem Gitter G
$\left\{\rho_{\mathrm{m}}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic magnetic charge density vector / Algebraischer magnetischer Ladungsdichtevektor
$\left\{\mathbf{Q}_{\mathrm{m}}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic magnetic charge vector / Algebraischer magnetischer Ladungsvektor

FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Governing Analytic Equations

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

FIT Grid Equations

Maxwell's grid equations /
Maxwellsche Gittergleichungen

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}(t) = \left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$$

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\left\{\boldsymbol{\rho}_{\mathrm{m}}\right\}(t) = \left\{\mathbf{Q}_{\mathrm{m}}\right\}(t)$$

FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

Dispersion relation for a plane wave propagation in 1-D und 3-D: /
Dispersionsrelation für die Ausbreitung einer ebenen Welle in 1D und 3D

$$\text{1D: } E_x(z,t) = E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)}$$

$$\text{3D: } \underline{\mathbf{E}}(\underline{\mathbf{R}},t) = E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})}$$

where k_z is the z component of the wave vector $\underline{\mathbf{k}}$ /
wobei k_z die z-Komponente des Wellenvektors $\underline{\mathbf{k}}$ ist

$$\text{1D: } \underline{\mathbf{k}} = k_z \underline{\mathbf{e}}_z$$

$$\text{3D: } \underline{\mathbf{k}} = k_x \underline{\mathbf{e}}_x + k_y \underline{\mathbf{e}}_y + k_z \underline{\mathbf{e}}_z$$

with magnitude / mit dem Betrag

$$\text{1D: } |\underline{\mathbf{k}}| = k = \sqrt{\underline{\mathbf{k}} \bullet \underline{\mathbf{k}}} = \sqrt{k_z^2} = |k_z|$$

$$\text{3D: } |\underline{\mathbf{k}}| = k = \sqrt{\underline{\mathbf{k}} \bullet \underline{\mathbf{k}}} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

Insert the plane wave ansatz into the scalar 1-D and vector 3-D wave equation /
Setze den ebenen Wellenansatz in die skalare 1D und vektorielle 3D Wellengleichung ein

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

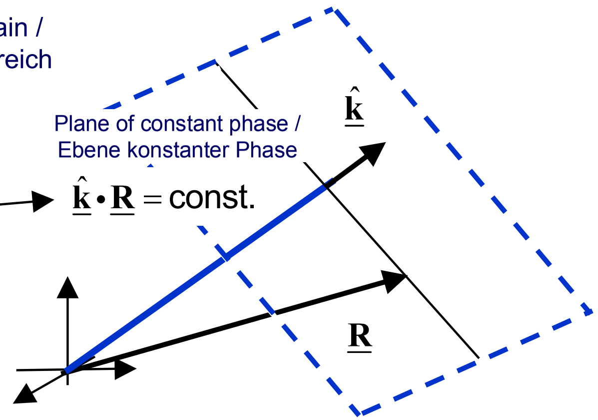
$$\Delta \underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}(\underline{\mathbf{R}},t) = 0$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

Complex Monofrequent (monochromatic) plane wave in the time domain /
Komplexe monofrequente (monochromatische) ebene Welle im Zeitbereich

$$E_x(\underline{\mathbf{R}},t) = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\left(\omega_0 t - k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}\right)}$$

$$= E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\omega_0 t}\; \mathrm{e}^{\mathrm{j}k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}}$$



Wave vector / Wellenvektor	$\underline{\mathbf{k}} = \mathrm{k}_x\underline{\mathbf{e}}_x + \mathrm{k}_z\underline{\mathbf{e}}_z + \mathrm{k}_z\underline{\mathbf{e}}_z = \mathrm{k}_z\underline{\mathbf{e}}_z$
Magnitude of the wave vector / Betrag des Wellenvektors	$\lvert\underline{\mathbf{k}}\rvert = \sqrt{\underline{\mathbf{k}}\bullet\underline{\mathbf{k}}} = \sqrt{\mathrm{k}_x^2+\mathrm{k}_y^2+\mathrm{k}_z^2} = \sqrt{\mathrm{k}_z^2} = \lvert k_z\rvert = k$
Wavenumber / Wellenzahl	$k = \dfrac{\omega_0}{c}$
Circular frequency / Kreisfrequenz	$\omega_0 = 2\pi f_0$
Propagation direction / Ausbreitungsrichtung	$\hat{\underline{\mathbf{k}}} = \dfrac{\underline{\mathbf{k}}}{\lvert\underline{\mathbf{k}}\rvert} = \dfrac{\mathrm{k}_z\underline{\mathbf{e}}_z}{\lvert\mathrm{k}_z\rvert} = \dfrac{\mathrm{sgn}(\mathrm{k}_z)\lvert\mathrm{k}_z\rvert\underline{\mathbf{e}}_z}{k} = \dfrac{\mathrm{sgn}(\mathrm{k}_z)k\underline{\mathbf{e}}_z}{k} = \mathrm{sgn}(\mathrm{k}_z)\underline{\mathbf{e}}_z$
Phase of the plane wave / Phase der ebenen Welle	$k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}} = k\,\mathrm{sgn}(\mathrm{k}_z)\underline{\mathbf{e}}_z\bullet(x\underline{\mathbf{e}}_x + y\underline{\mathbf{e}}_z + z\underline{\mathbf{e}}_z) = k\,\mathrm{sgn}(\mathrm{k}_z)z\underline{\mathbf{e}}_z\bullet\underline{\mathbf{e}}_z = k\,\mathrm{sgn}(\mathrm{k}_z)z$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

We compute / Wir berechnen

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

$$\frac{\partial^2}{\partial z^2} E_0(\omega_0,\underline{\hat{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_0(\omega_0,\underline{\hat{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} = 0$$

$$(\mathrm{j}k_z)^2 E_0(\omega_0,\underline{\hat{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} - \frac{1}{c_0^2}(\mathrm{j}\omega_0)^2 E_0(\omega_0,\underline{\hat{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} = 0$$

$$\underbrace{\left[-k_z^2 + \frac{\omega_0^2}{c_0^2}\right]}_{\text{Dispersion relation / Dispersionsrelation}} E_0(\omega_0,\underline{\hat{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} = 0$$

Dispersion relation / Dispersionsrelation

$$k_z^2 = k^2 = \frac{\omega_0^2}{c_0^2}$$

Dispersion relation of a monochromatic plane wave / Dispersionsrelation einer monochromatischen ebenen Welle

$$k = \frac{\omega_0}{c_0} \quad \rightarrow \quad \omega_0(k) = c_0 k \quad \rightarrow \quad \omega_0 \rightarrow \omega \quad \omega(k) = c_0 k$$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

We compute in the 1-D case / Wir berechnen im 1D-Fall

$$\Delta \underline{\mathbf{E}}(\underline{\mathbf{R}}, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}(\underline{\mathbf{R}}, t) = 0$$

$$\Delta E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} = 0$$

$$\mathrm{j}^2 (k_x^2 + k_y^2 + k_z^2) E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} - \frac{1}{c_0^2} (\mathrm{j}\omega_0)^2 E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} = 0$$

$$\underbrace{\left[-(k_x^2 + k_y^2 + k_z^2) + \frac{\omega_0^2}{c_0^2} \right]} E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} = 0$$

Dispersion relation /
Dispersionsrelation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \frac{\omega_0^2}{c_0^2}$$

Dispersion relation of a monochromatic plane wave /
Dispersionsrelation einer monochromatischen ebenen Welle

$$k = \frac{\omega_0}{c_0} \quad \rightarrow \quad \omega_0(k) = c_0 k \quad \rightarrow \quad \omega_0 \rightarrow \omega \quad \omega(k) = c_0 k$$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

Dispersion relation for a plane wave / Dispersionsrelation für eine ebene Welle

$$k = \frac{\omega_0}{c_0} \to k(\omega) = \frac{\omega}{c_0}$$

$$\omega(k) = k\, c_0$$

This means that the circular frequency is a function of k /
Dies bedeutet, dass die Kreisfrequenz eine Funktion von k ist

Dispersion relation / Dispersionsrelation

$$\omega(k)$$

We define now / Wir definieren nun:

- Phase velocity / Phasengeschwindigkeit
- Phase velocity vector / Phasengeschwindigkeitsvektor
- Group or energy velocity / Gruppen- oder Energiegeschwindigkeit
- Group or energy velocity vector / Gruppen- oder Energiegeschwindigkeitsvektor

Dispersion Relation and Phase, Group, and Energy Velocities / Dispersionsrelation und Phasen-, Gruppen- und Energiegeschwindigkeiten

Dispersion relation / Dispersionsrelation $\omega(k)$

- Phase velocity / Phasengeschwindigkeit

$$c_{\mathrm{ph}}(\omega,k)=\frac{\omega(k)}{k}$$

- Phase velocity vector / Phasengeschwindigkeitsvektor

$$\text{1D:}\quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k)=\frac{\omega(k)}{k}\operatorname{sgn}(k_z)$$

$$\text{3D:}\quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k)=\frac{\omega(k)}{k}\hat{\underline{\mathbf{k}}}$$

- Group or energy velocity / Gruppen- oder Energiegeschwindigkeit

$$\text{1D:}\quad c_{\mathrm{gr}}(\omega,k)=c_{\mathrm{E}}(\omega,k)=\frac{d}{dk}\omega(k)$$

$$\text{3D:}\quad c_{\mathrm{gr}}(\omega,\underline{\mathbf{k}})=c_{\mathrm{E}}(\omega,\underline{\mathbf{k}})=\left|\nabla_{\underline{\mathbf{k}}}\omega(k)\right|$$

- Group or energy velocity vector / Gruppen- oder Energiegeschwindigkeitsvektor

$$\text{1D:}\quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,k)=\underline{\mathbf{c}}_{\mathrm{E}}(\omega,k)=\frac{d}{dk}\omega(k)\hat{\underline{\mathbf{k}}}$$

$$\text{3D:}\quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,\underline{\mathbf{k}})=\underline{\mathbf{c}}_{\mathrm{E}}(\omega,\underline{\mathbf{k}})=\nabla_{\underline{\mathbf{k}}}\omega(k)$$

Gradient with regard to the wave vector $\underline{\mathbf{k}}$ /
Gradient bezüglich des Wellenvektors $\underline{\mathbf{k}}$

$$\nabla_{\underline{\mathbf{k}}}=\underline{\mathbf{e}}_x\frac{\partial}{\partial k_x}+\underline{\mathbf{e}}_y\frac{\partial}{\partial k_y}+\underline{\mathbf{e}}_z\frac{\partial}{\partial k_z}$$

Dispersion Relation and Phase, Group, and Energy Velocities for a Monochromatic Plane Wave / Dispersionsrelation und Phasen-, Gruppen- und Energiegeschwindigkeiten für eine monochromatische ebene Welle

Dispersion relation for a monochromatic plane wave / Dispersionsrelation für eine monochromatische ebene Welle

$$\omega(k) = kc_0$$

- Phase velocity / Phasengeschwindigkeit

$$c_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k} = \frac{kc_0}{k} = c_0$$

- Phase velocity vector / Phasengeschwindigkeitsvektor

$$\text{1D:}\quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}\operatorname{sgn}(k_z) = c_0 \operatorname{sgn}(k_z)$$

$$\text{3D:}\quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}\hat{\underline{\mathbf{k}}} = c_0\hat{\underline{\mathbf{k}}}$$

- Group or energy velocity / Gruppen- oder Energiegeschwindigkeit

$$\text{1D:}\quad c_{\mathrm{gr}}(\omega,k) = c_{\mathrm{E}}(\omega,k) = \frac{d}{dk}\omega(k) = \frac{d}{dk}kc_0 = c_0$$

$$\text{3D:}\quad c_{\mathrm{gr}}(\omega,\underline{\mathbf{k}}) = c_{\mathrm{E}}(\omega,\underline{\mathbf{k}}) = \left|\nabla_{\underline{\mathbf{k}}}\omega(k)\right| = \left|\nabla_{\underline{\mathbf{k}}}kc_0\right| = c_0$$

- Group or energy velocity vector / Gruppen- oder Energiegeschwindigkeitsvektor

$$\text{1D:}\quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,k) = \underline{\mathbf{c}}_{\mathrm{E}}(\omega,k) = \frac{d}{dk}\omega(k)\hat{\underline{\mathbf{k}}} = c_0\hat{\underline{\mathbf{k}}}$$

$$\text{3D:}\quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,\underline{\mathbf{k}}) = \underline{\mathbf{c}}_{\mathrm{E}}(\omega,\underline{\mathbf{k}}) = \nabla_{\underline{\mathbf{k}}}\omega(k) = c_0\hat{\underline{\mathbf{k}}}$$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

$$\nabla_{\underline{\mathbf{k}}} = \underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z}$$

$$\begin{aligned}
\nabla_{\underline{\mathbf{k}}} k c_0 &= \left(\underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z} \right) \sqrt{k_x^2 + k_y^2 + k_z^2}\, c_0 \\
&= \left(\underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z} \right) \sqrt{k_x^2 + k_y^2 + k_z^2}\, c_0 \\
&= c_0 \left(\underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z} \right) \sqrt{k_x^2 + k_y^2 + k_z^2} \\
&= c_0 \left[\frac{1}{2} \frac{2 k_x \underline{\mathbf{e}}_x}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{1}{2} \frac{2 k_y \underline{\mathbf{e}}_y}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{1}{2} \frac{2 k_z \underline{\mathbf{e}}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \right] \\
&= c_0 \left[\frac{k_x \underline{\mathbf{e}}_x}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{k_y \underline{\mathbf{e}}_y}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{k_z \underline{\mathbf{e}}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \right] \\
&= c_0 \frac{k_x \underline{\mathbf{e}}_x + k_y \underline{\mathbf{e}}_y + k_z \underline{\mathbf{e}}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \\
&= c_0 \hat{\underline{\mathbf{k}}}
\end{aligned}$$

Numerical Dispersion Relation / Numerische Dispersionsrelation

Numerical dispersion relation for the FD, FDTD, and FIT algorithms /
Numerische Dispersionsrelation für die FD-, FDTD- und FIT-Algorithmen

3-D cse / 3D-Fall

$$\frac{1}{(c_0\Delta t)^2}\sin^2\left(\frac{\omega_0\Delta t}{2}\right)=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)+\frac{1}{(\Delta z)^2}\sin^2\left(\frac{k_z\Delta z}{2}\right)$$

2-D cse / 2D-Fall

$$\frac{1}{(c_0\Delta t)^2}\sin^2\left(\frac{\omega_0\Delta t}{2}\right)=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta z}{2}\right)$$

1-D cse / 1D-Fall

$$\frac{1}{(c_0\Delta t)^2}\sin^2\left(\frac{\omega_0\Delta t}{2}\right)=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)$$

3-D cse / 3D-Fall

$$\omega(k_x,k_y,k_z,\Delta x,\Delta y,\Delta z,\Delta t)=\frac{2}{\Delta t}\arcsin\left\{c_0\Delta t\sqrt{\left[\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)+\frac{1}{(\Delta z)^2}\sin^2\left(\frac{k_z\Delta z}{2}\right)\right]}\right\}$$

2-D cse / 2D-Fall

$$\omega(k_x,k_y,\Delta x,\Delta y,\Delta t)=\frac{2}{\Delta t}\arcsin\left\{c_0\Delta t\sqrt{\left[\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)\right]}\right\}$$

1-D cse / 1D-Fall

$$\omega(k_x,\Delta x,\Delta t)=\frac{2}{\Delta t}\arcsin\left\{\frac{c_0\Delta t}{\Delta x}\sqrt{\sin^2\left(\frac{k_x\Delta x}{2}\right)}\right\}$$

Numerical Dispersion Relation / Numerische Dispersionsrelation

1-D case
1D-Fall

$$\omega\left(k_x, \Delta x, \Delta t\right) = \frac{2}{\Delta t} \arcsin\left\{ \frac{c_0 \Delta t}{\Delta x} \sin\left(\frac{|k_x| \Delta x}{2} \right) \right\}$$

$$\frac{c_0 \Delta t}{\Delta x} = \widehat{\Delta t}$$

$$|k_x| = k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{|k_x|}$$

$$G = \frac{\lambda}{\Delta x} = \frac{2\pi}{|k_x| \Delta x}$$

1-D case
1D-Fall

$$\omega\left(k_x, \Delta x, \Delta t\right) = \frac{2}{\Delta t} \arcsin\left\{ \underbrace{\frac{c_0 \Delta t}{\Delta x}}_{=\widehat{\Delta t}} \sin\left(\underbrace{\frac{|k_x| \Delta x}{2}}_{=\frac{\pi}{G}} \right) \right\}$$

$$= \frac{2}{\Delta t} \arcsin\left\{ \widehat{\Delta t} \sin\left(\frac{\pi}{G} \right) \right\}$$

1-D Numerical Dispersion Relation / 1D Numerische Dispersionsrelation

Numerical dispersion relation / Numerische Dispersionsrelation

$$\omega(k_x, \Delta x, \Delta t) = \frac{2}{\Delta t} \arcsin\left\{ \frac{c_0 \Delta t}{\Delta x} \sin\left(\frac{|k_x| \Delta x}{2} \right) \right\}$$

$$\omega(k_x, \Delta x, \Delta t) = \frac{2}{\Delta t} \arcsin\left\{ \widehat{\Delta t} \sin\left(\frac{\pi}{G} \right) \right\}$$

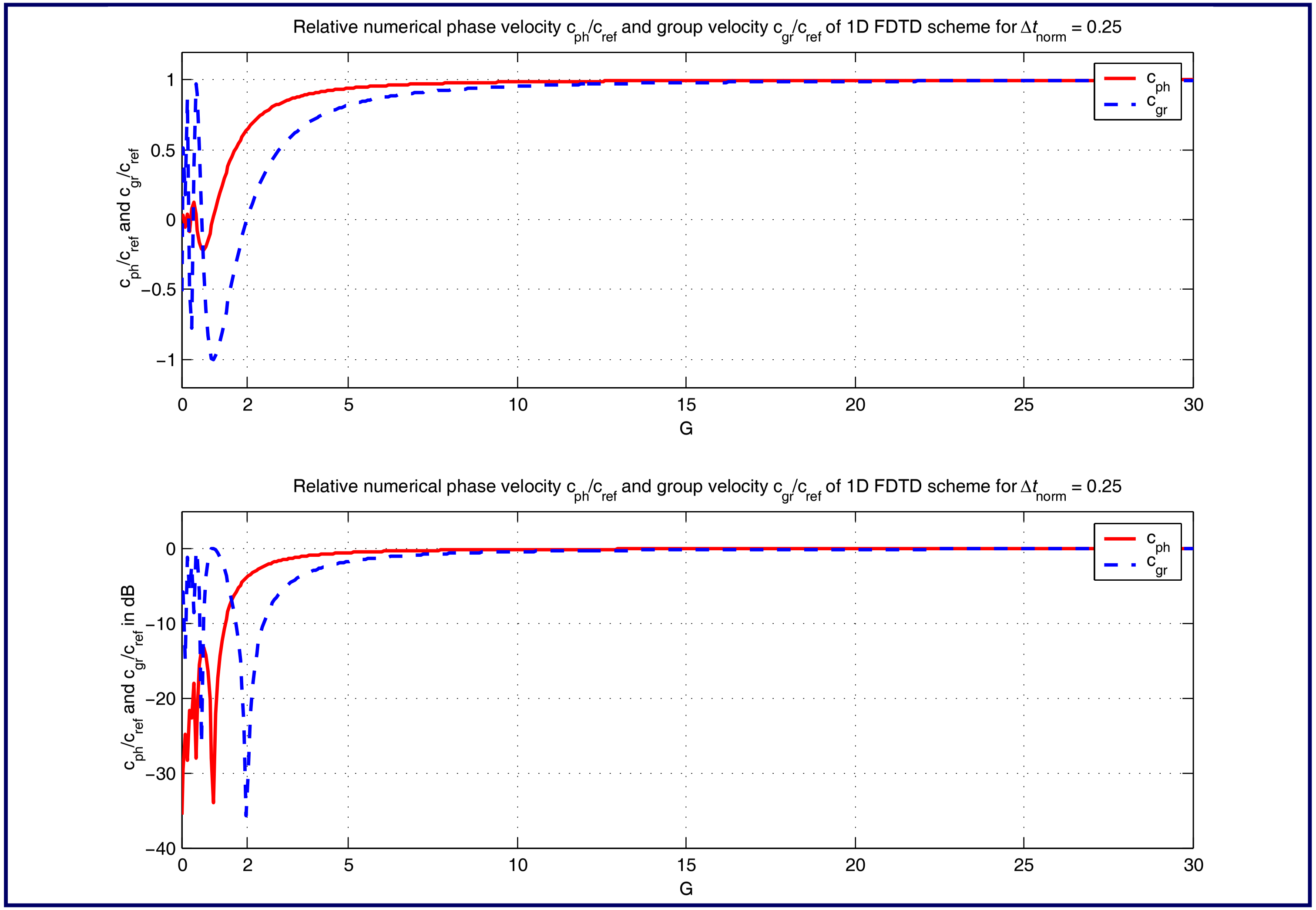
Numerical phase velocity / Numerische Phasengeschwindigkeit

$$\frac{c_{\mathrm{ph}}\left(G, \widehat{\Delta t}\right)}{c_0} = \frac{G}{\pi \widehat{\Delta t}} \arcsin\left\{ \widehat{\Delta t} \sin\left(\frac{\pi}{G} \right) \right\}$$

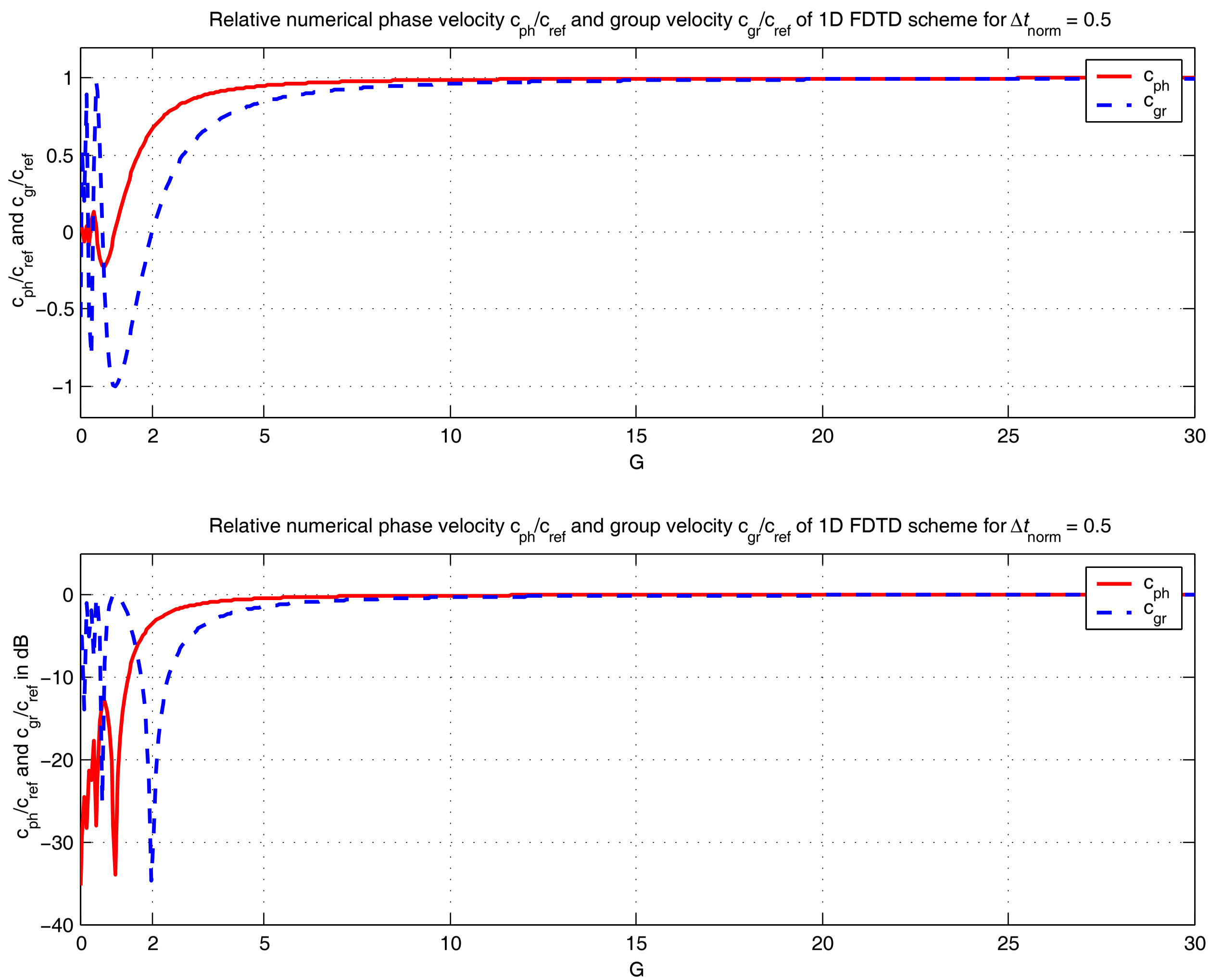
Numerical group velocity / Numerische Gruppengeschwindigkeit

$$\frac{c_{\mathrm{gr}}\left(G, \widehat{\Delta t}\right)}{c_0} = \frac{\cos\left(\frac{\pi}{G} \right)}{\sqrt{1 - \widehat{\Delta t}^2 \sin^2\left(\frac{\pi}{G} \right)}}$$

1-D Numerical Dispersion Relation / 1D Numerische Dispersionsrelation



1-D Numerical Dispersion Relation / 1D Numerische Dispersionsrelation



1-D Numerical Dispersion Relation – Magic Time Step / 1D Numerische Dispersionsrelation – Magische Zeitschrittweite

$$\widehat{\Delta t} = 1$$

Numerical phase velocity / Numerische Phasengeschwindigkeit

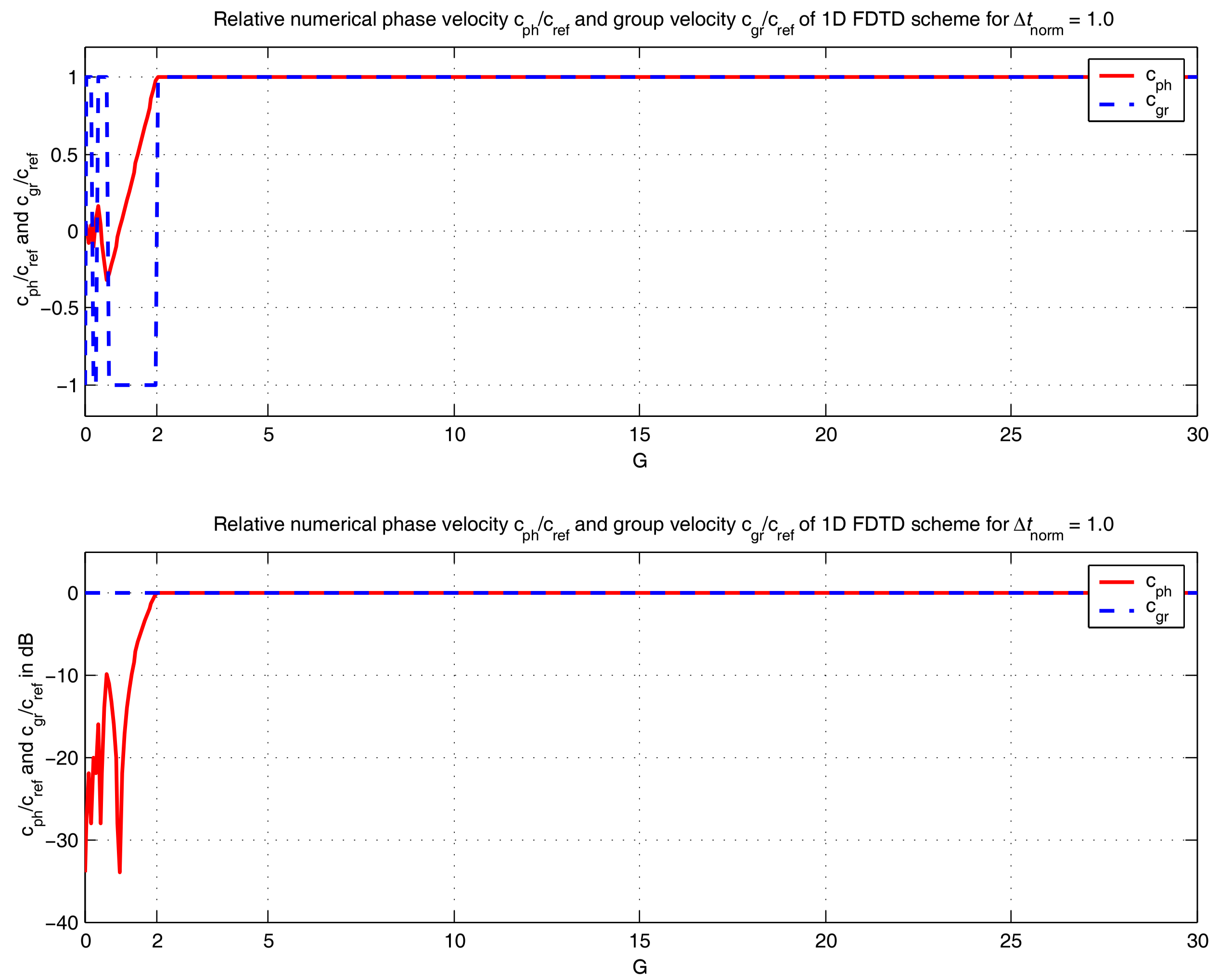
$$\frac{c_{\mathrm{ph}}\left(G, \widehat{\Delta t}\right)}{c_0} = \frac{G}{\pi \widehat{\Delta t}} \arcsin\left\{\widehat{\Delta t} \sin\left(\frac{\pi}{G}\right)\right\} = \frac{G}{\pi} \arcsin\left\{\sin\left(\frac{\pi}{G}\right)\right\} = \frac{G}{\pi}\frac{\pi}{G} = 1$$

Numerical group velocity / Numerische Gruppengeschwindigkeit

$$\frac{c_{\mathrm{gr}}\left(G, \widehat{\Delta t}\right)}{c_0} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{1 - \widehat{\Delta t}^2 \sin^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{1 - \sin^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{\cos^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{\cos^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\cos\left(\frac{\pi}{G}\right)} = 1$$

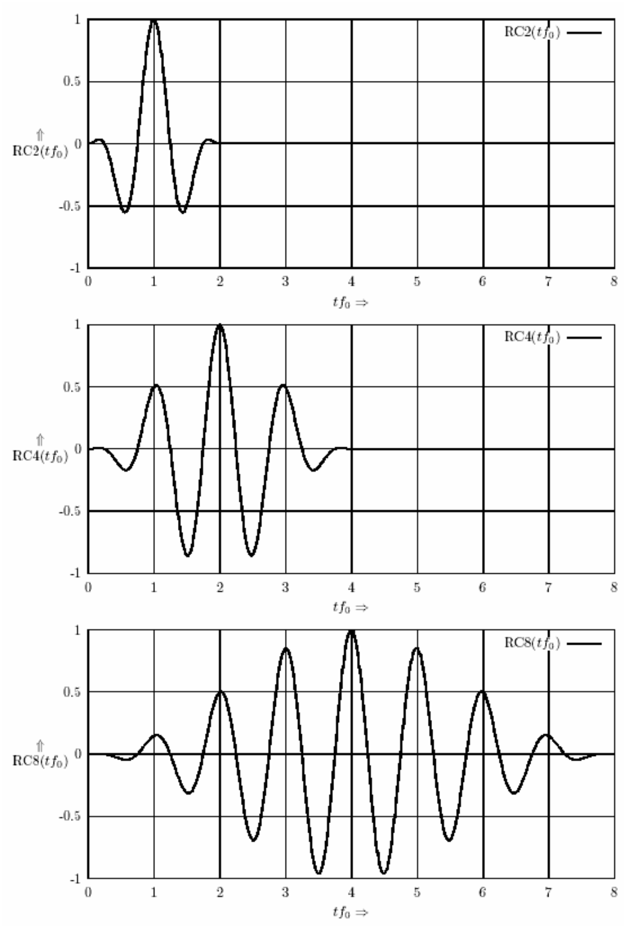
⇨ No numerical dispersion / Keine numerische Dispersion !

1-D Numerical Dispersion Relation – Magic Time Step / 1D Numerische Dispersionsrelation – Magische Zeitschrittweite

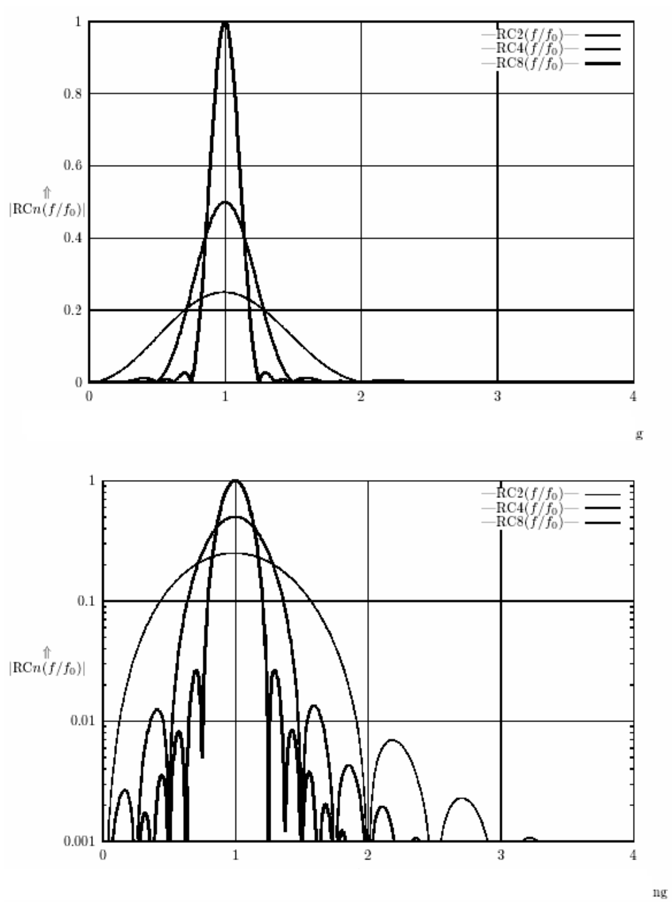


Numerical Dispersion – Test Function / Numerische Dispersion – Testfunktion

Time History of the RC2, RC4, and RC8 Excitation Function / Zeitfunktion des RC2-, RC4- und RC8-Anregungsimpulses



Time Spectrum of the RC2, RC4, and RC8 Excitation Function / Zeitspektrum des RC2-, RC4- und RC8-Anregungsimpulses



Numerical Dispersion of an RC2 Pulse / Numerische Dispersion eines RC2-Impulses

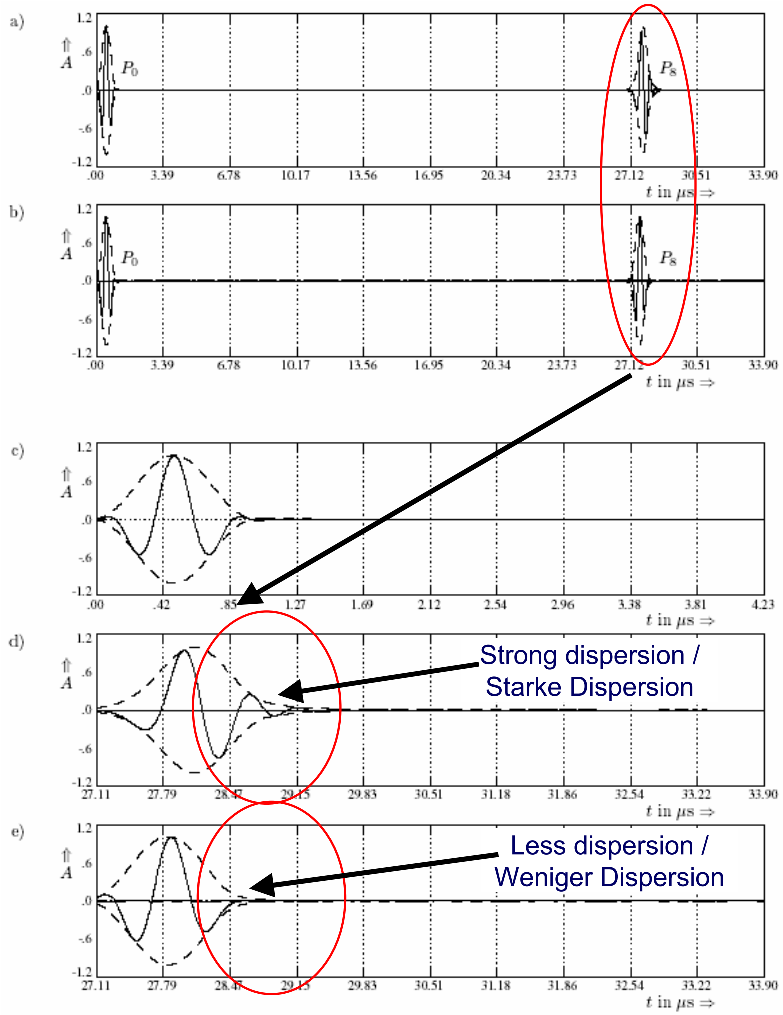
Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordung in der Zeit (4S2T)

Original RC2 Pulse / Originaler RC2-Impuls

Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordnung in der Zeit (4S2T)



Numerical Dispersion of an RC4 Pulse / Numerische Dispersion eines RC4-Impulses

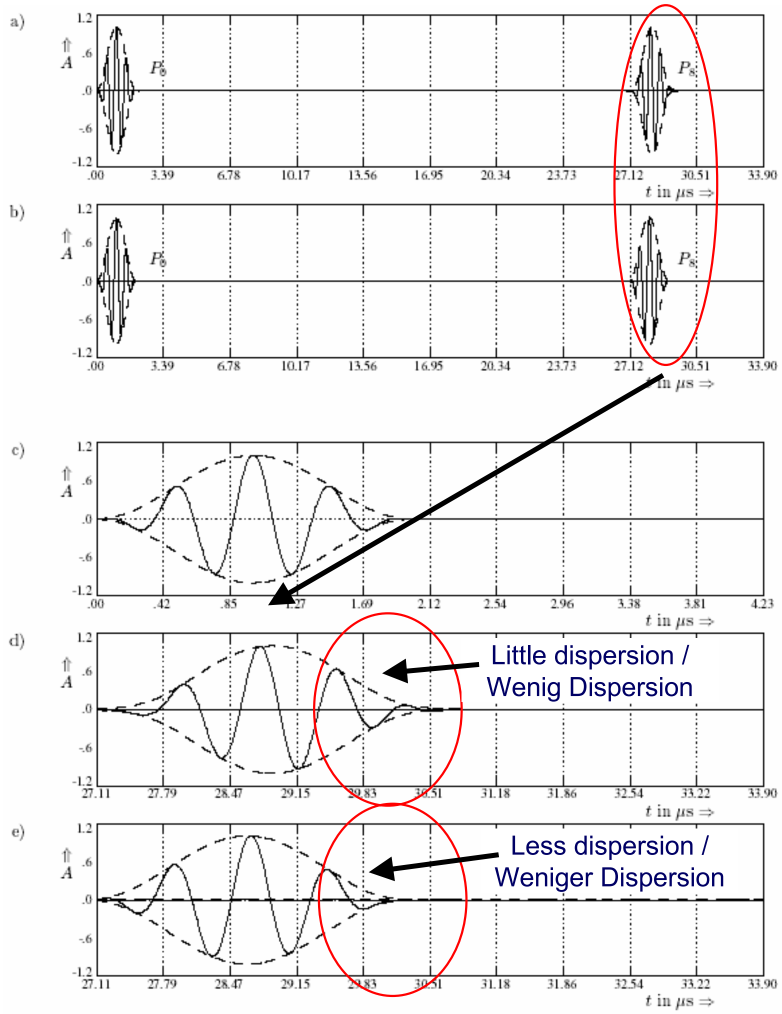
Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordung in der Zeit (4S2T)

Original RC4 Pulse / Originaler RC4-Impuls

Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordnung in der Zeit (4S2T)



Numerical Dispersion of an RC8 Pulse / Numerische Dispersion eines RC8-Impulses

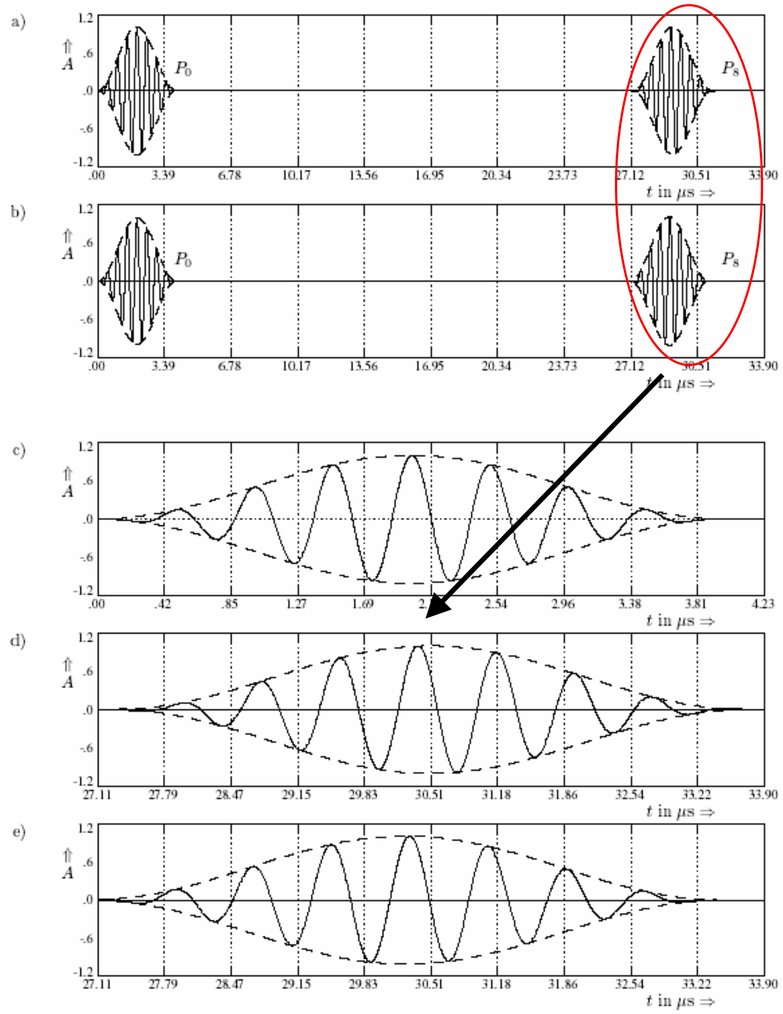
Second Order Scheme in Space and Time (2S2T) /
Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordung in der Zeit (4S2T)

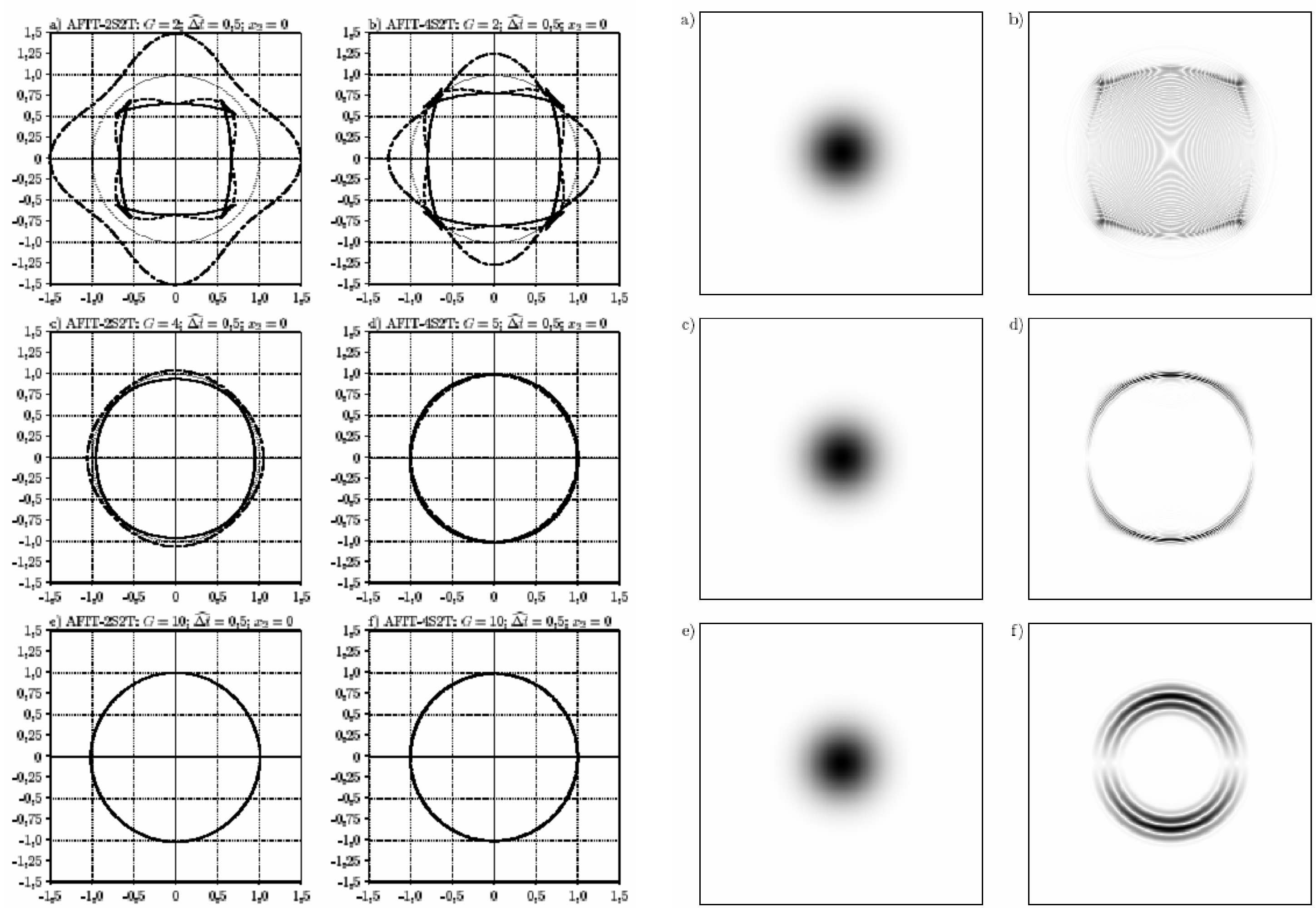
Original RC8 Pulse /
Originaler RC8-Impuls

Second Order Scheme in Space and Time (2S2T) /
Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordnung in der Zeit (4S2T)



2-D Numerical Dispersion Relation – Numerical Anisotropy / 2D Numerische Dispersionsrelation – Numerische Anisotropie



FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Derivation of the Numerical Dispersion Relation for the 1-D FD Scheme of 2nd Order / Ableitung der numerischen Dispersionsrelation für das 1D-FD-Schema 2ter Ordnung

Stability by the *von Neumann*'s method (Fourier series method):

Insert a complex monofrequent (monochromatic) plane wave into the discrete FD equations and analyze the spectral radius of the amplification matrix, where the spectral radius must be smaller equal one.

Stabilität durch die *von Neumann*sche Methode (Fourier-Reihen-Methode):

Setze eine komplex monofrequente (monochromatische) ebene Welle in die diskreten FD-Gleichungen ein und analysiere den spektralen Radius der Verstärkungsmatrix, wobei der spektrale Radius kleinergleich Eins sein muss.

Complex monofrequent (monochromatic) plane wave / Komplex monofrequente (monochromatische) ebene Welle

$$E_x(\underline{\mathbf{R}},t) = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\left(\omega_0 t - k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}\right)} = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 t}\;\mathrm{e}^{\mathrm{j}k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}}$$

$$\{\mathbf{W}\}^{(n_t+1)} = [\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\{\mathbf{W}\}^{(n_t)}$$

$[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$: Amplification matrix / Verstärkungsmatrix

Spectral radius / Spektraler Radius $\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) \le 1$ of the matrix / der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

where / wobei $\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) = \max\limits_{n=1,\ldots,N}\left|\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)\right|$

$\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)$: nth eigenvalue of the matrix / n-ter Eigenwert der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Consistency / Konsistenz

Consistency

Consistency means that the discrete equations result in the analytical equations by calculating the limit $\{\Delta z, \Delta t\} \to 0$.

We can prove the consistency of the 1-D FD scheme using the above numerical dispersion relation. We show, that the grid dispersion relation reaches in the limit $\{\Delta z, \Delta t\} \to 0$ the analytical dispersion relation for a plane wave as a solution of the homogeneous Helmholtz equation.

Konsistenz

Konsistenz bedeutet, dass die diskreten Gleichungen bei dem Grenzübergang $\{\Delta z, \Delta t\} \to 0$ in die analytischen Gleichungen übergehen.

Wir können die Konsistenz des 1D-FD-Schemas anhand der numerischen Dispersionsrelation überprüfen. Wir zeigen, dass die numerische Dispersionsrelation bei dem Grenzübergang $\{\Delta z, \Delta t\} \to 0$ in die analytische Dispersionsrelation einer ebene Welle als Lösung der homogenen Helmholtz-Gleichung übergeht.

Consistency / Konsistenz

$$\lim_{\{\Delta z, \Delta t\} \to 0} \left[\frac{1}{(c_0 \Delta t)^2} \sin^2\left(\frac{\omega_0 \Delta t}{2}\right) = \frac{1}{(\Delta z)^2} \sin^2\left(\frac{\mathrm{k}_z \Delta z}{2}\right) \right]$$

$$\lim_{\{\Delta z, \Delta t\} \to 0} \left[\frac{1}{c_0 \Delta t} \sin\left(\frac{\omega_0 \Delta t}{2}\right) = \frac{1}{\Delta z} \sin\left(\frac{|\mathrm{k}_z| \Delta z}{2}\right) \right]$$

$$\lim_{\Delta t \to 0} \sin\left(\frac{\omega_0 \Delta t}{2}\right) = \frac{\omega_0 \Delta t}{2}$$

$$\lim_{\Delta z \to 0} \sin\left(\frac{|\mathrm{k}_z| \Delta z}{2}\right) = \frac{|\mathrm{k}_z| \Delta z}{2}$$

$$\frac{1}{c_0 \Delta t} \frac{\omega_0 \Delta t}{2} = \frac{1}{\Delta z} \frac{|\mathrm{k}_z| \Delta z}{2}$$

$$\underbrace{\left(\frac{\omega_0}{c_0}\right)}_{= k_0} = |\mathrm{k}_z|$$

$$k_0 = |\mathrm{k}_z|$$

FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung**

  $\Delta z = ?$
  $\Delta t = ?$

- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung**

  $\Delta t = f(\Delta z)$

- **Convergence / Konvergenz**

Convergence / Konvergenz

Convergence

The importance of the concept of consistency and stability is seen in the Lax-Richtmyer equivalence theorem, which is the fundamental theorem in the theory of finite difference schemes for initial value problems.

Lax-Richtmyer Equivalence Theorem

A consistent finite difference scheme for a partial differential equations of which the initial value problem is well-posed is convergent if and only if it is stable.

Konvergenz

Die Wichtigkeit des Konzeptes der Konsistenz und Stabilität kann man an dem Lax-Richtmyer-Äquivalenztheorem sehen, welches ein fundamentales Theorem in der Theorie der Finite Differenzen zur Lösung eines Anfangswertproblems darstellt.

Lax-Richtmyer Äquivalenztheorem

Ein konsistentes Finite Differenzen Schema für eine partielle Differentialgleichung eines gut gestellten Anfangswertproblems ist konvergent, wenn und nur wenn es stabil ist.

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form /
Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}(t) = \{\mathbf{Q}_\mathrm{e}\}(t)$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \equiv 0$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\} = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\} = \{\mathbf{Q}_\mathrm{e}\}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned} \nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \rho_\mathrm{e}(\underline{\mathbf{R}}) \\ &= \nabla \bullet \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \\ &= \nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} \\ &= -\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} \end{aligned}$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} = -\rho_\mathrm{e}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta}\Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_e(\underline{\mathbf{R}})$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \rho_e(\underline{\mathbf{R}}) \\ &= \nabla \bullet \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \\ &= \nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \\ &= -\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dS}} &= \iiint_V \rho_e(\underline{\mathbf{R}})\, \mathrm{d}V \\ &= \oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dS}} \\ &= \oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}}\nabla \bullet \\ &= -\oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}}\nabla \bullet\end{aligned}$$

$$-\oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}})\, \mathrm{d}V$$

Inhomogeneous, anisotropic case / Inhomogener anisotroper Fall

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} = -\rho_e(\underline{\mathbf{R}})$$

Homogeneous, isotropic case / Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta} \Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Differential form / Differentialform

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_e(\underline{\mathbf{R}})$$

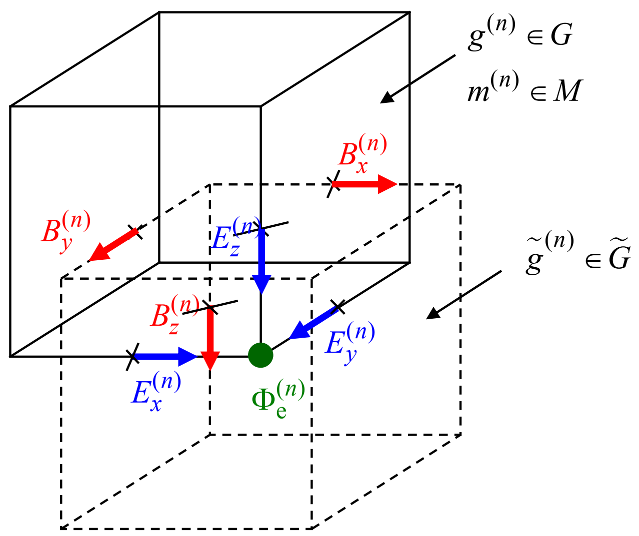
Integral form / Integralform

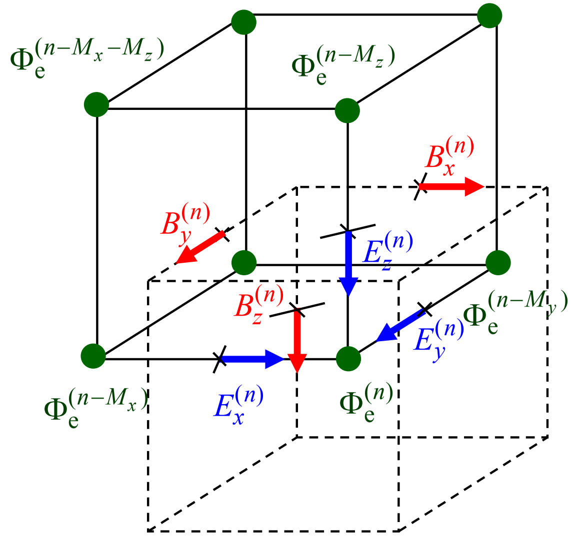
$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}} = -\int_C \nabla\Phi_e(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}}$$
$$= -\left[\Phi_e(\underline{\mathbf{R}}_2) - \Phi_e(\underline{\mathbf{R}}_1)\right]$$

FIT grid equation / FIT-Gittergleichung

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\{\Phi_e\}^{(n)}$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_e\}$$





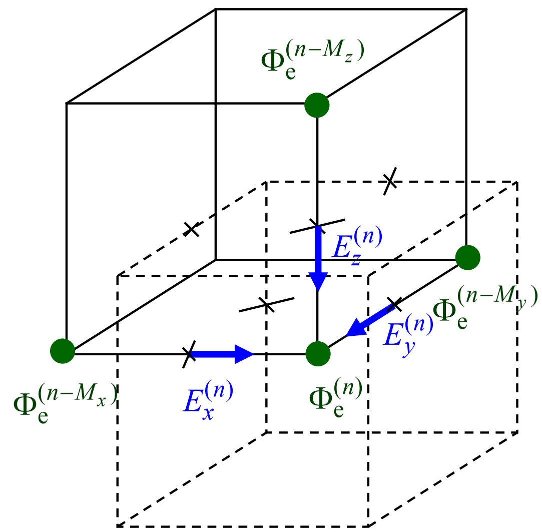
FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Integral form / Integralform

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = -\int_C \left[\nabla \Phi_e(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dR}} = -\left[\Phi_e(\underline{\mathbf{R}}_2) - \Phi_e(\underline{\mathbf{R}}_1)\right]$$

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z) \bullet \underline{\mathbf{dR}} = \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z) \bullet \underline{\mathbf{e}}_x \, \mathrm{d}x = \int_{x=x_0}^{x_0+\Delta x} E_x(x,y,z) \, \mathrm{d}x = E_x^{(n)} \int_{x=x_0}^{x_0+\Delta x} \mathrm{d}x = E_x^{(n)} \Delta x$$

$$\int_C \left[\nabla \Phi_e(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dR}} = \int_{x=x_0}^{x_0+\Delta x} \left[\nabla \Phi_e(x,y,z)\right] \bullet \underline{\mathbf{dR}} = \int_{x=x_0}^{x_0+\Delta x} \left[\nabla \Phi_e(x,y,z)\right] \bullet \underline{\mathbf{e}}_x \, \mathrm{d}x = \int_{x=x_0}^{x_0+\Delta x} \frac{\partial}{\partial x} \Phi_e(x,y,z) \, \mathrm{d}x = \Phi_e(x_0,y,z) - \Phi_e(x_0+\Delta x,y,z) = \Phi_e^{(n-M_x)} - \Phi_e^{(n)}$$



$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = -\int_C \left[\nabla \Phi_e(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dR}} = -\left[\Phi_e(\underline{\mathbf{R}}_2) - \Phi_e(\underline{\mathbf{R}}_1)\right]$$

$$E_x^{(n)} \Delta x = -\Phi_e^{(n)} - \Phi_e^{(n-M_x)} = -(I - S_{-M_x})\, \Phi_e^{(n)}$$

$$E_y^{(n)} \Delta y = -\Phi_e^{(n)} - \Phi_e^{(n-M_y)} = -(I - S_{-M_y})\, \Phi_e^{(n)}$$

$$E_z^{(n)} \Delta z = -\Phi_e^{(n)} - \Phi_e^{(n-M_z)} = -(I - S_{-M_z})\, \Phi_e^{(n)}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \mathbf{d}\underline{\mathbf{R}} = -\int_C \left[\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right] \cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= -\left[\Phi_{\mathrm{e}}(\underline{\mathbf{R}}_2) - \Phi_{\mathrm{e}}(\underline{\mathbf{R}}_1)\right]$$

$$\underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}} = -\underbrace{\begin{bmatrix} (I - S_{-M_x}) \\ (I - S_{-M_y}) \\ (I - S_{-M_z}) \end{bmatrix}}_{=-[\mathrm{grad}]} \Phi_{\mathrm{e}}^{(n)}$$

$$E_x^{(n)} \Delta x = -\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_x)}$$
$$= -(I - S_{-M_x})\, \Phi_{\mathrm{e}}^{(n)}$$

$$[R]\{E\}^{(n)} = -[\mathrm{grad}]\Phi_{\mathrm{e}}^{(n)}$$

$$E_y^{(n)} \Delta y = -\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_y)}$$
$$= -(I - S_{-M_y})\, \Phi_{\mathrm{e}}^{(n)}$$

$$E_z^{(n)} \Delta z = -\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_z)}$$
$$= -(I - S_{-M_z})\, \Phi_{\mathrm{e}}^{(n)}$$

$$[R] = \begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix} \rightarrow [R]^{-1} = \begin{bmatrix} \frac{1}{\Delta x} & & \\ & \frac{1}{\Delta y} & \\ & & \frac{1}{\Delta z} \end{bmatrix}$$

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\}$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\} = -[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\}$$

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} = -\rho_\mathrm{e}(\underline{\mathbf{R}})$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\widetilde{\boldsymbol{\rho}}_\mathrm{e}\}$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta}\Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Downarrow$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

with / mit

$$\Delta\Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

$$[\mathbf{A}] = [\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]$$

$$\{\mathbf{x}\} = \{\Phi_\mathrm{e}\}$$

$$\{\mathbf{b}\} = -[\widetilde{\mathbf{V}}]\{\widetilde{\boldsymbol{\rho}}_\mathrm{e}\}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$\widetilde{[\mathbf{div}]}\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -\widetilde{[\mathbf{V}]}\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$\{\Phi_\mathrm{e}\} = \begin{Bmatrix} \Phi_\mathrm{e}^{(1)}(t) \\ \Phi_\mathrm{e}^{(2)}(t) \\ \vdots \\ \Phi_\mathrm{e}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

$$\widetilde{[\mathbf{S}]} = \begin{bmatrix} [\mathrm{diag}\{\Delta y \Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x \Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x \Delta y\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{R}] = \begin{bmatrix} [\mathrm{diag}\{\Delta x\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta y\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta z\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{R}]^{-1} = \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\varepsilon}\right]\left[\widetilde{\mathbf{S}}\right]\left[\mathbf{R}\right]^{-1}\left[\mathbf{grad}\right]\left\{\Phi_{\mathrm{e}}\right\} = -\left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}$$

$$\left[\widetilde{\mathbf{S}}\right]\left[\mathbf{R}\right]^{-1} = \begin{bmatrix} \left[\mathrm{diag}\{\Delta y \Delta z\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\{\Delta x \Delta z\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\{\Delta x \Delta y\}\right]_{N\times N} \end{bmatrix} \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{\Delta y \Delta z}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{\Delta x \Delta z}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{\Delta x \Delta y}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)D_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)D_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)D_z^{(n)}(t)\Delta x\Delta y$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}E_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}E_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}E_z^{(n)}(t)\Delta x\Delta y$$

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$E_x^{(n)} = -\frac{1}{\Delta x}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)}\right) = -\frac{1}{\Delta x}(I - S_{-M_x})\,\Phi_\mathrm{e}^{(n)}$$

$$E_y^{(n)} = -\frac{1}{\Delta y}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)}\right) = -\frac{1}{\Delta y}(I - S_{-M_y})\,\Phi_\mathrm{e}^{(n)}$$

$$E_z^{(n)} = -\frac{1}{\Delta z}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)}\right) = -\frac{1}{\Delta z}(I - S_{-M_z})\,\Phi_\mathrm{e}^{(n)}$$

$$\begin{aligned}
\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} &= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}E_x^{(n)}\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}E_y^{(n)}\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}E_z^{(n)}\Delta x\Delta y \\
&= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I - S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\right]\Delta y\Delta z \\
&\quad + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I - S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta z \\
&\quad + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I - S_{-M_z})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta y
\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\begin{aligned}\oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = &\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\right]\Delta y\Delta z\\ &+\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta z\\ &+\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta y\end{aligned}$$

$$\begin{aligned}&\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\right]\Delta y\Delta z\\ &+\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta z\\ &+\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta y = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z\end{aligned}$$

$$\begin{aligned}&-\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\\ &-\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\\ &-\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)} = \rho_\mathrm{e}^{(n)}\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\left(\tilde{\varepsilon}_{xx}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}\right)$$

$$=\frac{1}{(\Delta x)^2}\left[S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}-\underbrace{S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\underbrace{S_{M_x}S_{-M_x}}_{=I}\Phi_{\mathrm{e}}^{(n)}}-\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}\right]$$

$$=\frac{1}{(\Delta x)^2}\left[\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n+M_x)}\underbrace{-\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}}_{=\left[\tilde{\varepsilon}_{xx}^{(n)}+\tilde{\varepsilon}_{xx}^{(n+M_x)}\right]\Phi_{\mathrm{e}}^{(n)}}+\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n-M_x)}\right]$$

$$=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[\underbrace{(I+S_{M_x})\tilde{\varepsilon}_{xx}^{(n)}}_{=2A_{M_x}}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta y)^2}\left\{\tilde{\varepsilon}_{yy}^{(n+M_y)}\Phi_{\mathrm{e}}^{(n+M_y)}-\left[2S_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{yy}^{(n)}\Phi_{\mathrm{e}}^{(n-M_y)}\right\}$$

$$=\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}$$

$$\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta z)^2}\left\{\tilde{\varepsilon}_{zz}^{(n+M_z)}\Phi_{\mathrm{e}}^{(n+M_z)}-\left[(I-S_{M_z})\tilde{\varepsilon}_{zz}^{(n)}\right]\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{zz}^{(n)}\Phi_{\mathrm{e}}^{(n-M_z)}\right\}$$

$$=\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n)}}S_{-M_z}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n)}}S_{-M_y}+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n)}}S_{-M_x}-\underbrace{\left\{\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}+\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}+\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}\right\}}_{=\alpha^{(n)}}I+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n+M_x)}}S_{M_x}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n+M_y)}}S_{M_y}+\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n+M_z)}}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\alpha_{zz}^{(n)}S_{-M_z}+\alpha_{yy}^{(n)}S_{-M_y}+\alpha_{xx}^{(n)}S_{-M_x}-\alpha^{(n)}I+\alpha_{xx}^{(n+M_x)}S_{M_x}+\alpha_{yy}^{(n+M_y)}S_{M_y}+\alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \oint\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \oint\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}}$$

$$= -\left\{\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)} + \frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)} + \frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\right\}\Phi_\mathrm{e}^{(n)}$$

$$= -\left\{\alpha_{zz}^{(n)}S_{-M_z} + \alpha_{yy}^{(n)}S_{-M_y} + \alpha_{xx}^{(n)}S_{-M_x} - \alpha^{(n)}I + \alpha_{xx}^{(n+M_x)}S_{M_x} + \alpha_{yy}^{(n+M_y)}S_{M_y} + \alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_\mathrm{e}^{(n)}$$

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$-\left(\alpha_{zz}^{(n)}S_{-M_z} + \alpha_{yy}^{(n)}S_{-M_y} + \alpha_{xx}^{(n)}S_{-M_x} - \alpha^{(n)}I + \alpha_{xx}^{(n+M_x)}S_{M_x} + \alpha_{yy}^{(n+M_y)}S_{M_y} + \alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\alpha_{zz}^{(n)}S_{-M_z} - \alpha_{yy}^{(n)}S_{-M_y} - \alpha_{xx}^{(n)}S_{-M_x} + \alpha^{(n)}I - \alpha_{xx}^{(n+M_x)}S_{M_x} - \alpha_{yy}^{(n+M_y)}S_{M_y} - \alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\underbrace{\frac{\alpha_{zz}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n)}}S_{-M_z} - \underbrace{\frac{\alpha_{yy}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n)}}S_{-M_y} - \underbrace{\frac{\alpha_{xx}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n)}}S_{-M_x} + I - \underbrace{\frac{\alpha_{xx}^{(n+M_x)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n+M_x)}}S_{M_x} - \underbrace{\frac{\alpha_{yy}^{(n+M_y)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n+M_y)}}S_{M_y} - \underbrace{\frac{\alpha_{zz}^{(n+M_z)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n+M_z)}}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \frac{\rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

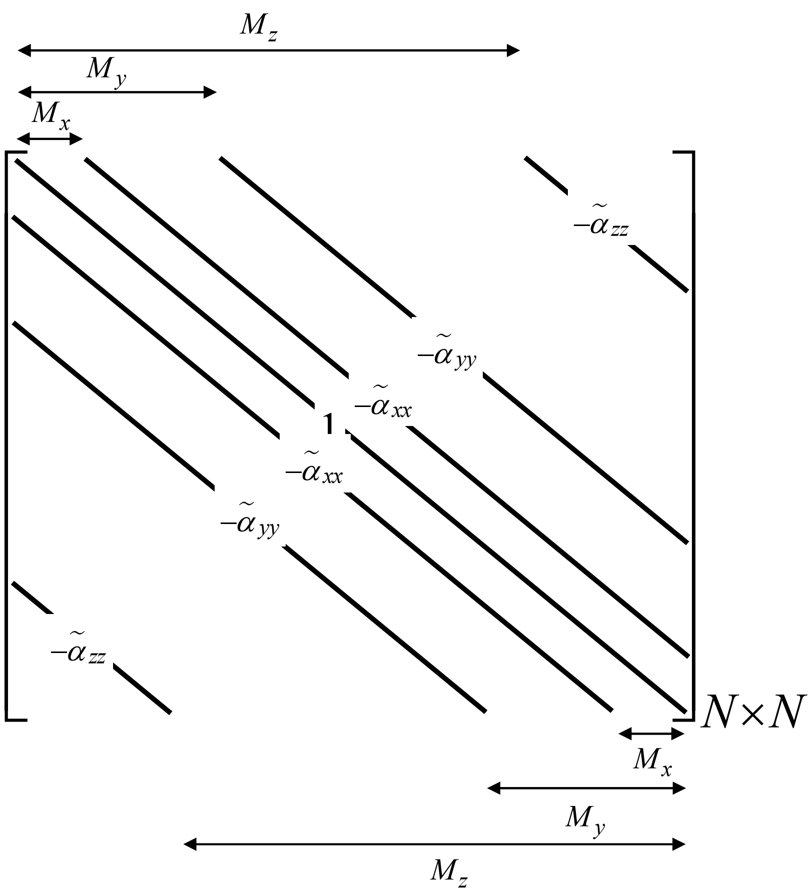
$$\left(-\tilde{\alpha}_{zz}^{(n)}S_{-M_z} - \tilde{\alpha}_{yy}^{(n)}S_{-M_y} - \tilde{\alpha}_{xx}^{(n)}S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)}S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)}S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \frac{\rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

Discrete Poisson's Grid Equation / Diskrete Poissonsche Gittergleichung

$$\left(-\tilde{\alpha}_{zz}^{(n)} S_{-M_z} - \tilde{\alpha}_{yy}^{(n)} S_{-M_y} - \tilde{\alpha}_{xx}^{(n)} S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)} S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)} S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)} S_{M_z}\right) \Phi_{\mathrm{e}}^{(n)} = \frac{\rho_{\mathrm{e}}^{(n)}(t)\Delta x \Delta y \Delta z}{\alpha^{(n)}}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

$[\mathbf{A}] =$



3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson’s grid equation / Elektrostatische Poissonsche Gittergleichung

$$\widetilde{[\mathbf{div}]}\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -\widetilde{[\mathbf{V}]}\{\boldsymbol{\rho}_\mathrm{e}\}$$

Homogeneous isotropic case / Homogener isotroper Fall

$$\widetilde{[\boldsymbol{\varepsilon}]}_{3N\times 3N} = \varepsilon_0\varepsilon_\mathrm{r}[\mathbf{I}]_{3N\times 3N}$$

$$\widetilde{[\mathbf{S}]} = (\Delta x)^2[\mathbf{I}]_{3N\times 3N}$$

$$[\mathbf{R}]^{-1} = \frac{1}{\Delta x}[\mathbf{I}]_{3N\times 3N}$$

$$\widetilde{[\mathbf{V}]} = (\Delta x)^3[\mathbf{I}]_{N\times N}$$

$$\widetilde{[\mathbf{div}]}\varepsilon_0\varepsilon_\mathrm{r}[\mathbf{I}](\Delta x)^2[\mathbf{I}]\frac{1}{\Delta x}[\mathbf{I}][\mathbf{grad}]\{\Phi_\mathrm{e}\} = -(\Delta x)^3[\mathbf{I}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$\widetilde{[\mathbf{div}]}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$-\widetilde{[\mathbf{div}]}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = \frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}$$

Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

One of the simplest types of iterative methods for solving the linear system /
Einer der einfachsten Typen von iterativen Methoden zur Lösung des linearen Gleichungssystems

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

is the stationary iterative method / ist die stationäre iterative Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}$$

where the matrix $[\mathbf{G}]$ and the vector $\{\mathbf{c}\}$ are chosen so that fixed point of the function /
wobei die Matrix $[\mathbf{G}]$ und der Vektor $\{\mathbf{c}\}$ so gewählt werden, dass der Fixpunkt der Funktion

$$\{\mathbf{g}\}(\{\mathbf{x}\}) = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

is solution to $[\mathbf{A}]\ \{\mathbf{x}\} = \{\mathbf{b}\}$. /
Lösung von $[\mathbf{A}]\ \{\mathbf{x}\} = \{\mathbf{b}\}$ ist.

The method is stationary if $[\mathbf{G}]$ and **{c}** are constant over all iterations *l*.
Die Methode ist stationär, wenn $[\mathbf{G}]$ und **{c}** für alle Iterationen konstant sind.

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- ⇨ Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

LU decomposition of matrix $[\mathbf{A}]$ / LU-Zerlegung der Matrix $[\mathbf{A}]$

$$[\mathbf{A}] = [\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]$$

Lower triangular matrix / Untere Dreiecksmatrix

$$[\mathbf{L}] = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ A_{21} & 0 & \cdots & & \vdots \\ A_{31} & A_{32} & \ddots & & 0 \\ \vdots & & & 0 & 0 \\ A_{N1} & A_{N2} & \cdots & A_{N(N-1)} & 0 \end{bmatrix}_{N\times N}$$

Main diagonal matrix / Hauptdiagonalmatrix

$$[\mathbf{D}] = \left[\operatorname{diag}\{A_{11}, A_{22}, \cdots, A_{NN}\}\right]_{N\times N}$$

Upper triangular matrix / Obere Dreiecksmatrix

$$[\mathbf{U}] = \begin{bmatrix} 0 & A_{12} & \cdots & \cdots & A_{1N} \\ 0 & 0 & \cdots & & \vdots \\ 0 & 0 & \ddots & & A_{(N-2)N} \\ \vdots & & & 0 & A_{(N-1)N} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times N}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\} \quad \rightarrow \quad \{[\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]\}\{\mathbf{x}\} = \{\mathbf{b}\} \quad \rightarrow \quad [\mathbf{D}]\{\mathbf{x}\} = -\{[\mathbf{L}] + [\mathbf{U}]\}\{\mathbf{x}\} + \{\mathbf{b}\}$$

Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$\{[\mathbf{L}]+[\mathbf{D}]+[\mathbf{U}]\}\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$[\mathbf{D}]\{\mathbf{x}\} = -\{[\mathbf{L}]+[\mathbf{U}]\}\{\mathbf{x}\} + \{\mathbf{b}\}$$

$$\{\mathbf{x}\} = \underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}}_{=[\mathbf{G}]}\{\mathbf{x}\} + \underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}}$$

$$= [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\} = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\} \quad l = 1, 2, \ldots, L$$

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- ⇨ Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

Jacobi Method / Jacobi-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_\mathrm{J}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_\mathrm{J} \quad l = 1,2,\ldots,L$$

$$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}$$

$$\{\mathbf{c}\}_\mathrm{J} = [\mathbf{D}]^{-1}\{\mathbf{b}\}$$

$$x_i^{(l+1)} = \sum_{j=1}^{N} G_{J,ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1,2,\ldots,L$$

It follows with the LU decomposition of matrix $[\mathbf{A}]$ /
Mit der LU-Zerlegung der Matrix $[\mathbf{A}]$ folgt

$$x_i^{(l+1)} = \sum_{j=1}^{i-1} G_{J,ij}\, x_j^{(l)} + \sum_{j=i+1}^{N} G_{J,ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1,2,\ldots,L$$

Gauss-Seidel Method / Gauß-Seidel-Methode

Iterative process of the Jacobi method /
Iterativer Prozess der Jacobi-Methode

$$x_i^{(l+1)} = \sum_{j=1}^{i-1} G_{\mathrm{J},ij} \boxed{x_j^{(l)}} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1,2,\dots,L$$

Iterative process of the Gauss-Seidel method /
Iterativer Prozess der Gauß-Seidel-Methode

$$x_i^{(l+1)} = \sum_{j=1}^{i-1} G_{\mathrm{J},ij} \boxed{x_j^{(l+1)}} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1,2,\dots,L$$

In the Gauss-Seidel method we make use of the already updated field values /
In der Gauß-Seidel-Methode beziehen wir die bereits aktualisierten Feldwerte in den iterativen Prozess mit ein.

Gauss-Seidel Method / Gauß-Seidel-Methode

$$\{\mathbf{x}\}^{(l+1)} = -[\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+1)} - [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} + [\mathbf{D}]^{-1}\{\mathbf{b}\} \qquad l = 1,2,\ldots,L$$

$$[\mathbf{D}]\{\mathbf{x}\}^{(l+1)} = -[\mathbf{L}]\{\mathbf{x}\}^{(l+1)} - [\mathbf{U}]\{\mathbf{x}\}^{(l)} + \{\mathbf{b}\} \qquad l = 1,2,\ldots,L$$

$$[\mathbf{D}]\{\mathbf{x}\}^{(l+1)} + [\mathbf{L}]\{\mathbf{x}\}^{(l+1)} = -[\mathbf{U}]\{\mathbf{x}\}^{(l)} + \{\mathbf{b}\} \qquad l = 1,2,\ldots,L$$

$$\left([\mathbf{D}] + [\mathbf{L}]\right)\{\mathbf{x}\}^{(l+1)} = -[\mathbf{U}]\{\mathbf{x}\}^{(l)} + \{\mathbf{b}\} \qquad l = 1,2,\ldots,L$$

$$\{\mathbf{x}\}^{(l+1)} = -\left([\mathbf{D}] + [\mathbf{L}]\right)^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} + \left([\mathbf{D}] + [\mathbf{L}]\right)^{-1}\{\mathbf{b}\} \qquad l = 1,2,\ldots,L$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{GS}}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{GS}} \qquad l = 1,2,\ldots,L$$

$$[\mathbf{G}]_{\mathrm{GS}} = -\left([\mathbf{D}] + [\mathbf{L}]\right)^{-1}[\mathbf{U}]$$

$$\{\mathbf{c}\}_{\mathrm{GS}} = \left([\mathbf{D}] + [\mathbf{L}]\right)^{-1}\{\mathbf{b}\}$$

End of Lecture 7 /
Ende der 7. Vorlesung