Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

8th Lecture / 8. Vorlesung

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3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in lokaler und globaler Matrixform

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{R}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}}$$

$$\frac{d}{dt}\iint_S \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}} = \oint_{C=\partial S} \underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{R}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet d\underline{\mathbf{S}}$$

Discrete grid equations in local matrix form /
Diskrete Gittergleichungen in lokaler Matrixform

$$[S]\frac{d}{dt}\{B\}^{(n)}(t) = -[\mathrm{curl}][R]\{E\}^{(n)}(t) - [S]\{J_{\mathrm{m}}\}^{(n)}(t)$$

$$\widetilde{[\varepsilon]}^{(n)}\widetilde{[S]}\frac{d}{dt}\{E\}^{(n)}(t) = \widetilde{[\mathrm{curl}]}\widetilde{[\nu]}^{(n)}\widetilde{[R]}\{B\}^{(n)}(t) - \widetilde{[S]}\{J_{\mathrm{e}}\}^{(n)}(t)$$

$n = 1, 2, \ldots, N$

Discrete grid equations in global matrix form /
Diskrete Gittergleichungen in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) - [\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\boldsymbol{\nu}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_{\mathrm{e}}\}(t)$$

3-D FIT – … Discrete Grid Equations in Gobal Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Faraday's induction law in global matrix form /
Faradaysches Induktionsgesetz in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$[\mathbf{S}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\{\mathbf{B}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$[\mathbf{curl}]$	$\in \mathbb{R}^{3N\times 3N}$	Topological curl operator in matrix form on the grid G / Topologischer Rotationsoperator in Matrixform auf dem Gitter G
$[\mathbf{R}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary lines on the grid G / Diagonalmatrix der Elementarstrecken auf dem Gitter G
$\{\mathbf{E}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrische Feldstärkevektor
$\{\mathbf{J}_\mathrm{m}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic current density vector / Algebraischer magnetischer Stromdichtevektor

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Faraday's induction law in global matrix form /
Faradaysches Induktionsgesetz in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$[\mathbf{S}] = \begin{bmatrix} \Delta y\Delta z & & & & & & & \\ & \ddots & & & & & & \\ & & \Delta y\Delta z & & & & & \\ & & & \Delta x\Delta y & & & & \\ & & & & \ddots & & & \\ & & & & & \Delta x\Delta y & & \\ & & & & & & \Delta x\Delta y & \\ & & & & & & & \ddots \\ & & & & & & & & \Delta x\Delta y \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} [\mathrm{diag}\{\Delta y\Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x\Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x\Delta y\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^\mathrm{T} & -[\mathbf{P}_y]^\mathrm{T} \\ -[\mathbf{P}_z]^\mathrm{T} & [\mathbf{0}] & [\mathbf{P}_x]^\mathrm{T} \\ [\mathbf{P}_y]^\mathrm{T} & -[\mathbf{P}_x]^\mathrm{T} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{R}] = \begin{bmatrix} \Delta x & & & & & & & & \\ & \ddots & & & & & & & \\ & & \Delta x & & & & & & \\ & & & \Delta y & & & & & \\ & & & & \ddots & & & & \\ & & & & & \Delta y & & & \\ & & & & & & \Delta z & & \\ & & & & & & & \ddots & \\ & & & & & & & & \Delta z \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} [\mathrm{diag}\{\Delta x\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta y\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta z\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$\{\mathbf{B}\}(t) = \begin{Bmatrix} \{B_x\}(t) \\ \{B_y\}(t) \\ \{B_z\}(t) \end{Bmatrix}_{3N} \qquad \{B_i\}(t) = \begin{Bmatrix} B_i^{(1)}(t) \\ B_i^{(2)}(t) \\ \vdots \\ B_i^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Faraday's induction law in global matrix form /
Faradaysches Induktionsgesetz in globaler Matrixform

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$\{\mathbf{E}\}(t) = \begin{Bmatrix} \{E_x\}(t) \\ \{E_y\}(t) \\ \{E_z\}(t) \end{Bmatrix}_{3N} \qquad \{E_i\}(t) = \begin{Bmatrix} E_i^{(1)}(t) \\ E_i^{(2)}(t) \\ \vdots \\ E_i^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

$$\{\mathbf{J}_\mathrm{m}\}(t) = \begin{Bmatrix} \{J_{\mathrm{m}x}\}(t) \\ \{J_{\mathrm{m}y}\}(t) \\ \{J_{\mathrm{m}z}\}(t) \end{Bmatrix}_{3N} \qquad \{J_{\mathrm{m}i}\}(t) = \begin{Bmatrix} J_{\mathrm{m}i}^{(1)}(t) \\ J_{\mathrm{m}i}^{(2)}(t) \\ \vdots \\ J_{\mathrm{m}i}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form /
3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$[\widetilde{\boldsymbol{\varepsilon}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of permittivities on the grid $\widetilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\widetilde{G}$
$[\widetilde{\mathbf{S}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\widetilde{G}$
$\{\mathbf{E}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$[\widetilde{\mathrm{curl}}]$	$\in \mathbb{R}^{3N\times 3N}$	Topological curl operator in matrix form on the grid $\widetilde{G}$ / Topologischer Rotationsoperator in Matrixform auf dem Gitter $\widetilde{G}$
$[\widetilde{\boldsymbol{\nu}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of impermeabilities on the grid $\widetilde{G}$ / Diagonalmatrix der Impermeabilitäten auf dem Gitter $\widetilde{G}$
$[\widetilde{\mathbf{R}}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary lines on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarstrecken auf dem Gitter $\widetilde{G}$
$\{\mathbf{B}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$\{\mathbf{J}_\mathrm{e}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric current density vector / Algebraischer elektrischer Stromdichtevektor

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\mathbf{v}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_\mathrm{e}\}(t)$$

$$\widetilde{[\varepsilon]} = \begin{bmatrix} \tilde{\varepsilon}_{xx}^{(1)} & & & & & & & & \\ & \ddots & & & & & & & \\ & & \tilde{\varepsilon}_{xx}^{(N)} & & & & & & \\ & & & \tilde{\varepsilon}_{yy}^{(1)} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{\varepsilon}_{yy}^{(N)} & & & \\ & & & & & & \tilde{\varepsilon}_{zz}^{(1)} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \tilde{\varepsilon}_{zz}^{(N)} \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\{\tilde{\varepsilon}_{xx}^{(1)}, \tilde{\varepsilon}_{xx}^{(2)}, \ldots, \tilde{\varepsilon}_{xx}^{(N)}\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\{\tilde{\varepsilon}_{yy}^{(1)}, \tilde{\varepsilon}_{yy}^{(2)}, \ldots, \tilde{\varepsilon}_{yy}^{(N)}\}_N\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\{\tilde{\varepsilon}_{zz}^{(1)}, \tilde{\varepsilon}_{zz}^{(2)}, \ldots, \tilde{\varepsilon}_{zz}^{(N)}\}_N\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$[\tilde{\boldsymbol{\varepsilon}}][\tilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\tilde{\mathbf{v}}][\tilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\tilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$$[\tilde{\mathbf{S}}] = \begin{bmatrix} \Delta y\Delta z & & & & & & & \\ & \ddots & & & & & & \\ & & \Delta y\Delta z & & & & & \\ & & & \Delta x\Delta z & & & & \\ & & & & \ddots & & & \\ & & & & & \Delta x\Delta z & & \\ & & & & & & \Delta x\Delta y & \\ & & & & & & & \ddots \\ & & & & & & & & \Delta y\Delta z \end{bmatrix}_{3N\times 3N} = \begin{bmatrix} [\mathrm{diag}\{\Delta y\Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x\Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x\Delta y\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\tilde{\mathbf{R}}] = \begin{bmatrix} \Delta x & & & & & & & \\ & \ddots & & & & & & \\ & & \Delta x & & & & & \\ & & & \Delta y & & & & \\ & & & & \ddots & & & \\ & & & & & \Delta y & & \\ & & & & & & \Delta z & \\ & & & & & & & \ddots \\ & & & & & & & & \Delta z \end{bmatrix}_{3N\times 3N} = \begin{bmatrix} [\mathrm{diag}\{\Delta x\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta y\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta z\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$\{\mathbf{J}_\mathrm{e}\}(t) = \begin{Bmatrix} \{J_\mathrm{ex}\}(t) \\ \{J_\mathrm{ey}\}(t) \\ \{J_\mathrm{ez}\}(t) \end{Bmatrix}_{3N} \qquad \{J_{\mathrm{e}i}\}(t) = \begin{Bmatrix} J_{\mathrm{e}i}^{(1)}(t) \\ J_{\mathrm{e}i}^{(2)}(t) \\ \vdots \\ J_{\mathrm{e}i}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

3-D FIT – … Discrete Grid Equations in Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in globaler Matrixform

Ampère-Maxwell's circuital law in global matrix form /
Ampère-Maxwellsches Durchflutungsgesetz in globaler Matrixform

$$\widetilde{[\boldsymbol{\varepsilon}]}\widetilde{[\mathbf{S}]}\frac{d}{dt}\{\mathbf{E}\}(t) = \widetilde{[\mathbf{curl}]}\widetilde{[\mathbf{v}]}\widetilde{[\mathbf{R}]}\{\mathbf{B}\}(t) - \widetilde{[\mathbf{S}]}\{\mathbf{J}_\mathrm{e}\}(t)$$

$$\widetilde{[\nu]} = \begin{bmatrix} \tilde{\nu}_{xx}^{(1)} & & & & & & & & \\ & \ddots & & & & & & & \\ & & \tilde{\nu}_{xx}^{(N)} & & & & & & \\ & & & \tilde{\nu}_{yy}^{(1)} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{\nu}_{yy}^{(N)} & & & \\ & & & & & & \tilde{\nu}_{zz}^{(1)} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \tilde{\nu}_{zz}^{(N)} \end{bmatrix}_{3N\times 3N}$$

$$= \begin{bmatrix} \left[\operatorname{diag}\{\tilde{\nu}_{xx}^{(1)}, \tilde{\nu}_{xx}^{(2)}, \ldots, \tilde{\nu}_{xx}^{(N)}\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\operatorname{diag}\{\tilde{\nu}_{yy}^{(1)}, \tilde{\nu}_{yy}^{(2)}, \ldots, \tilde{\nu}_{yy}^{(N)}\}_N\right]_{N\times N} & [0] \\ [0] & [0] & \left[\operatorname{diag}\{\tilde{\nu}_{zz}^{(1)}, \tilde{\nu}_{zz}^{(2)}, \ldots, \tilde{\nu}_{zz}^{(N)}\}_N\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in lokaler und globaler Matrixform

The two discrete grid equations in global matrix form read /
Die beiden diskreten Gittergleichungen in globaler Matrixform lauten

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

We arrange the last equations in the form /
Wir bringen die letzten beiden Gleichungen in die Form

$$\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{S}]^{-1} \quad [\mathbf{curl}] \quad [\mathbf{R}]\{\mathbf{E}\}(t) - [\mathbf{S}]^{-1} \quad [\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$$[\mathbf{S}]^{-1}[\mathbf{S}] = [\mathbf{I}]$$

$$[\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{S}}] = \underbrace{[\widetilde{\mathbf{S}}]^{-1}[\widetilde{\mathbf{S}}]}_{=[\mathbf{I}]}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} = [\widetilde{\boldsymbol{\varepsilon}}]^{-1}$$

$$\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{S}]^{-1} \quad [\mathbf{curl}] \quad [\mathbf{R}]\{\mathbf{E}\}(t) - \quad \{\mathbf{J}_\mathrm{m}\}(t)$$

$$\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1} [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}(t)$$

3-D FIT – … Discrete Grid Equations in Local and Global Matrix Form / 3D-FIT – ... diskrete Gittergleichungen in lokaler und globaler Matrixform

The two discrete grid equations in global matrix form read /
Die beiden diskreten Gittergleichungen in globaler Matrixform lauten

$$\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{S}]^{-1} \quad [\mathbf{curl}] \quad [\mathbf{R}]\{\mathbf{E}\}(t) - \quad \{\mathbf{J}_\mathrm{m}\}(t)$$

$$\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}(t)$$

Now we write these two matrix equations in matrix form and find a first-order system of differential equations /
Nun schreiben wir die beiden Matrixgleichungen in Matrixform und finden das folgende System von Differentialgleichungen erster Ordnung

$$\frac{d}{dt}\{\mathbf{y}\}(t) = [\mathbf{A}]\{\mathbf{y}\}(t) + \{\mathbf{q}\}(t)$$

with / mit

Solution vector / Lösungsvektor
$$\{\mathbf{y}\}(t) = \begin{Bmatrix} \{\mathbf{B}\}(t) \\ \{\mathbf{E}\}(t) \end{Bmatrix}$$

System matrix / Systemmatrix
$$[\mathbf{A}] = \begin{bmatrix} [0] & [\mathbf{S}]^{-1}[\mathbf{curl}][\mathbf{R}] \\ [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}] & [0] \end{bmatrix}$$

Source vector / Quellvektor
$$\{\mathbf{q}\}(t) = \begin{Bmatrix} - \quad \{\mathbf{J}_\mathrm{m}\}(t) \\ -[\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}(t) \end{Bmatrix}$$

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

A general solution of the initial value problem (IVP) with the initial value $\{\mathbf{y}\}(t_0)$ is /
Eine allgemeine Lösung des Anfangswertproblems (AWP) mit dem Anfangswert $\{\mathbf{y}\}(t_0)$ ist

$$\{\mathbf{y}\}(t) = \{\mathbf{y}\}(t_0) + \underbrace{\int_{t'=t_0}^{t} \underbrace{\{[\mathbf{A}]\{\mathbf{y}\}(t) + \{\mathbf{q}\}(t)\}}_{=\{\dot{\mathbf{y}}\}(t)} \,\mathrm{d}t}_{\text{time integration / zeitliche Integration}}$$

- implicit time integration / implizierte Zeitintegration
- explicit time integration / explizite Zeitintegration

Explicit time integration / Explizite Zeitintegration

$$\{\mathbf{B}\}(t) = \boxed{\{\mathbf{B}\}(t_0)} + \int_{t'=t_0}^{t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{E}\}(t) = \boxed{\{\mathbf{E}\}(t_0)} + \int_{t'=t_0}^{t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t'$$

Initial value /
Anfangswert

$t = [0, T];\quad T:$ time interval to be simulated / zu simulierendes Zeitinterval

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

Discretization in time on a staggered grid in time /
Diskretisierung in der Zeit auf einem versetzten Gitter in der Zeit

$$\{\mathbf{B}\}(t) \to \{\mathbf{B}\}(n_t \Delta t) \to \{\mathbf{B}\}^{(n_t)}$$

$$\{\mathbf{E}\}(t) \to \{\mathbf{E}\}\left[\left(n_t + \frac{1}{2}\right)\Delta t\right] \to \{\mathbf{E}\}^{(n_t+1/2)}$$

$$\{\mathbf{B}\}(t) = \{\mathbf{B}\}(t_0) + \int_{t'=t_0}^{t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{E}\}(t) = \{\mathbf{E}\}(t_0) + \int_{t'=t_0}^{t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \int_{t'=(n_t-1)\Delta t}^{n_t\Delta t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t'$$

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \int_{t'=(n_t-1/2)\Delta t}^{(n_t+1/2)\Delta t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t'$$

Mid point rule /
Mittelpunktsregel

$$\int_{t'=(n_t-1)\Delta t}^{n_t\Delta t} \{\dot{\mathbf{B}}\}(t')\,\mathrm{d}t' = \{\dot{\mathbf{B}}\}\left[(n_t-1/2)\Delta t\right]\Delta t = \{\dot{\mathbf{B}}\}^{(n_t-1/2)}\Delta t$$

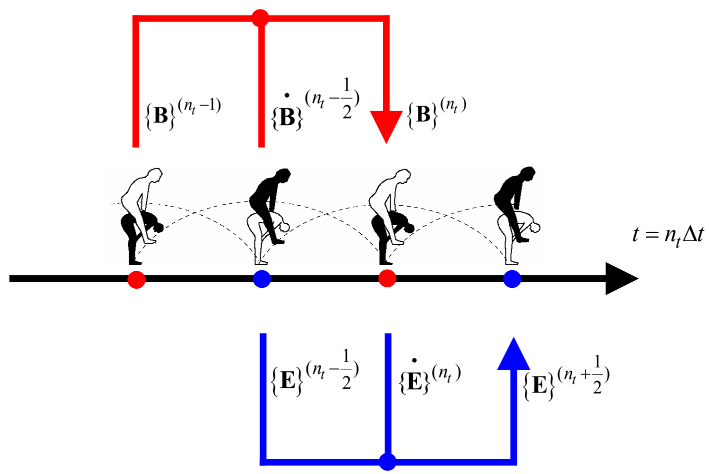
$$\int_{t'=(n_t-1/2)\Delta t}^{(n_t+1/2)\Delta t} \{\dot{\mathbf{E}}\}(t')\,\mathrm{d}t' = \{\dot{\mathbf{E}}\}(n_t\Delta t)\Delta t = \{\dot{\mathbf{E}}\}^{(n_t)}\Delta t$$

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

The leapfrog structure of the algorithm in time /
Die Bocksprung-Struktur des Algorithmus in der Zeit

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \Delta t \{\dot{\mathbf{B}}\}^{(n_t-1/2)}$$

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \Delta t \{\dot{\mathbf{E}}\}^{(n_t)}$$



3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

Electromagnetic grid equations (EMGE) of the so-called
Electromagnetic Finite Integration Technique (EMFIT) algorithm /
Elektromagnetische Gittergleichungen (EMGG) des so genannten
Elektromagnetischen Finite Integrationstechnik (EMFIT) Algorithmus

Faraday's induction grid equation / Faradaysche Induktionsgittergleichung

$$\{\dot{\mathbf{B}}\}^{(n_t-1/2)} = -[\mathbf{S}]^{-1}[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}^{(n_t-1/2)} - \{\mathbf{J}_\mathrm{m}\}^{(n_t-1/2)}$$

Time integration / Zeitintegration

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \Delta t\{\dot{\mathbf{B}}\}^{(n_t-1/2)}$$

Ampère-Maxwell's circuital grid equation / Ampère-Maxwellsche Durchflutungsgittergleichung

$$\{\dot{\mathbf{E}}\}^{(n_t} = [\widetilde{\mathbf{S}}]^{-1}[\widetilde{\boldsymbol{\varepsilon}}]^{-1}[\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}^{(n_t)} - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}^{(n_t)}$$

Time integration / Zeitintegration

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \Delta t\{\dot{\mathbf{E}}\}^{(n_t)}$$

3-D FIT – … Solution of the Initial Value Problem (IVP) / 3D-FIT – Lösung des Anfangswertproblems (AWP)

Electromagnetic grid equations (EMGE) of the so-called EMFIT algorithm /
Elektromagnetische Gittergleichungen (EMGG) des so genannten EMFIT-Algorithmus

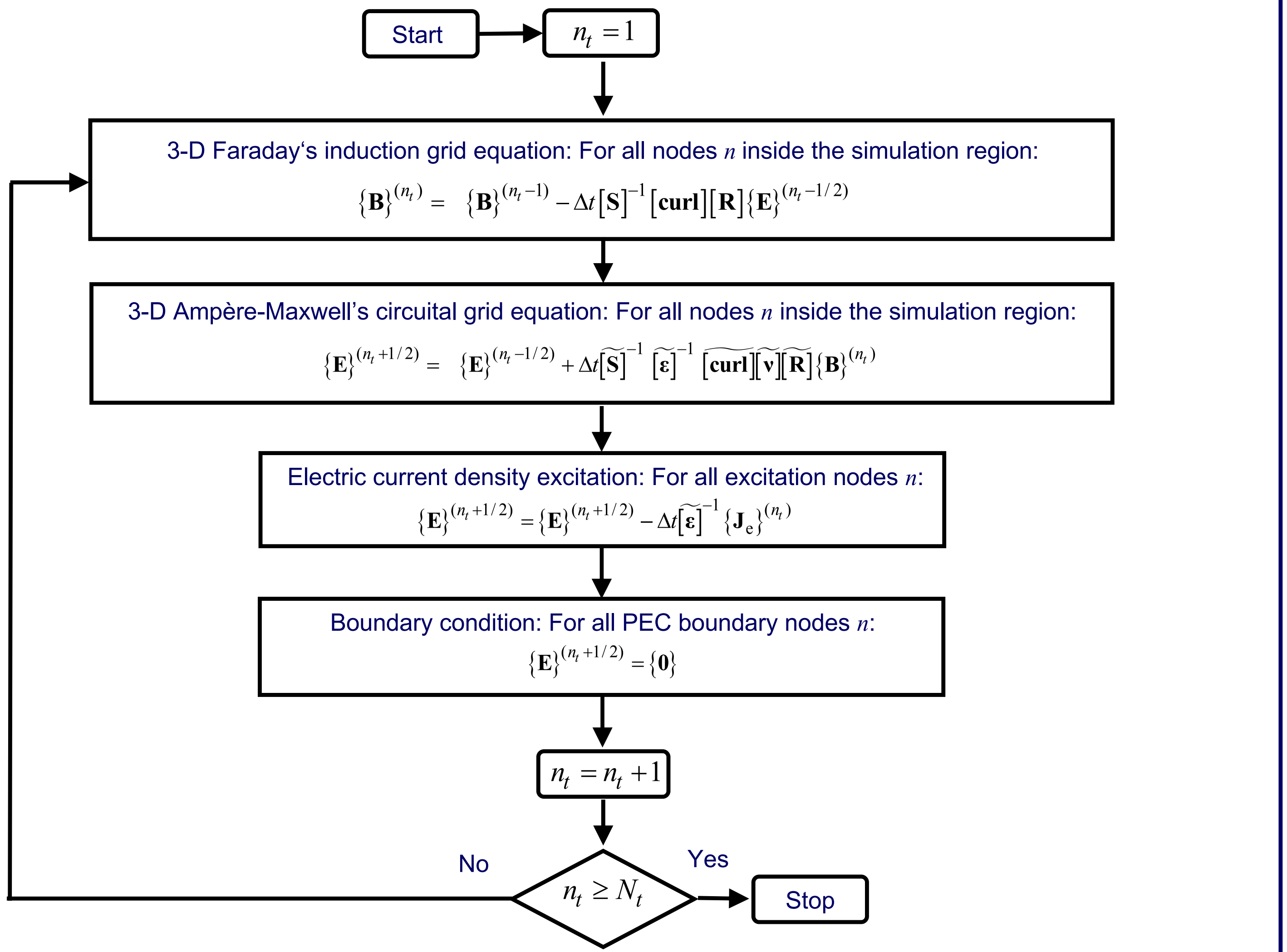
Time-integrated Faraday's induction grid equation /
Zeitlich integrierte Faradaysche Induktionsgittergleichung

$$\{\mathbf{B}\}^{(n_t)} = \{\mathbf{B}\}^{(n_t-1)} + \Delta t\left[-[\mathbf{S}]^{-1}[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}^{(n_t-1/2)} - \{\mathbf{J}_\mathrm{m}\}^{(n_t-1/2)}\right]$$

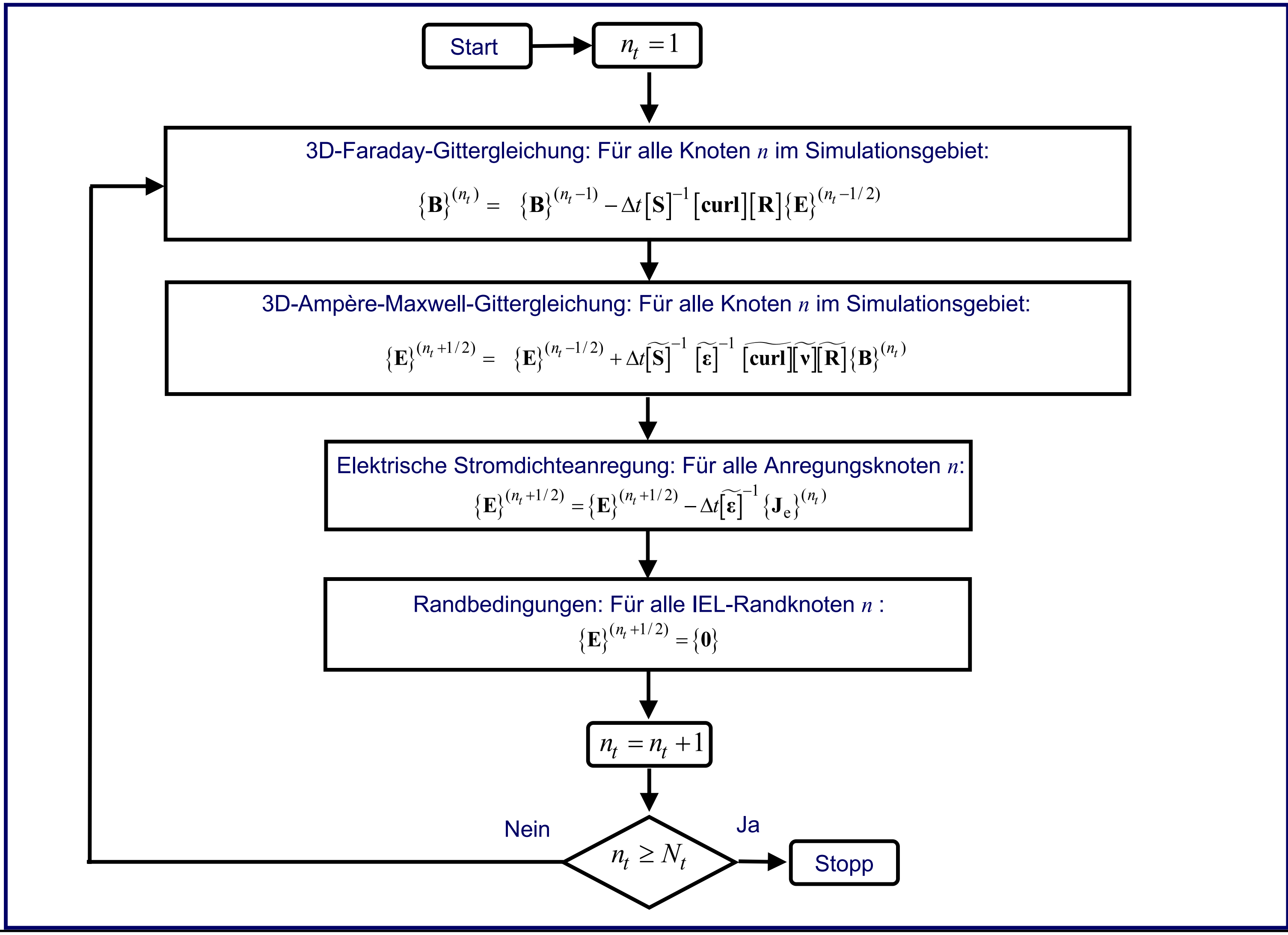
Time-integrated Ampère-Maxwell's circuital grid equation /
Zeitlich integrierte Ampère-Maxwellsche Durchflutungsgittergleichung

$$\{\mathbf{E}\}^{(n_t+1/2)} = \{\mathbf{E}\}^{(n_t-1/2)} + \Delta t\left[[\widetilde{\mathbf{S}}]^{-1}\,[\widetilde{\boldsymbol{\varepsilon}}]^{-1}\,[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}^{(n_t)} - [\widetilde{\boldsymbol{\varepsilon}}]^{-1}\{\mathbf{J}_\mathrm{e}\}^{(n_t)}\right]$$

3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



3-D FIT – … Normalized … Grid Equations / 3D-FIT – ... normierte ... Gittergleichungen

Normalized electromagnetic grid equations (EMGE) of the so-called EMFIT algorithm /
Normierte elektromagnetische Gittergleichungen (EMGG) des so genannten EMFIT-Algorithmus

Normalized time-integrated Faraday's induction grid equation /
Normierte zeitlich integrierte Faradaysche Induktionsgittergleichung

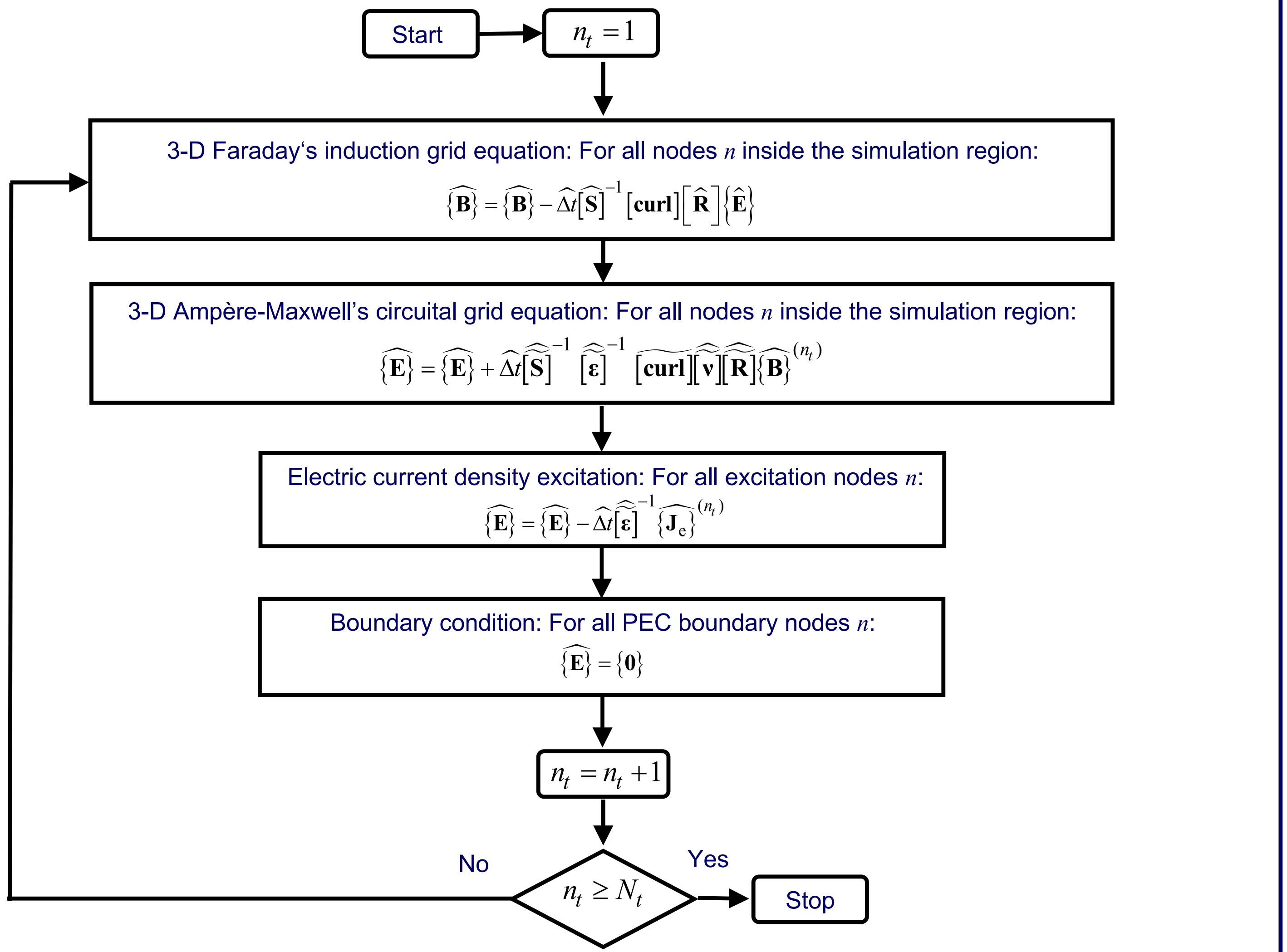
$$\widehat{\{\mathbf{B}\}}^{(n_t)} = \widehat{\{\mathbf{B}\}}^{(n_t-1)} + \widehat{\Delta t}\left[-\left[\widehat{\mathbf{S}}\right]^{-1}[\mathbf{curl}]\left[\widehat{\mathbf{R}}\right]\left\{\widehat{\mathbf{E}}\right\}^{(n_t-1/2)} - \widehat{\{\mathbf{J}_\mathrm{m}\}}^{(n_t-1/2)}\right]$$

Normalized time-integrated Ampère-Maxwell's circuital grid equation /
Normierte zeitlich integrierte Ampère-Maxwellsche Durchflutungsgittergleichung

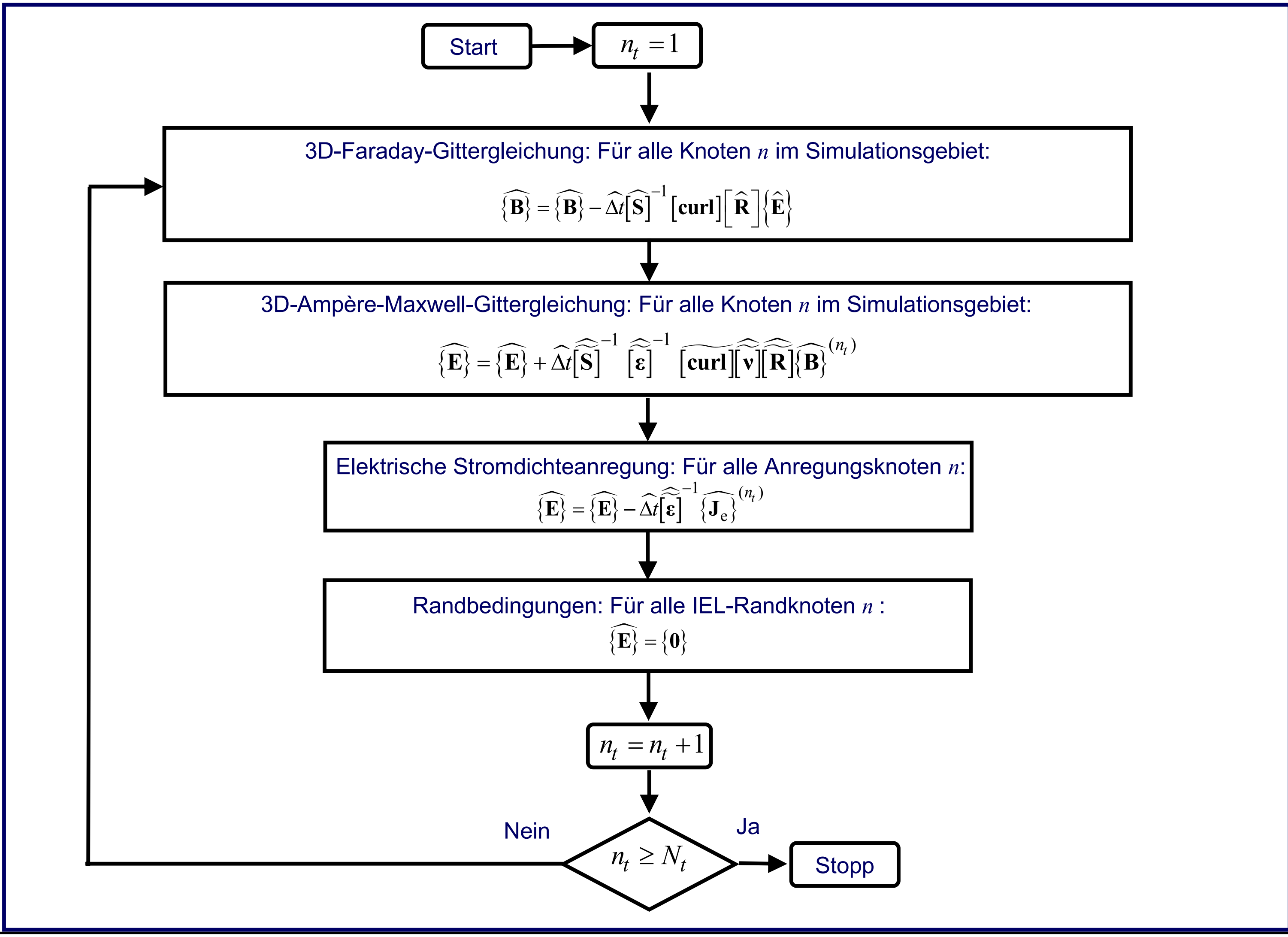
$$\widehat{\{\mathbf{E}\}}^{(n_t+1/2)} = \widehat{\{\mathbf{E}\}}^{(n_t-1/2)} + \widehat{\Delta t}\left[\widehat{\widehat{[\mathbf{S}]}}^{-1}\widehat{\widehat{[\boldsymbol{\varepsilon}]}}^{-1}\widetilde{[\mathbf{curl}]}\widehat{\widehat{[\boldsymbol{\nu}]}}\widehat{\widehat{[\mathbf{R}]}}\widehat{\{\mathbf{B}\}}^{(n_t)} - \widehat{\widehat{[\boldsymbol{\varepsilon}]}}^{-1}\widehat{\{\mathbf{J}_\mathrm{e}\}}^{(n_t)}\right]$$

In a computer implementation we can neglect the integer time step counter n_t. /
In der Rechnerimplementierung kann der ganzzahlige Zeitschrittzähler n_t unterdrückt werden.

3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



3-D FIT Algorithm – Flow Chart / 3D-FIT-Algorithmus – Flussdiagramm



FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

?

FIT

Maxwell's grid equations /
Maxwellsche Gittergleichungen

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

FIT Discretization of the 3rd Maxwell Equation / FIT-Diskretisierung der 3. Maxwellschen Gleichung

Integral form / Integralform

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

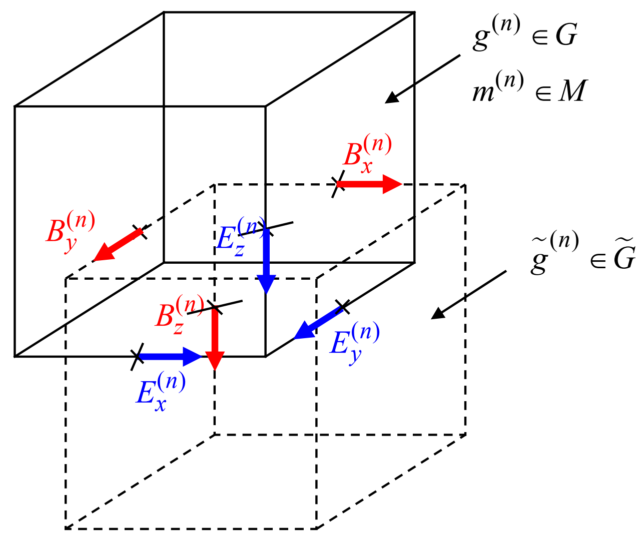
$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\cdot \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

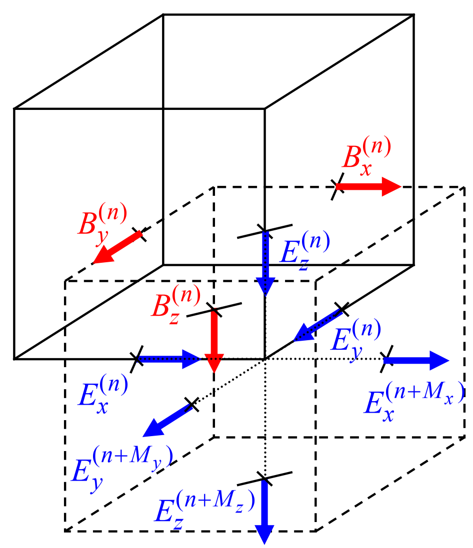
Differential form / Differentialform

$$\nabla\cdot \underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_\mathrm{e}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\nabla\cdot\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] = \rho_\mathrm{e}(\underline{\mathbf{R}},t)$$

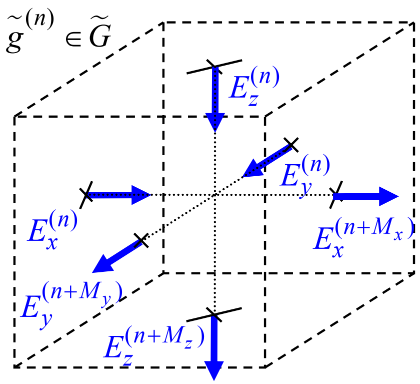


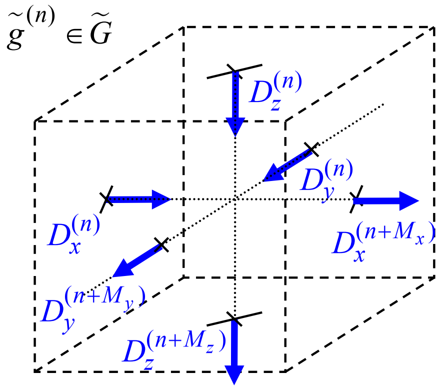


FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

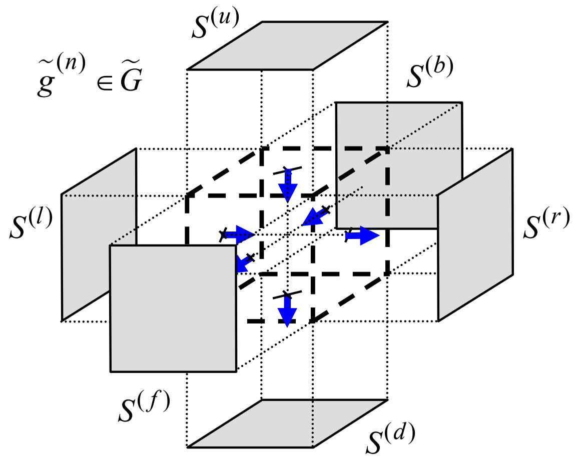
$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\cdot\underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV$$





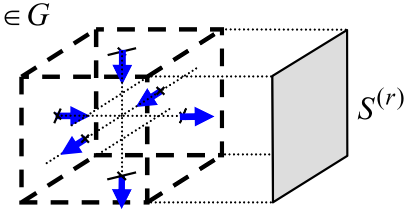
$$D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$
$$D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$
$$D_y^{(n)} = \tilde{\varepsilon}_{yy}^{(n)} E_x^{(n)}$$
$$D_y^{(n+M_y)} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}$$
$$D_z^{(n)} = \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}$$
$$D_z^{(n+M_z)} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}$$



$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} = \iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} + \iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$
$$+\iint_{S^{(f)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} + \iint_{S^{(b)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$
$$+\iint_{S^{(d)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}} + \iint_{S^{(u)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot\underline{\mathbf{dS}}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$\tilde{g}^{(n)} \in \tilde{G}$



$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

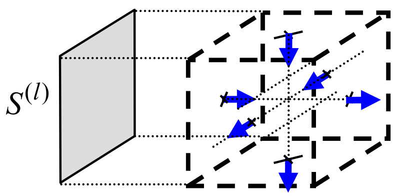
$$S^{(r)}: \quad D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\begin{aligned}
\iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} &= \iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathrm{e}}_x\,\mathrm{d}y\mathrm{d}z \\
&= \iint_{S^{(r)}} D_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= \iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})E_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= E_x^{(n+M_x)}(t)\underbrace{\iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}y\mathrm{d}z}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right] \\
&= \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right] \\
&= D_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]
\end{aligned}$$

$$\begin{aligned}
&\iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S \\
&= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n+M_x)} + \varepsilon_{xx}^{(n+M_x+M_y)} + \varepsilon_{xx}^{(n+M_x+M_z)} + \varepsilon_{xx}^{(n+M_x+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}}\Delta y\Delta z \\
&= \tilde{\varepsilon}_{xx}^{(n+M_x)}\Delta y\Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$\tilde{g}^{(n)} \in \widetilde{G}$

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

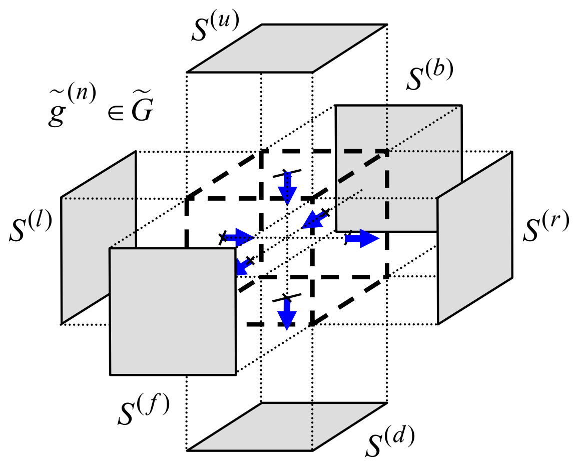
$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = -\underline{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

$$S^{(l)}:\quad D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

$$\begin{aligned}
\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} &= -\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathrm{e}}_x\,\mathrm{d}y\mathrm{d}z \\
&= -\iint_{S^{(l)}} D_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= -\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}}) E_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= -E_x^{(n)}(t) \underbrace{\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}y\mathrm{d}z}_{=\tilde{\varepsilon}_{xx}^{(n)}\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right] \\
&= -\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right] \\
&= -D_x^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\,\Delta z + \Delta y(\Delta z)^3\right]
\end{aligned}$$

$$\begin{aligned}
&\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S \\
&= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n)} + \varepsilon_{xx}^{(n+M_y)} + \varepsilon_{xx}^{(n+M_z)} + \varepsilon_{xx}^{(n+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n)}} \Delta y\Delta z \\
&= \tilde{\varepsilon}_{xx}^{(n)}\Delta y\Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$$S^{(l)}: \quad D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

$$S^{(r)}: \quad D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$S^{(b)}: \quad D_y^{(n)} = \tilde{\varepsilon}_{yy}^{(n)} E_x^{(n)}$$

$$S^{(f)}: \quad D_y^{(n+M_y)} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}$$

$$S^{(u)}: \quad D_z^{(n)} = \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}$$

$$S^{(d)}: \quad D_z^{(n+M_z)} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}$$

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$\iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$
$$= D_x^{(n+M_x)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$

$$\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$
$$= -D_x^{(n)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y(\Delta z)^3\right]$$

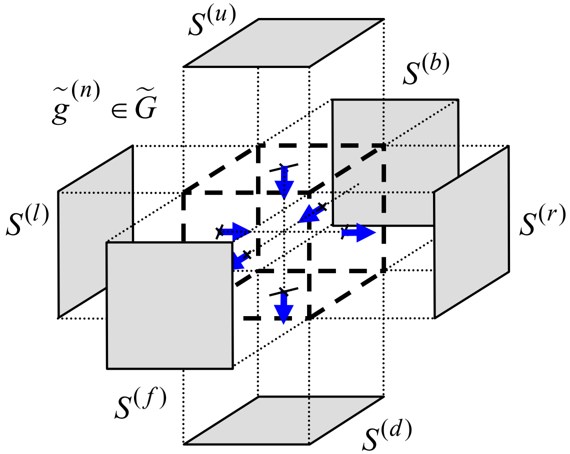
$$\iint_{S^{(f)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$
$$= D_y^{(n+M_y)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$

$$\iint_{S^{(b)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$
$$= -D_y^{(n)}(t)\Delta x\Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x(\Delta z)^3\right]$$

$$\iint_{S^{(d)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$
$$= D_z^{(n+M_z)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$

$$\iint_{S^{(u)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$
$$= -D_z^{(n)}(t)\Delta x\Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x(\Delta y)^3\right]$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$$\tilde{g}^{(n)} \in \tilde{G}$$

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$= \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z - \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z$$

$$+ \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t)\Delta x\Delta z - \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z$$

$$+ \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t)\Delta x\Delta y - \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$

$$= D_x^{(n+M_x)}(t)\Delta y\Delta z - D_x^{(n)}(t)\Delta y\Delta z$$

$$+ D_y^{(n+M_y)}(t)\Delta x\Delta z - D_y^{(n)}(t)\Delta x\Delta z$$

$$+ D_z^{(n+M_z)}(t)\Delta x\Delta y - D_z^{(n)}(t)\Delta x\Delta y$$

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$= \left[\tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t) - \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t) \right] \Delta y\Delta z$$

$$+ \left[\tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t) - \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t) \right] \Delta x\Delta z$$

$$+ \left[\tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t) - \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t) \right] \Delta x\Delta y$$

$$= \left[D_x^{(n+M_x)}(t) - D_x^{(n)}(t) \right] \Delta y\Delta z$$

$$+ \left[D_y^{(n+M_y)}(t) - D_y^{(n)}(t) \right] \Delta x\Delta z$$

$$+ \left[D_z^{(n+M_z)}(t) - D_z^{(n)}(t) \right] \Delta x\Delta y$$

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$= \left(S_{M_x} - I \right) \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z$$

$$+ \left(S_{M_y} - I \right) \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z$$

$$+ \left(S_{M_z} - I \right) \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$

$$= \left(S_{M_x} - I \right) D_x^{(n)}(t)\Delta y\Delta z$$

$$+ \left(S_{M_y} - I \right) D_y^{(n)}(t)\Delta x\Delta z$$

$$+ \left(S_{M_z} - I \right) D_z^{(n)}(t)\Delta x\Delta y$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$

$$= \underbrace{\begin{bmatrix} S_{M_x}-I & S_{M_y}-I & S_{M_z}-I \end{bmatrix}}_{=\widetilde{[\mathrm{div}]}} \underbrace{\begin{bmatrix} \tilde{\varepsilon}_{zz}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)} \end{bmatrix}}_{=\widetilde{[\varepsilon]}^{(n)}} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta z \end{bmatrix}}_{=\widetilde{[S]}} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}}_{=\{E\}^{(n)}(t)}$$

$$= \widetilde{[\mathrm{div}]}\widetilde{[\varepsilon]}^{(n)}\widetilde{[S]}\{E\}^{(n)}(t)$$

$$= \left(S_{M_x}-I\right) D_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right) D_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right) D_z^{(n)}(t)\Delta x\Delta y$$

$$= \underbrace{\begin{bmatrix} S_{M_x}-I & S_{M_y}-I & S_{M_z}-I \end{bmatrix}}_{=\widetilde{[\mathrm{div}]}} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta z \end{bmatrix}}_{=\widetilde{[S]}} \underbrace{\begin{Bmatrix} D_x^{(n)}(t) \\ D_y^{(n)}(t) \\ D_z^{(n)}(t) \end{Bmatrix}}_{=\{D\}^{(n)}(t)}$$

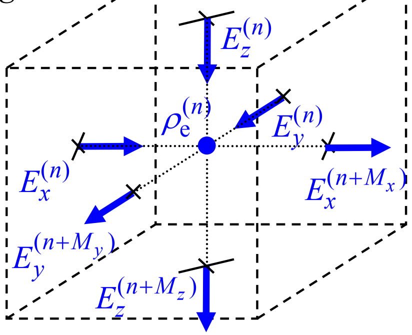
$$= \widetilde{[\mathrm{div}]}\widetilde{[S]}\{D\}^{(n)}(t)$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

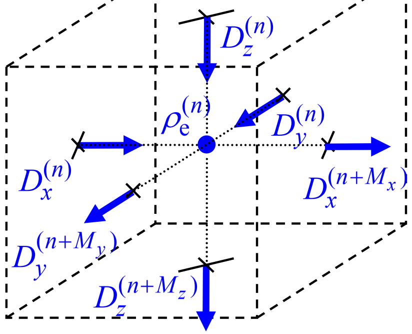
$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\cdot \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

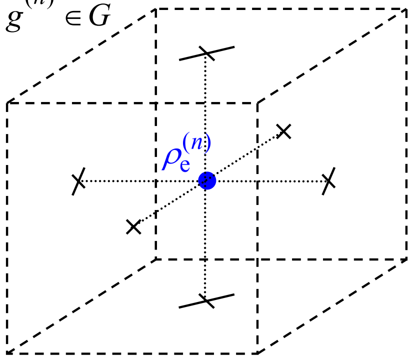
$V = \tilde{g}^{(n)} \in \tilde{G}$



$V = \tilde{g}^{(n)} \in \tilde{G}$



$V = \tilde{g}^{(n)} \in \tilde{G}$



$$\begin{aligned}
\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV &= \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z + \bigcirc\left[(\Delta x)^3\,\Delta y\Delta z + \Delta x(\Delta y)^3\,\Delta z + \Delta x\Delta y(\Delta z)^3\right] \\
&= Q_\mathrm{e}^{(n)}(t) + \bigcirc\left[(\Delta x)^3\,\Delta y\Delta z + \Delta x(\Delta y)^3\,\Delta z + \Delta x\Delta y(\Delta z)^3\right] \\
&= \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z + \bigcirc\left[(\Delta x)^5\right] \qquad \text{if} \quad \Delta x \approx \Delta y \approx \Delta z \\
&= Q_\mathrm{e}^{(n)}(t) + \bigcirc\left[(\Delta x)^5\right] \qquad \text{if} \quad \Delta x \approx \Delta y \approx \Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (…) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV = Q_e(t)$$

$$\oint\!\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV = Q_e(t)$$

Discrete grid equations in local matrix form /
Diskrete Gittergleichungen in lokaler Matrixform

$$[\widetilde{\mathrm{div}}][\widetilde{\varepsilon}]^{(n)}[\widetilde{S}]\{E\}^{(n)}(t) = \rho_e^{(n)}(t)\Delta x \Delta y \Delta z = Q_e^{(n)}(t)$$

$$[\widetilde{\mathrm{div}}][\widetilde{S}]\{D\}^{(n)}(t) = \rho_e^{(n)}(t)\Delta x \Delta y \Delta z = Q_e^{(n)}(t)$$

Discrete grid equations in global matrix form /
Diskrete Gittergleichungen in globaler Matrixform

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_e\}(t) = \{\mathbf{Q}_e\}(t)$$

$$[\widetilde{\mathbf{div}}][\widetilde{\mathbf{S}}]\{\mathbf{D}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_e\}(t) = \{\mathbf{Q}_e\}(t)$$

with / mit $[\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x],[\mathbf{P}_y],[\mathbf{P}_z]\right]_{N\times 3N}$

Discrete Local and Global Gradient, Divergence, and Curl Operators / Diskrete lokale und globale Gradienten-, Divergenz- und Rotationsoperatoren

Discrete gradient operator / Diskreter Gradientenoperator

$$[\mathbf{grad}] = \begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}_{3N\times N}$$

$$[\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix}_{3N\times N}$$

Discrete curl operator / Diskreter Rotationsoperator

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^{\mathrm{T}} & -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} & [\mathbf{0}] & [\mathbf{P}_x]^{\mathrm{T}} \\ [\mathbf{P}_y]^{\mathrm{T}} & -[\mathbf{P}_x]^{\mathrm{T}} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

Discrete divergence operator / Diskreter Divergenzoperator

$$[\mathbf{div}] := \left[-[\mathbf{P}_x]^{\mathrm{T}}, -[\mathbf{P}_y]^{\mathrm{T}}, -[\mathbf{P}_z]^{\mathrm{T}} \right]_{N\times 3N}$$

$$[\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x], [\mathbf{P}_y], [\mathbf{P}_z] \right]_{N\times 3N}$$

$$\text{curl grad} = \nabla\times\nabla = \underline{\mathbf{0}}$$
$$\text{div curl} = \nabla\bullet\nabla = 0$$

$$-[\widetilde{\mathbf{div}}] = [\mathbf{grad}]^{\mathrm{T}}$$
$$[\widetilde{\mathbf{grad}}]^{\mathrm{T}} = [\mathbf{div}]$$
$$[\mathbf{curl}] = [\widetilde{\mathbf{curl}}]^{\mathrm{T}}$$

$$[\mathbf{curl}][\mathbf{grad}] = [\mathbf{0}]$$
$$[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{grad}}] = [\mathbf{0}]$$
$$[\mathbf{div}][\mathbf{curl}] = [\mathbf{0}]$$
$$[\widetilde{\mathbf{div}}][\widetilde{\mathbf{curl}}] = [\mathbf{0}]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form /
Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{E}\right\}(t) = \left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_\mathrm{e}\right\}(t) = \left\{\mathbf{Q}_\mathrm{e}\right\}(t)$$

$\left[\widetilde{\mathbf{div}}\right]$	$\in \mathbb{R}^{N\times 3N}$	Topological divergence operator in matrix form on the grid $\widetilde{G}$ / Topologischer Divergenzoperator in Matrixform auf dem Gitter $\widetilde{G}$
$\left[\widetilde{\boldsymbol{\varepsilon}}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of permittivities on the grid $\widetilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\widetilde{G}$
$\left[\widetilde{\mathbf{S}}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\widetilde{G}$
$\left\{\mathbf{E}\right\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$\left[\widetilde{\mathbf{V}}\right]$	$\in \mathbb{R}^{N\times N}$	Diagonal matrix of elementary volumes on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarvolumina auf dem Gitter $\widetilde{G}$
$\left\{\rho_\mathrm{e}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic electric charge density vector / Algebraischer elektrischer Ladungsdichtevektor
$\left\{\mathbf{Q}_\mathrm{e}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic electric charge vector / Algebraischer elektrischer Ladungsvektor

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Magnetic Gauss' grid equation – 4th Maxwell's grid equation in global matrix form /
Magnetische Gaußsche Gittergleichung – 4. Maxwellsche Gittergleichung in globaler Matrixform

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\boldsymbol{\rho}_\mathrm{m}\}(t) = \{\mathbf{Q}_\mathrm{m}\}(t)$$

$[\mathbf{div}]$	$\in \mathbb{R}^{N\times 3N}$	Topological divergence operator in matrix form on the grid G / Topologischer Divergenzoperator in Matrixform auf dem Gitter G
$[\mathbf{S}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\{\mathbf{B}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$[\mathbf{V}]$	$\in \mathbb{R}^{N\times N}$	Diagonal matrix of elementary volumes on the grid G / Diagonalmatrix der Elementarvolumina auf dem Gitter G
$\{\rho_\mathrm{m}\}(t)$	$\in \mathbb{R}^{N}$	Algebraic magnetic charge density vector / Algebraischer magnetischer Ladungsdichtevektor
$\{\mathbf{Q}_\mathrm{m}\}(t)$	$\in \mathbb{R}^{N}$	Algebraic magnetic charge vector / Algebraischer magnetischer Ladungsvektor

FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Governing Analytic Equations

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

FIT Grid Equations

Maxwell's grid equations /
Maxwellsche Gittergleichungen

$$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}(t) = \left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$$

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\left\{\boldsymbol{\rho}_{\mathrm{m}}\right\}(t) = \left\{\mathbf{Q}_{\mathrm{m}}\right\}(t)$$

FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

Dispersion relation for a plane wave propagation in 1-D und 3-D: /
Dispersionsrelation für die Ausbreitung einer ebenen Welle in 1D und 3D

$$\text{1D: } E_x(z,t) = E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)}$$

$$\text{3D: } \underline{\mathbf{E}}(\underline{\mathbf{R}},t) = E_0(\omega_0, \hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})}$$

where k_z is the z component of the wave vector $\underline{\mathbf{k}}$ /
wobei k_z die z-Komponente des Wellenvektors $\underline{\mathbf{k}}$ ist

$$\text{1D: } \underline{\mathbf{k}} = k_z \underline{\mathbf{e}}_z$$

$$\text{3D: } \underline{\mathbf{k}} = k_x \underline{\mathbf{e}}_x + k_y \underline{\mathbf{e}}_y + k_z \underline{\mathbf{e}}_z$$

with magnitude / mit dem Betrag

$$\text{1D: } |\underline{\mathbf{k}}| = k = \sqrt{\underline{\mathbf{k}} \bullet \underline{\mathbf{k}}} = \sqrt{k_z^2} = |k_z|$$

$$\text{3D: } |\underline{\mathbf{k}}| = k = \sqrt{\underline{\mathbf{k}} \bullet \underline{\mathbf{k}}} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

Insert the plane wave ansatz into the scalar 1-D and vector 3-D wave equation /
Setze den ebenen Wellenansatz in die skalare 1D und vektorielle 3D Wellengleichung ein

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

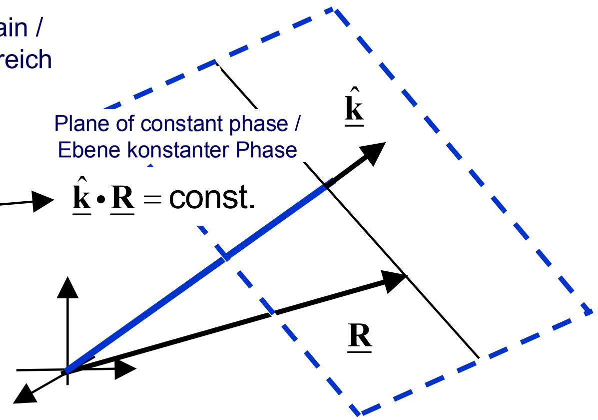
$$\Delta \underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}(\underline{\mathbf{R}},t) = 0$$

Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order / Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung

Complex Monofrequent (monochromatic) plane wave in the time domain / Komplexe monofrequente (monochromatische) ebene Welle im Zeitbereich

$$E_x(\underline{\mathbf{R}},t) = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\left(\omega_0 t - k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}\right)}$$

$$= E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}\omega_0 t}\, \mathrm{e}^{\mathrm{j}k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}}$$



Wave vector / Wellenvektor	$\underline{\mathbf{k}} = \mathrm{k}_x\underline{\mathbf{e}}_x + \mathrm{k}_z\underline{\mathbf{e}}_z + \mathrm{k}_z\underline{\mathbf{e}}_z = \mathrm{k}_z\underline{\mathbf{e}}_z$
Magnitude of the wave vector / Betrag des Wellenvektors	$\lvert\underline{\mathbf{k}}\rvert = \sqrt{\underline{\mathbf{k}}\bullet\underline{\mathbf{k}}} = \sqrt{\mathrm{k}_x^2 + \mathrm{k}_y^2 + \mathrm{k}_z^2} = \sqrt{\mathrm{k}_z^2} = \lvert k_z\rvert = k$
Wavenumber / Wellenzahl	$k = \dfrac{\omega_0}{c}$
Circular frequency / Kreisfrequenz	$\omega_0 = 2\pi f_0$
Propagation direction / Ausbreitungsrichtung	$\hat{\underline{\mathbf{k}}} = \dfrac{\underline{\mathbf{k}}}{\lvert\underline{\mathbf{k}}\rvert} = \dfrac{\mathrm{k}_z\underline{\mathbf{e}}_z}{\lvert\mathrm{k}_z\rvert} = \dfrac{\mathrm{sgn}(\mathrm{k}_z)\lvert\mathrm{k}_z\rvert\underline{\mathbf{e}}_z}{k} = \dfrac{\mathrm{sgn}(\mathrm{k}_z)k\underline{\mathbf{e}}_z}{k} = \mathrm{sgn}(\mathrm{k}_z)\underline{\mathbf{e}}_z$
Phase of the plane wave / Phase der ebenen Welle	$k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}} = k\,\mathrm{sgn}(\mathrm{k}_z)\underline{\mathbf{e}}_z\bullet(x\underline{\mathbf{e}}_x + y\underline{\mathbf{e}}_z + z\underline{\mathbf{e}}_z) = k\,\mathrm{sgn}(\mathrm{k}_z)z\underline{\mathbf{e}}_z\bullet\underline{\mathbf{e}}_z = k\,\mathrm{sgn}(\mathrm{k}_z)z$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

We compute / Wir berechnen

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

$$\frac{\partial^2}{\partial z^2} E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} = 0$$

$$(\mathrm{j}k_z)^2 E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} - \frac{1}{c_0^2}(\mathrm{j}\omega_0)^2 E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} = 0$$

$$\underbrace{\left[-k_z^2 + \frac{\omega_0^2}{c_0^2}\right]}_{\text{Dispersion relation / Dispersionsrelation}} E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}(\omega_0 t - k_z z)} = 0$$

$$k_z^2 = k^2 = \frac{\omega_0^2}{c_0^2}$$

Dispersion relation of a monochromatic plane wave / Dispersionsrelation einer monochromatischen ebenen Welle

$$k = \frac{\omega_0}{c_0} \quad\rightarrow\quad \omega_0(k) = c_0 k \quad\rightarrow\quad \omega_0 \rightarrow \omega \quad \omega(k) = c_0 k$$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

We compute in the 1-D case / Wir berechnen im 1D-Fall

$$\Delta \underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = 0$$

$$\Delta E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} = 0$$

$$\mathrm{j}^2 (k_x^2 + k_y^2 + k_z^2) E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} - \frac{1}{c_0^2}(\mathrm{j}\omega_0)^2 E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} = 0$$

$$\underbrace{\left[-(k_x^2 + k_y^2 + k_z^2) + \frac{\omega_0^2}{c_0^2}\right]}_{\text{Dispersion relation / Dispersionsrelation}} E_0(\omega_0,\hat{\underline{\mathbf{k}}})\, \mathrm{e}^{-\mathrm{j}(\omega_0 t - \underline{\mathbf{k}} \bullet \underline{\mathbf{R}})} = 0$$

Dispersion relation /
Dispersionsrelation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \frac{\omega_0^2}{c_0^2}$$

Dispersion relation of a monochromatic plane wave /
Dispersionsrelation einer monochromatischen ebenen Welle

$$k = \frac{\omega_0}{c_0} \quad \rightarrow \quad \omega_0(k) = c_0 k \quad \rightarrow \quad \omega_0 \rightarrow \omega \quad \omega(k) = c_0 k$$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

Dispersion relation for a plane wave /
Dispersionsrelation für eine ebene Welle

$$k = \frac{\omega_0}{c_0} \rightarrow k(\omega) = \frac{\omega}{c_0}$$

$$\omega(k) = k\,c_0$$

This means that the circular frequency is a function of k /
Dies bedeutet, dass die Kreisfrequenz eine Funktion von k ist

Dispersion relation / Dispersionsrelation

$$\omega(k)$$

We define now / Wir definieren nun:

- Phase velocity / Phasengeschwindigkeit
- Phase velocity vector / Phasengeschwindigkeitsvektor
- Group or energy velocity / Gruppen- oder Energiegeschwindigkeit
- Group or energy velocity vector / Gruppen- oder Energiegeschwindigkeitsvektor

Dispersion Relation and Phase, Group, and Energy Velocities / Dispersionsrelation und Phasen-, Gruppen- und Energiegeschwindigkeiten

Dispersion relation / Dispersionsrelation $\omega(k)$

- Phase velocity / Phasengeschwindigkeit

$$c_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}$$

- Phase velocity vector / Phasengeschwindigkeitsvektor

$$\text{1D:} \quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}\,\mathrm{sgn}(k_z)$$

$$\text{3D:} \quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}\,\hat{\underline{\mathbf{k}}}$$

- Group or energy velocity / Gruppen- oder Energiegeschwindigkeit

$$\text{1D:} \quad c_{\mathrm{gr}}(\omega,k) = c_{\mathrm{E}}(\omega,k) = \frac{d}{dk}\omega(k)$$

$$\text{3D:} \quad c_{\mathrm{gr}}(\omega,\underline{\mathbf{k}}) = c_{\mathrm{E}}(\omega,\underline{\mathbf{k}}) = \left|\nabla_{\underline{\mathbf{k}}}\omega(k)\right|$$

- Group or energy velocity vector / Gruppen- oder Energiegeschwindigkeitsvektor

$$\text{1D:} \quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,k) = \underline{\mathbf{c}}_{\mathrm{E}}(\omega,k) = \frac{d}{dk}\omega(k)\hat{\underline{\mathbf{k}}}$$

$$\text{3D:} \quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,\underline{\mathbf{k}}) = \underline{\mathbf{c}}_{\mathrm{E}}(\omega,\underline{\mathbf{k}}) = \nabla_{\underline{\mathbf{k}}}\omega(k)$$

Gradient with regard to the wave vector $\underline{\mathbf{k}}$ /
Gradient bezüglich des Wellenvektors $\underline{\mathbf{k}}$

$$\nabla_{\underline{\mathbf{k}}} = \underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z}$$

Dispersion Relation and Phase, Group, and Energy Velocities for a Monochromatic Plane Wave / Dispersionsrelation und Phasen-, Gruppen- und Energiegeschwindigkeiten für eine monochromatische ebene Welle

Dispersion relation for a monochromatic plane wave / Dispersionsrelation für eine monochromatische ebene Welle

$$\omega(k) = kc_0$$

- Phase velocity / Phasengeschwindigkeit

$$c_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k} = \frac{kc_0}{k} = c_0$$

- Phase velocity vector / Phasengeschwindigkeitsvektor

$$\text{1D:}\quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}\,\mathrm{sgn}(k_z) = c_0\,\mathrm{sgn}(k_z)$$

$$\text{3D:}\quad \underline{\mathbf{c}}_{\mathrm{ph}}(\omega,k) = \frac{\omega(k)}{k}\,\hat{\underline{\mathbf{k}}} = c_0\hat{\underline{\mathbf{k}}}$$

- Group or energy velocity / Gruppen- oder Energiegeschwindigkeit

$$\text{1D:}\quad c_{\mathrm{gr}}(\omega,k) = c_{\mathrm{E}}(\omega,k) = \frac{d}{dk}\omega(k) = \frac{d}{dk}kc_0 = c_0$$

$$\text{3D:}\quad c_{\mathrm{gr}}(\omega,\underline{\mathbf{k}}) = c_{\mathrm{E}}(\omega,\underline{\mathbf{k}}) = \left|\nabla_{\underline{\mathbf{k}}}\omega(k)\right| = \left|\nabla_{\underline{\mathbf{k}}}kc_0\right| = c_0$$

- Group or energy velocity vector / Gruppen- oder Energiegeschwindigkeitsvektor

$$\text{1D:}\quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,k) = \underline{\mathbf{c}}_{\mathrm{E}}(\omega,k) = \frac{d}{dk}\omega(k)\hat{\underline{\mathbf{k}}} = c_0\hat{\underline{\mathbf{k}}}$$

$$\text{3D:}\quad \underline{\mathbf{c}}_{\mathrm{gr}}(\omega,\underline{\mathbf{k}}) = \underline{\mathbf{c}}_{\mathrm{E}}(\omega,\underline{\mathbf{k}}) = \nabla_{\underline{\mathbf{k}}}\omega(k) = c_0\hat{\underline{\mathbf{k}}}$$

Analytical and Numerical Dispersion Relation
Analytische und Numerische Dispersionsrelation

$$\nabla_{\underline{\mathbf{k}}} = \underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z}$$

$$\begin{aligned}
\nabla_{\underline{\mathbf{k}}} k c_0 &= \left(\underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z} \right) \sqrt{k_x^2 + k_y^2 + k_z^2}\, c_0 \\
&= \left(\underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z} \right) \sqrt{k_x^2 + k_y^2 + k_z^2}\, c_0 \\
&= c_0 \left(\underline{\mathbf{e}}_x \frac{\partial}{\partial k_x} + \underline{\mathbf{e}}_y \frac{\partial}{\partial k_y} + \underline{\mathbf{e}}_z \frac{\partial}{\partial k_z} \right) \sqrt{k_x^2 + k_y^2 + k_z^2} \\
&= c_0 \left[\frac{1}{2} \frac{2 k_x \underline{\mathbf{e}}_x}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{1}{2} \frac{2 k_y \underline{\mathbf{e}}_y}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{1}{2} \frac{2 k_z \underline{\mathbf{e}}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \right] \\
&= c_0 \left[\frac{k_x \underline{\mathbf{e}}_x}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{k_y \underline{\mathbf{e}}_y}{\sqrt{k_x^2 + k_y^2 + k_z^2}} + \frac{k_z \underline{\mathbf{e}}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \right] \\
&= c_0 \frac{k_x \underline{\mathbf{e}}_x + k_y \underline{\mathbf{e}}_y + k_z \underline{\mathbf{e}}_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \\
&= c_0 \hat{\underline{\mathbf{k}}}
\end{aligned}$$

Numerical Dispersion Relation / Numerische Dispersionsrelation

Numerical dispersion relation for the FD, FDTD, and FIT algorithms /
Numerische Dispersionsrelation für die FD-, FDTD- und FIT-Algorithmen

3-D cse / 3D-Fall

$$\frac{1}{(c_0\Delta t)^2}\sin^2\left(\frac{\omega_0\Delta t}{2}\right)=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)+\frac{1}{(\Delta z)^2}\sin^2\left(\frac{k_z\Delta z}{2}\right)$$

2-D cse / 2D-Fall

$$\frac{1}{(c_0\Delta t)^2}\sin^2\left(\frac{\omega_0\Delta t}{2}\right)=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta z}{2}\right)$$

1-D cse / 1D-Fall

$$\frac{1}{(c_0\Delta t)^2}\sin^2\left(\frac{\omega_0\Delta t}{2}\right)=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)$$

3-D cse / 3D-Fall

$$\omega\left(k_x,k_y,k_z,\Delta x,\Delta y,\Delta z,\Delta t\right)=\frac{2}{\Delta t}\arcsin\left\{c_0\Delta t\sqrt{\left[\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)+\frac{1}{(\Delta z)^2}\sin^2\left(\frac{k_z\Delta z}{2}\right)\right]}\right\}$$

2-D cse / 2D-Fall

$$\omega\left(k_x,k_y,\Delta x,\Delta y,\Delta t\right)=\frac{2}{\Delta t}\arcsin\left\{c_0\Delta t\sqrt{\left[\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)\right]}\right\}$$

1-D cse / 1D-Fall

$$\omega\left(k_x,\Delta x,\Delta t\right)=\frac{2}{\Delta t}\arcsin\left\{\frac{c_0\Delta t}{\Delta x}\sqrt{\sin^2\left(\frac{k_x\Delta x}{2}\right)}\right\}$$

Numerical Dispersion Relation / Numerische Dispersionsrelation

1-D case
1D-Fall

$$\omega\left(k_x, \Delta x, \Delta t\right) = \frac{2}{\Delta t} \arcsin\left\{ \frac{c_0 \Delta t}{\Delta x} \sin\left(\frac{\left|k_x\right| \Delta x}{2} \right) \right\}$$

$$\frac{c_0 \Delta t}{\Delta x} = \widehat{\Delta t}$$

$$\left|k_x\right| = k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{\left|k_x\right|}$$

$$G = \frac{\lambda}{\Delta x} = \frac{2\pi}{\left|k_x\right| \Delta x}$$

1-D case
1D-Fall

$$\omega\left(k_x, \Delta x, \Delta t\right) = \frac{2}{\Delta t} \arcsin\left\{ \underbrace{\frac{c_0 \Delta t}{\Delta x}}_{=\widehat{\Delta t}} \sin\left(\underbrace{\frac{\left|k_x\right| \Delta x}{2}}_{=\frac{\pi}{G}} \right) \right\}$$

$$= \frac{2}{\Delta t} \arcsin\left\{ \widehat{\Delta t} \sin\left(\frac{\pi}{G} \right) \right\}$$

1-D Numerical Dispersion Relation / 1D Numerische Dispersionsrelation

Numerical dispersion relation / Numerische Dispersionsrelation

$$\omega\left(k_x, \Delta x, \Delta t\right) = \frac{2}{\Delta t} \arcsin\left\{\frac{c_0 \Delta t}{\Delta x} \sin\left(\frac{|k_x| \Delta x}{2}\right)\right\}$$

$$\omega\left(k_x, \Delta x, \Delta t\right) = \frac{2}{\Delta t} \arcsin\left\{\widehat{\Delta t} \sin\left(\frac{\pi}{G}\right)\right\}$$

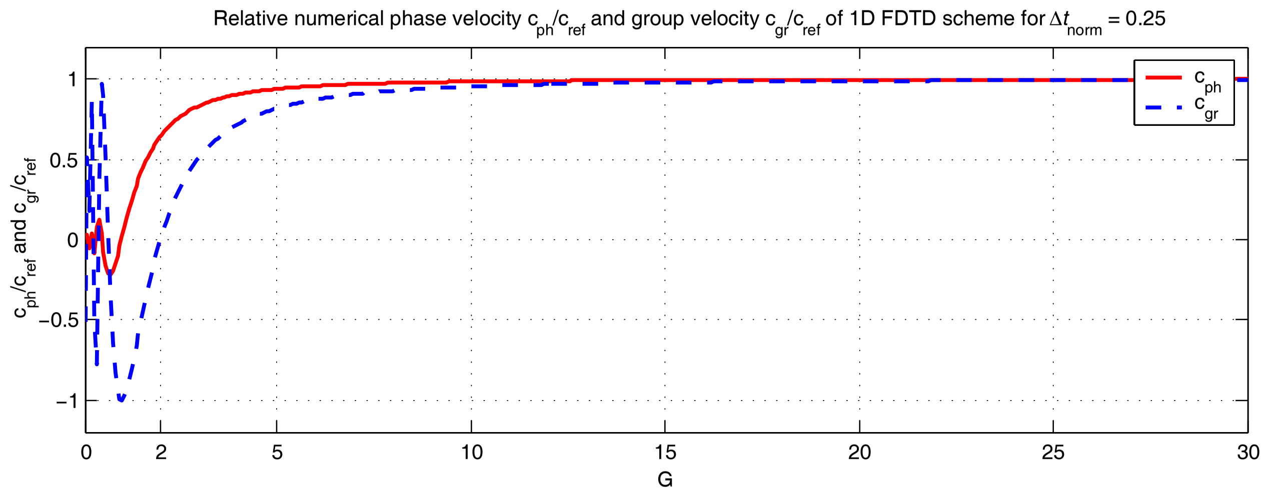
Numerical phase velocity / Numerische Phasengeschwindigkeit

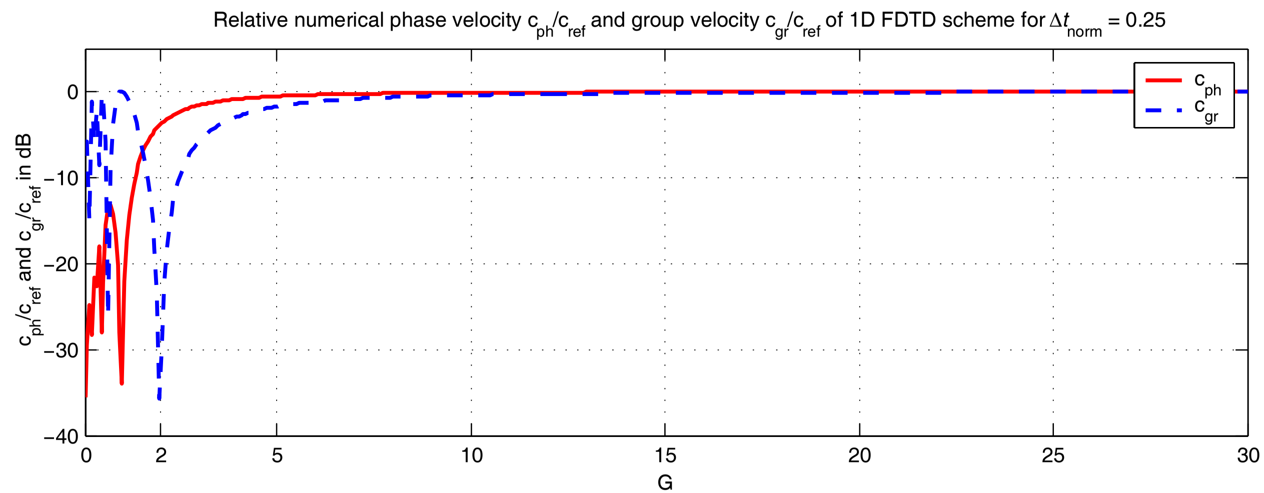
$$\frac{c_{\mathrm{ph}}\left(G, \widehat{\Delta t}\right)}{c_0} = \frac{G}{\pi \widehat{\Delta t}} \arcsin\left\{\widehat{\Delta t} \sin\left(\frac{\pi}{G}\right)\right\}$$

Numerical group velocity / Numerische Gruppengeschwindigkeit

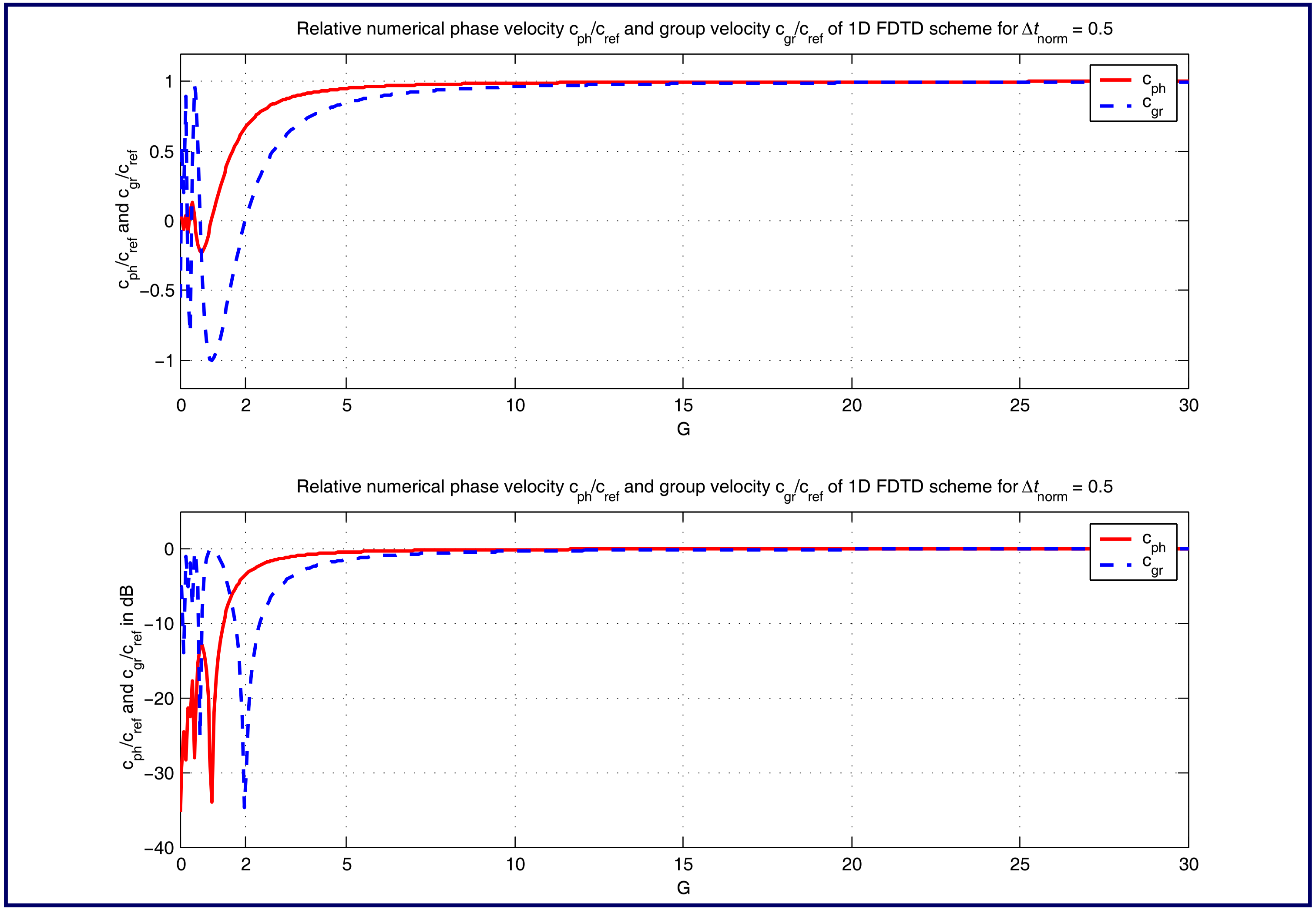
$$\frac{c_{\mathrm{gr}}\left(G, \widehat{\Delta t}\right)}{c_0} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{1 - \widehat{\Delta t}^2 \sin^2\left(\frac{\pi}{G}\right)}}$$

1-D Numerical Dispersion Relation /
1D Numerische Dispersionsrelation





1-D Numerical Dispersion Relation / 1D Numerische Dispersionsrelation



1-D Numerical Dispersion Relation – Magic Time Step / 1D Numerische Dispersionsrelation – Magische Zeitschrittweite

$$\widehat{\Delta t} = 1$$

Numerical phase velocity / Numerische Phasengeschwindigkeit

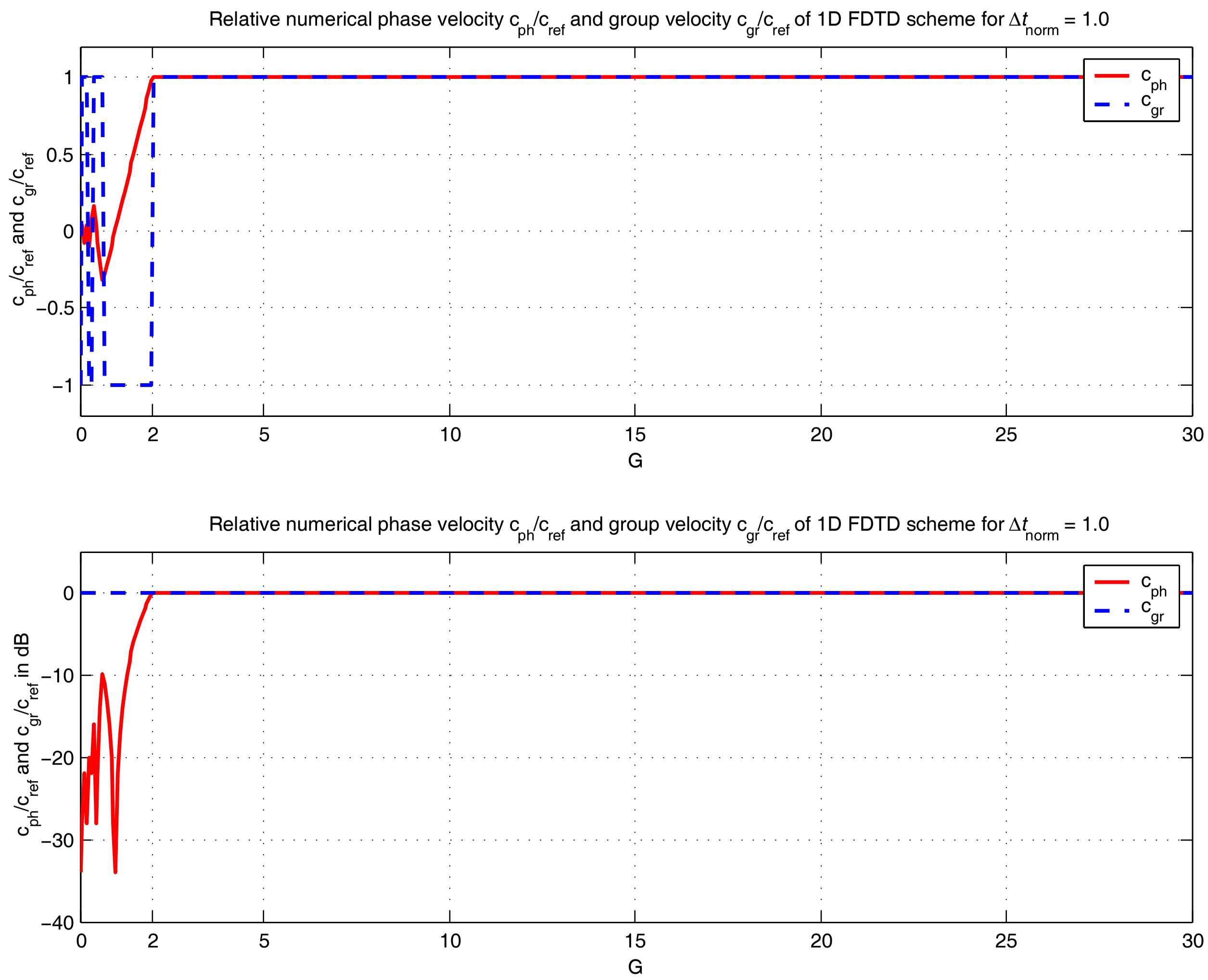
$$\frac{c_{\mathrm{ph}}\left(G,\widehat{\Delta t}\right)}{c_0} = \frac{G}{\pi\widehat{\Delta t}}\arcsin\left\{\widehat{\Delta t}\sin\left(\frac{\pi}{G}\right)\right\} = \frac{G}{\pi}\arcsin\left\{\sin\left(\frac{\pi}{G}\right)\right\} = \frac{G}{\pi}\frac{\pi}{G} = 1$$

Numerical group velocity / Numerische Gruppengeschwindigkeit

$$\frac{c_{\mathrm{gr}}\left(G,\widehat{\Delta t}\right)}{c_0} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{1-\widehat{\Delta t}^2\sin^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{1-\sin^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{\cos^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\sqrt{\cos^2\left(\frac{\pi}{G}\right)}} = \frac{\cos\left(\frac{\pi}{G}\right)}{\cos\left(\frac{\pi}{G}\right)} = 1$$

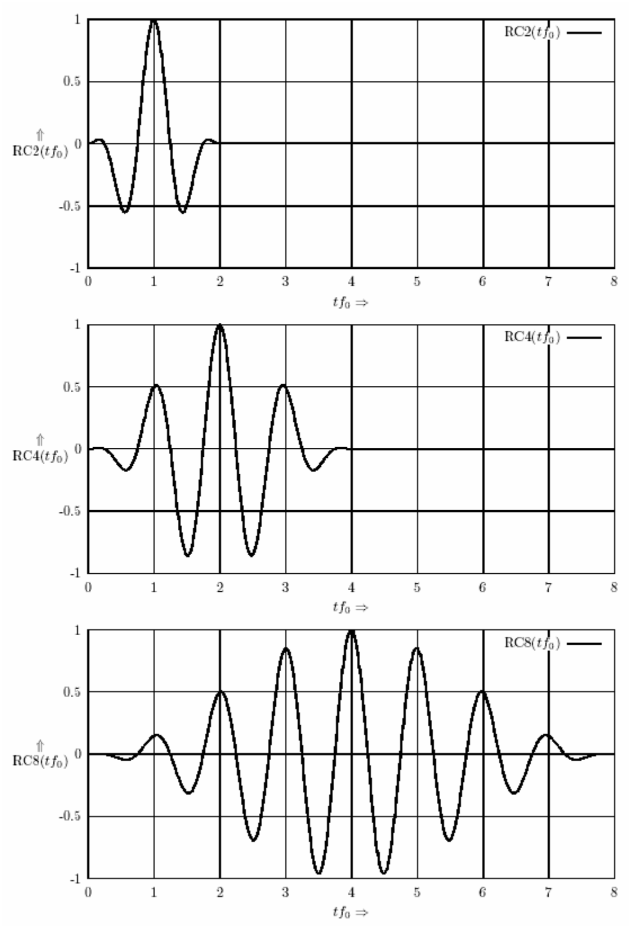
⇨ No numerical dispersion / Keine numerische Dispersion **!**

1-D Numerical Dispersion Relation – Magic Time Step / 1D Numerische Dispersionsrelation – Magische Zeitschrittweite

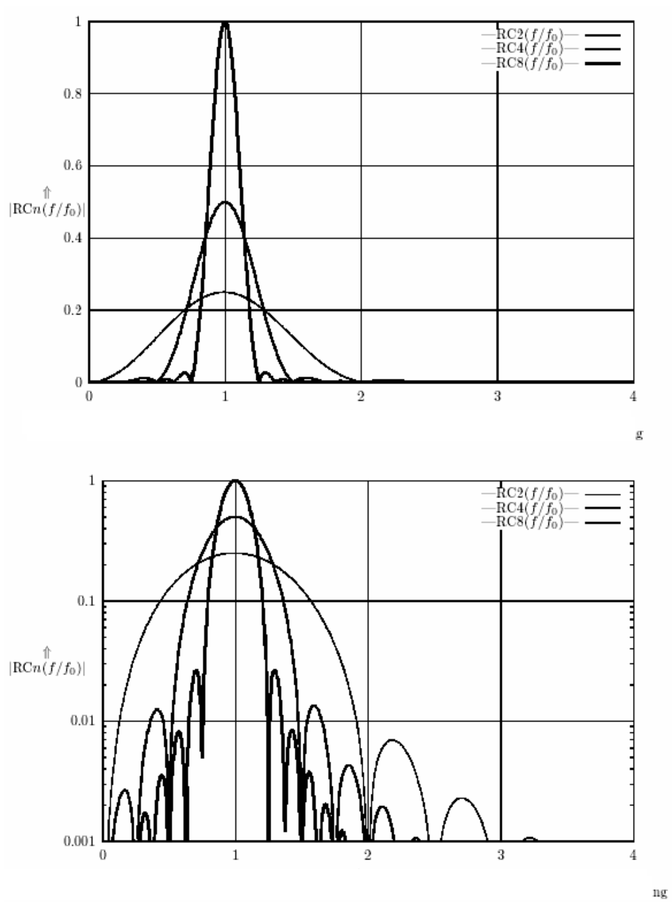


Numerical Dispersion – Test Function / Numerische Dispersion – Testfunktion

Time History of the RC2, RC4, and RC8 Excitation Function / Zeitfunktion des RC2-, RC4- und RC8-Anregungsimpulses



Time Spectrum of the RC2, RC4, and RC8 Excitation Function / Zeitspektrum des RC2-, RC4- und RC8-Anregungsimpulses



Numerical Dispersion of an RC2 Pulse / Numerische Dispersion eines RC2-Impulses

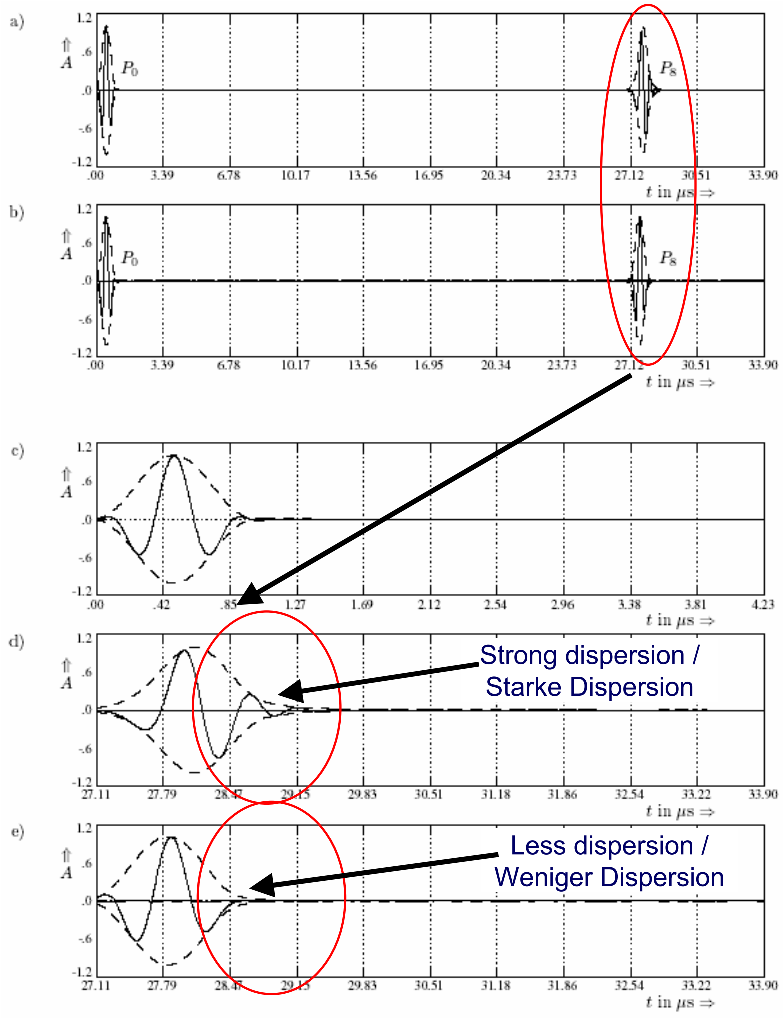
Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordung in der Zeit (4S2T)

Original RC2 Pulse / Originaler RC2-Impuls

Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordnung in der Zeit (4S2T)



Numerical Dispersion of an RC4 Pulse / Numerische Dispersion eines RC4-Impulses

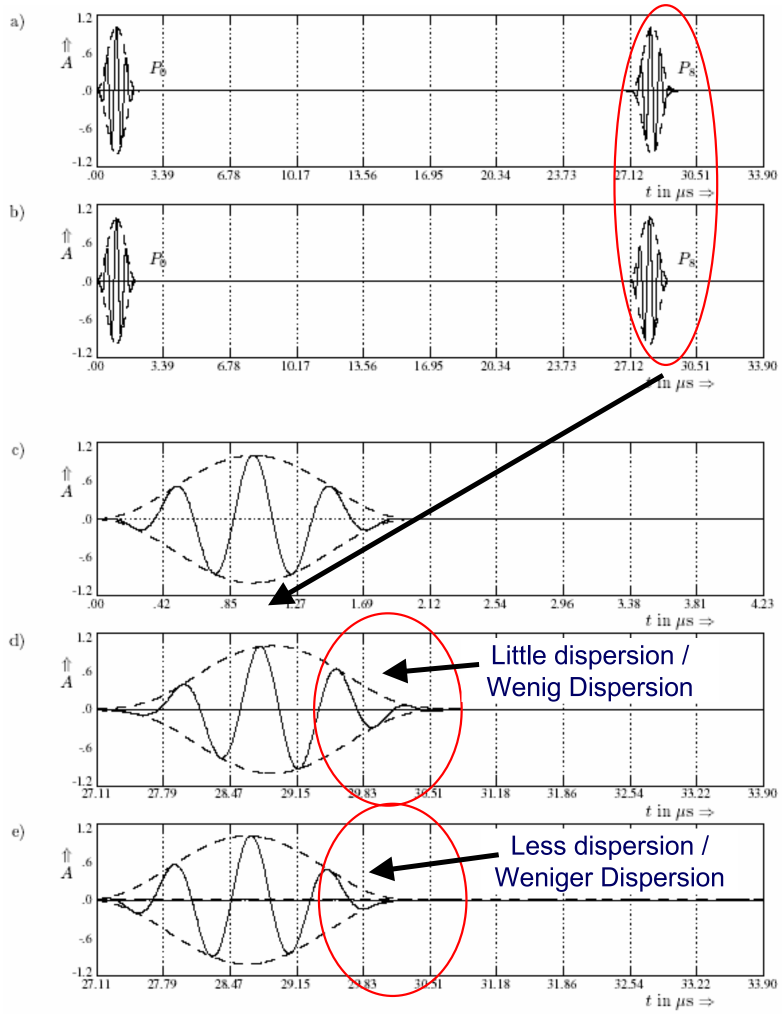
Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordung in der Zeit (4S2T)

Original RC4 Pulse / Originaler RC4-Impuls

Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordnung in der Zeit (4S2T)



Numerical Dispersion of an RC8 Pulse / Numerische Dispersion eines RC8-Impulses

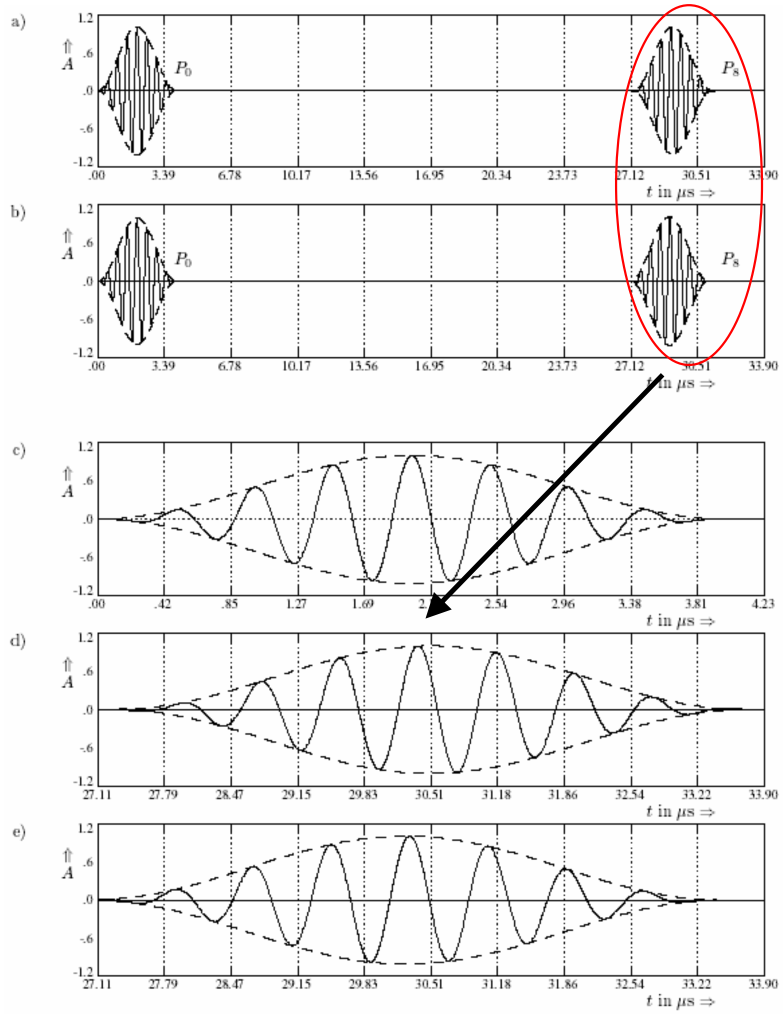
Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordung in der Zeit (4S2T)

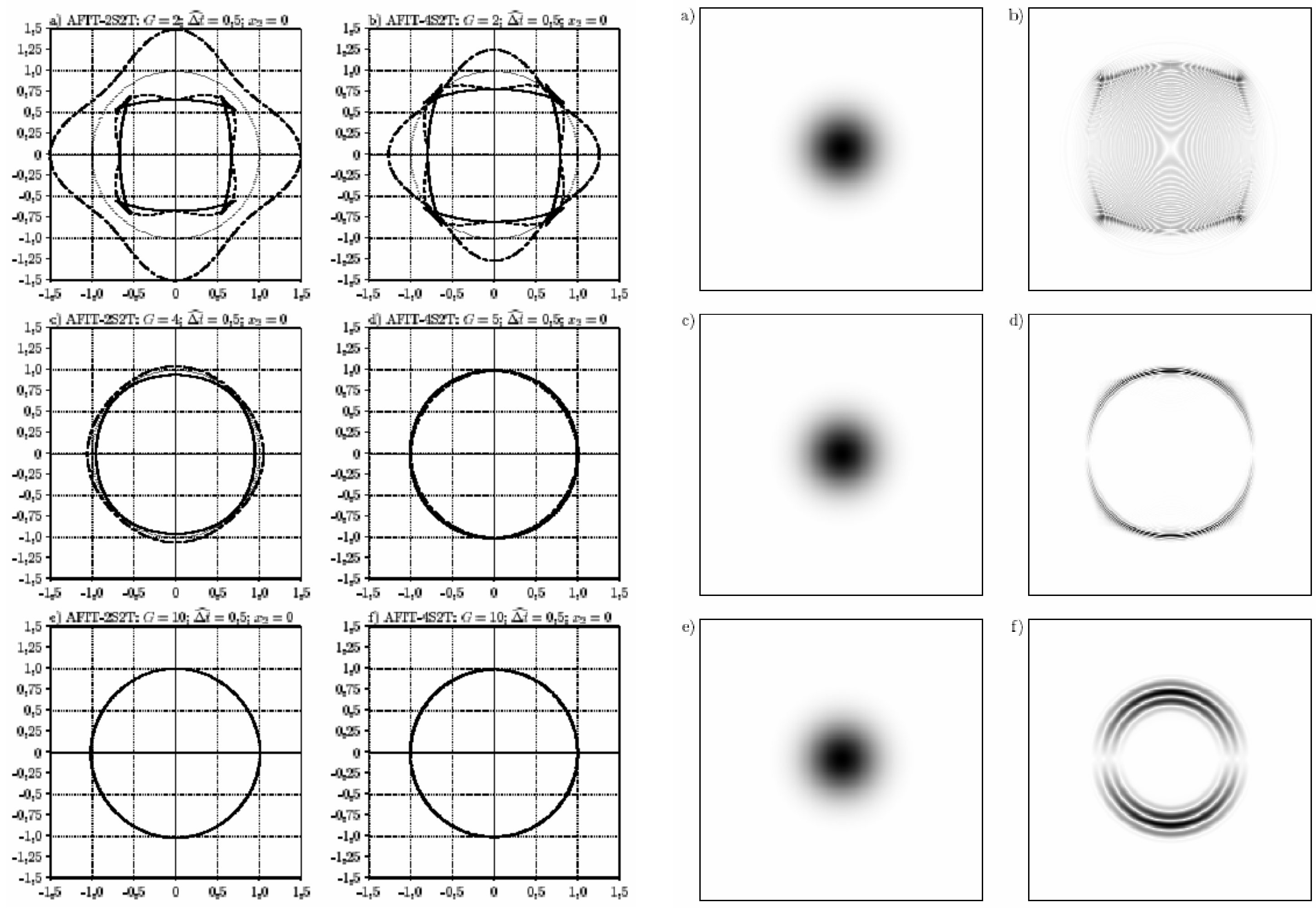
Original RC8 Pulse / Originaler RC8-Impuls

Second Order Scheme in Space and Time (2S2T) / Schema zweiter Ordnung im Raum und in der Zeit (2S2T)

Fourth Order Scheme in Space und Second Order Scheme in Time (4S2T) / Schema vierter Ordnung im Raum und zweiter Ordnung in der Zeit (4S2T)



2-D Numerical Dispersion Relation – Numerical Anisotropy / 2D Numerische Dispersionsrelation – Numerische Anisotropie



FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Derivation of the Numerical Dispersion Relation for the 1-D FD Scheme of 2nd Order / Ableitung der numerischen Dispersionsrelation für das 1D-FD-Schema 2ter Ordnung

Stability by the *von Neumann*'s method (Fourier series method):

Insert a complex monofrequent (monochromatic) plane wave into the discrete FD equations and analyze the spectral radius of the amplification matrix, where the spectral radius must be smaller equal one.

Stabilität durch die *von Neumann*sche Methode (Fourier-Reihen-Methode):

Setze eine komplex monofrequente (monochromatische) ebene Welle in die diskreten FD-Gleichungen ein und analysiere den spektralen Radius der Verstärkungsmatrix, wobei der spektrale Radius kleinergleich Eins sein muss.

Complex monofrequent (monochromatic) plane wave / Komplex monofrequente (monochromatische) ebene Welle

$$E_x(\underline{\mathbf{R}},t) = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\left(\omega_0 t - k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}\right)} = E_0(\omega_0,\hat{\underline{\mathbf{k}}})\,\mathrm{e}^{-\mathrm{j}\omega_0 t}\,\mathrm{e}^{\mathrm{j}k\hat{\underline{\mathbf{k}}}\bullet\underline{\mathbf{R}}}$$

$$\{\mathbf{W}\}^{(n_t+1)} = [\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\,\{\mathbf{W}\}^{(n_t)}$$

$[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$: Amplification matrix / Verstärkungsmatrix

Spectral radius / Spektraler Radius $\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) \le 1$ of the matrix / der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

where / wobei $\rho\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right) = \max\limits_{n=1,\dots,N} \left|\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)\right|$

$\nu_n\left([\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}\right)$: nth eigenvalue of the matrix / n-ter Eigenwert der Matrix $[\mathbf{G}]_{1\mathrm{D}}^{\mathrm{FD}}$

FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
- **Dispersion / Dispersion**
- **Stability Condition / Stabilitätsbedingung** $\Delta t = f(\Delta z)$
- **Convergence / Konvergenz**

Consistency / Konsistenz

Consistency

Consistency means that the discrete equations result in the analytical equations by calculating the limit $\{\Delta z, \Delta t\} \rightarrow 0$.

We can prove the consistency of the 1-D FD scheme using the above numerical dispersion relation. We show, that the grid dispersion relation reaches in the limit $\{\Delta z, \Delta t\} \rightarrow 0$ the analytical dispersion relation for a plane wave as a solution of the homogeneous Helmholtz equation.

Konsistenz

Konsistenz bedeutet, dass die diskreten Gleichungen bei dem Grenzübergang $\{\Delta z, \Delta t\} \rightarrow 0$ in die analytischen Gleichungen übergehen.

Wir können die Konsistenz des 1D-FD-Schemas anhand der numerischen Dispersionsrelation überprüfen. Wir zeigen, dass die numerische Dispersionsrelation bei dem Grenzübergang $\{\Delta z, \Delta t\} \rightarrow 0$ in die analytische Dispersionsrelation einer ebene Welle als Lösung der homogenen Helmholtz-Gleichung übergeht.

Consistency / Konsistenz

$$\lim_{\{\Delta z, \Delta t\} \to 0} \left[\frac{1}{(c_0 \Delta t)^2} \sin^2\left(\frac{\omega_0 \Delta t}{2}\right) = \frac{1}{(\Delta z)^2} \sin^2\left(\frac{\mathrm{k}_z \Delta z}{2}\right) \right]$$

$$\lim_{\{\Delta z, \Delta t\} \to 0} \left[\frac{1}{c_0 \Delta t} \sin\left(\frac{\omega_0 \Delta t}{2}\right) = \frac{1}{\Delta z} \sin\left(\frac{|\mathrm{k}_z| \Delta z}{2}\right) \right]$$

$$\lim_{\Delta t \to 0} \sin\left(\frac{\omega_0 \Delta t}{2}\right) = \frac{\omega_0 \Delta t}{2}$$

$$\lim_{\Delta z \to 0} \sin\left(\frac{|\mathrm{k}_z| \Delta z}{2}\right) = \frac{|\mathrm{k}_z| \Delta z}{2}$$

$$\frac{1}{c_0 \Delta t} \frac{\omega_0 \Delta t}{2} = \frac{1}{\Delta z} \frac{|\mathrm{k}_z| \Delta z}{2}$$

$$\underbrace{\left(\frac{\omega_0}{c_0}\right)}_{=k_0} = |\mathrm{k}_z|$$

$$k_0 = |\mathrm{k}_z|$$

FD Method – Properties / FD-Methode - Eigenschaften

- **Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung** $\Delta z = ?$ $\Delta t = ?$
- **Consistency / Konsistenz**
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- **Convergence / Konvergenz**

Convergence / Konvergenz

Convergence

The importance of the concept of consistency and stability is seen in the Lax-Richtmyer equivalence theorem, which is the fundamental theorem in the theory of finite difference schemes for initial value problems.

Lax-Richtmyer Equivalence Theorem

A consistent finite difference scheme for a partial differential equations of which the initial value problem is well-posed is convergent if and only if it is stable.

Konvergenz

Die Wichtigkeit des Konzeptes der Konsistenz und Stabilität kann man an dem Lax-Richtmyer-Äquivalenztheorem sehen, welches ein fundamentales Theorem in der Theorie der Finite Differenzen zur Lösung eines Anfangswertproblems darstellt.

Lax-Richtmyer Äquivalenztheorem

Ein konsistentes Finite Differenzen Schema für eine partielle Differentialgleichung eines gut gestellten Anfangswertproblems ist konvergent, wenn und nur wenn es stabil ist.

FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Governing Analytic Equations Maxwell's equations in integral form / Maxwellsche Gleichungen in Integralform	**FIT Grid Equations** Maxwell's grid equations / Maxwellsche Gittergleichungen
$\frac{d}{dt}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$	$[\mathbf{S}]\frac{d}{dt}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$
$\frac{d}{dt}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$	$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{d}{dt}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\mathbf{v}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$
$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$	$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}(t) = \{\mathbf{Q}_{\mathrm{e}}\}(t)$
$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$	$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\boldsymbol{\rho}_{\mathrm{m}}\}(t) = \{\mathbf{Q}_{\mathrm{m}}\}(t)$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form /
Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}(t) = \{\mathbf{Q}_\mathrm{e}\}(t)$$

$$\xrightarrow{\frac{\partial}{\partial t} \equiv 0}$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\} = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\} = \{\mathbf{Q}_\mathrm{e}\}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) = \rho_\mathrm{e}(\underline{\mathbf{R}}) &= \nabla \bullet \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \\ &= \nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} \\ &= -\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} = -\rho_\mathrm{e}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta} \Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_e(\underline{\mathbf{R}})$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \rho_e(\underline{\mathbf{R}}) \\ &= \nabla \bullet \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \\ &= \nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \\ &= -\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dS}} &= \iiint_V \rho_e(\underline{\mathbf{R}})\,\mathrm{d}V \\ &= \oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dS}} \\ &= \oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}}\nabla \bullet \\ &= -\oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}}\nabla \bullet\end{aligned}$$

$$-\oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}} = \iiint_V \rho_e(\underline{\mathbf{R}})\,\mathrm{d}V$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\right\} = -\rho_e(\underline{\mathbf{R}})$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta}\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Differential form / Differentialform

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_e(\underline{\mathbf{R}})$$

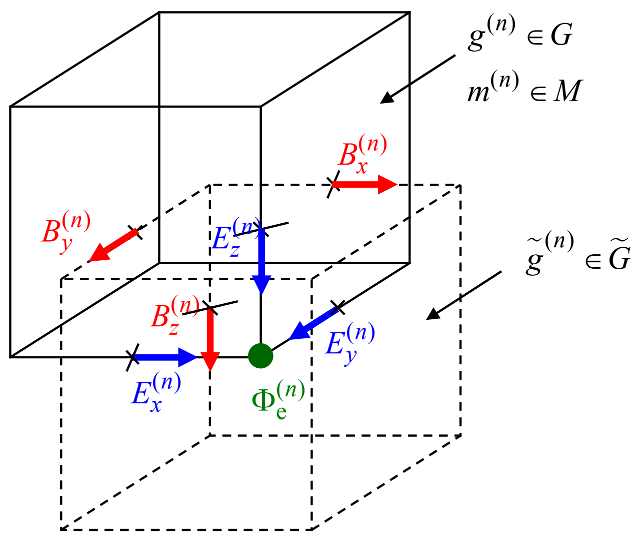
Integral form / Integralform

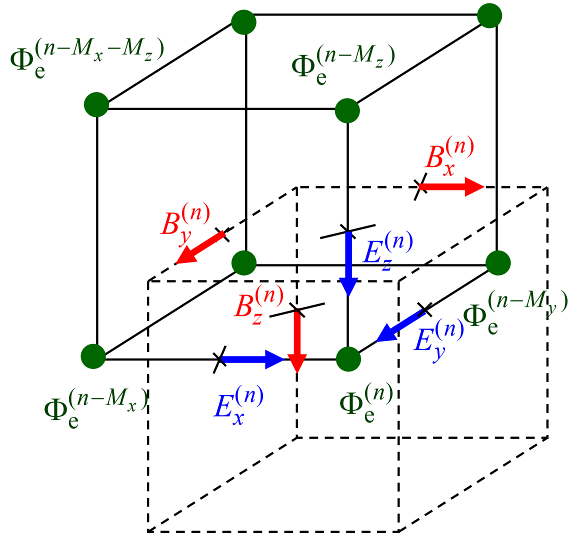
$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \mathbf{d}\underline{\mathbf{R}} = -\int_C \nabla\Phi_e(\underline{\mathbf{R}}) \cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= -\left[\Phi_e(\underline{\mathbf{R}}_2) - \Phi_e(\underline{\mathbf{R}}_1)\right]$$

FIT grid equation / FIT-Gittergleichung

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\{\Phi_e\}^{(n)}$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_e\}$$





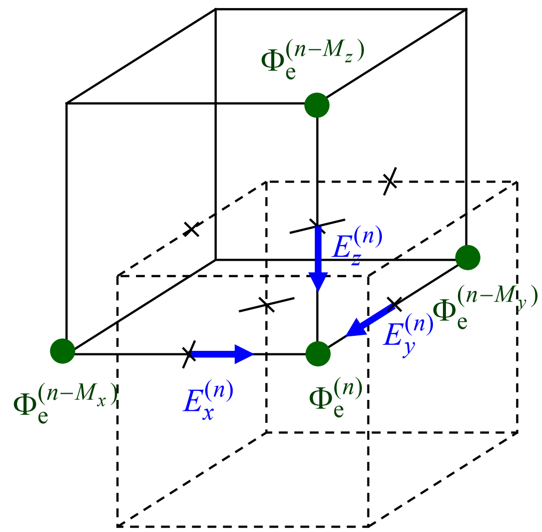
FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Integral form / Integralform

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}} = -\int_C \left[\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right] \cdot \underline{\mathbf{dR}} = -\left[\Phi_{\mathrm{e}}(\underline{\mathbf{R}}_2) - \Phi_{\mathrm{e}}(\underline{\mathbf{R}}_1)\right]$$

$$\begin{aligned}\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}} &= \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z) \cdot \underline{\mathbf{dR}} \\ &= \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z) \cdot \underline{\mathrm{e}}_x \, \mathrm{d}x \\ &= \int_{x=x_0}^{x_0+\Delta x} E_x(x,y,z) \, \mathrm{d}x \\ &= E_x^{(n)} \int_{x=x_0}^{x_0+\Delta x} \mathrm{d}x \\ &= E_x^{(n)} \Delta x\end{aligned}$$

$$\begin{aligned}\int_C \left[\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right] \cdot \underline{\mathbf{dR}} &= \int_{x=x_0}^{x_0+\Delta x} \left[\nabla \Phi_{\mathrm{e}}(x,y,z)\right] \cdot \underline{\mathbf{dR}} \\ &= \int_{x=x_0}^{x_0+\Delta x} \left[\nabla \Phi_{\mathrm{e}}(x,y,z)\right] \cdot \underline{\mathrm{e}}_x \, \mathrm{d}x \\ &= \int_{x=x_0}^{x_0+\Delta x} \frac{\partial}{\partial x} \Phi_{\mathrm{e}}(x,y,z) \, \mathrm{d}x \\ &= \Phi_{\mathrm{e}}(x_0,y,z) - \Phi_{\mathrm{e}}(x_0+\Delta x,y,z) \\ &= \Phi_{\mathrm{e}}^{(n-M_x)} - \Phi_{\mathrm{e}}^{(n)}\end{aligned}$$



$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}} = -\int_C \left[\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right] \cdot \underline{\mathbf{dR}} = -\left[\Phi_{\mathrm{e}}(\underline{\mathbf{R}}_2) - \Phi_{\mathrm{e}}(\underline{\mathbf{R}}_1)\right]$$

$$E_x^{(n)} \Delta x = -\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_x)} = -(I - S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}$$

$$E_y^{(n)} \Delta y = -\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_y)} = -(I - S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}$$

$$E_z^{(n)} \Delta z = -\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_z)} = -(I - S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \mathbf{d}\underline{\mathbf{R}} = -\int_C \left[\nabla \, \Phi_\mathrm{e}(\underline{\mathbf{R}}) \right] \cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1) \right]$$

$$\underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}} = -\underbrace{\begin{bmatrix} (I - S_{-M_x}) \\ (I - S_{-M_y}) \\ (I - S_{-M_z}) \end{bmatrix}}_{=-[\mathrm{grad}]} \Phi_\mathrm{e}^{(n)}$$

$$E_x^{(n)} \Delta x = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)}$$
$$= -(I - S_{-M_x}) \, \Phi_\mathrm{e}^{(n)}$$

$$E_y^{(n)} \Delta y = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)}$$
$$= -(I - S_{-M_y}) \, \Phi_\mathrm{e}^{(n)}$$

$$E_z^{(n)} \Delta z = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)}$$
$$= -(I - S_{-M_z}) \, \Phi_\mathrm{e}^{(n)}$$

$$[R]\{E\}^{(n)} = -[\mathrm{grad}]\Phi_\mathrm{e}^{(n)}$$

$$[R] = \begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix} \rightarrow [R]^{-1} = \begin{bmatrix} \frac{1}{\Delta x} & & \\ & \frac{1}{\Delta y} & \\ & & \frac{1}{\Delta z} \end{bmatrix}$$

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\Phi_\mathrm{e}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla \Phi_\mathrm{e}(\underline{\mathbf{R}})$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\}$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla \Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\right\} = -\rho_\mathrm{e}(\underline{\mathbf{R}})$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\} = -[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\}$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\widetilde{\boldsymbol{\rho}}_\mathrm{e}\}$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta} \Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta \Phi_\mathrm{e}(\underline{\mathbf{R}}) = -\frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Downarrow$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

with / mit

$$[\mathbf{A}] = [\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]$$

$$\{\mathbf{x}\} = \{\Phi_\mathrm{e}\}$$

$$\{\mathbf{b}\} = -[\widetilde{\mathbf{V}}]\{\widetilde{\boldsymbol{\rho}}_\mathrm{e}\}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left[\mathbf{R}\right]^{-1}\left[\mathbf{grad}\right]\left\{\Phi_{\mathrm{e}}\right\} = -\left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}$$

$$\left\{\Phi_{\mathrm{e}}\right\} = \begin{Bmatrix} \Phi_{\mathrm{e}}^{(1)}(t) \\ \Phi_{\mathrm{e}}^{(2)}(t) \\ \vdots \\ \Phi_{\mathrm{e}}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

$$\left[\widetilde{\mathbf{S}}\right] = \begin{bmatrix} \left[\mathrm{diag}\{\Delta y \Delta z\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\{\Delta x \Delta z\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\{\Delta x \Delta y\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$\left[\mathbf{R}\right] = \begin{bmatrix} \left[\mathrm{diag}\{\Delta x\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\{\Delta y\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\{\Delta z\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$\left[\mathbf{R}\right]^{-1} = \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$[\widetilde{\mathbf{S}}][\mathbf{R}]^{-1} = \begin{bmatrix} [\mathrm{diag}\{\Delta y \Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x \Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x \Delta y\}]_{N\times N} \end{bmatrix} \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{\Delta y \Delta z}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{\Delta x \Delta z}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{\Delta x \Delta y}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)D_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)D_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)D_z^{(n)}(t)\Delta x\Delta y$$

$$\oint\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}E_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}E_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}E_z^{(n)}(t)\Delta x\Delta y$$

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$E_x^{(n)} = -\frac{1}{\Delta x}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)}\right) = -\frac{1}{\Delta x}(I - S_{-M_x})\,\Phi_\mathrm{e}^{(n)}$$

$$E_y^{(n)} = -\frac{1}{\Delta y}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)}\right) = -\frac{1}{\Delta y}(I - S_{-M_y})\,\Phi_\mathrm{e}^{(n)}$$

$$E_z^{(n)} = -\frac{1}{\Delta z}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)}\right) = -\frac{1}{\Delta z}(I - S_{-M_z})\,\Phi_\mathrm{e}^{(n)}$$

$$\begin{aligned}\oint\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} &= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}E_x^{(n)}\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}E_y^{(n)}\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}E_z^{(n)}\Delta x\Delta y \\ &= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\right]\Delta y\Delta z \\ &\quad + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta z \\ &\quad + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta y\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\begin{aligned}\oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = {} & \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\right]\Delta y\Delta z\\ & +\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta z\\ & +\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta y\end{aligned}$$

$$\begin{aligned}&\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\right]\Delta y\Delta z\\ &+\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta z\\ &+\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)}\right]\Delta x\Delta y = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z\end{aligned}$$

$$\begin{aligned}&-\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)}\\ &-\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_\mathrm{e}^{(n)}\\ &-\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)} = \rho_\mathrm{e}^{(n)}\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)(\tilde{\varepsilon}_{xx}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)})$$

$$=\frac{1}{(\Delta x)^2}\left[S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}-\underbrace{S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\underbrace{S_{M_x}S_{-M_x}}_{=I}\Phi_{\mathrm{e}}^{(n)}}-\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}\right]$$

$$=\frac{1}{(\Delta x)^2}\left[\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n+M_x)}-\underbrace{\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}}_{=\left[\tilde{\varepsilon}_{xx}^{(n)}+\tilde{\varepsilon}_{xx}^{(n+M_x)}\right]\Phi_{\mathrm{e}}^{(n)}}+\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n-M_x)}\right]$$

$$=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[\underbrace{(I+S_{M_x})\tilde{\varepsilon}_{xx}^{(n)}}_{=2A_{M_x}}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_e^{(n)}=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_e^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_e^{(n)}$$

$$\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_e^{(n)}=\frac{1}{(\Delta y)^2}\left\{\tilde{\varepsilon}_{yy}^{(n+M_y)}\Phi_e^{(n+M_y)}-\left[2S_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]\Phi_e^{(n)}+\tilde{\varepsilon}_{yy}^{(n)}\Phi_e^{(n-M_y)}\right\}$$

$$=\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_e^{(n+)}$$

$$\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_e^{(n)}=\frac{1}{(\Delta z)^2}\left\{\tilde{\varepsilon}_{zz}^{(n+M_z)}\Phi_e^{(n+M_z)}-\left[(I-S_{M_z})\tilde{\varepsilon}_{zz}^{(n)}\right]\Phi_e^{(n)}+\tilde{\varepsilon}_{zz}^{(n)}\Phi_e^{(n-M_z)}\right\}$$

$$=\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_e^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n)}}S_{-M_z}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n)}}S_{-M_y}+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n)}}S_{-M_x}-\underbrace{\left\{\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}+\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}+\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}\right\}}_{=\alpha^{(n)}}I+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n+M_x)}}S_{M_x}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n+M_y)}}S_{M_y}+\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n+M_z)}}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\alpha_{zz}^{(n)}S_{-M_z}+\alpha_{yy}^{(n)}S_{-M_y}+\alpha_{xx}^{(n)}S_{-M_x}-\alpha^{(n)}I+\alpha_{xx}^{(n+M_x)}S_{M_x}+\alpha_{yy}^{(n+M_y)}S_{M_y}+\alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}$$
$$= \iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}$$
$$= -\left\{\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_e^{(n)} + \frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_e^{(n)} + \frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\right\}\Phi_e^{(n)}$$
$$= -\left\{\alpha_{zz}^{(n)}S_{-M_z} + \alpha_{yy}^{(n)}S_{-M_y} + \alpha_{xx}^{(n)}S_{-M_x} - \alpha^{(n)}I + \alpha_{xx}^{(n+M_x)}S_{M_x} + \alpha_{yy}^{(n+M_y)}S_{M_y} + \alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_e^{(n)}$$

$$\iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV = \rho_e^{(n)}(t)\Delta x\Delta y\Delta z$$

$$-\left(\alpha_{zz}^{(n)}S_{-M_z} + \alpha_{yy}^{(n)}S_{-M_y} + \alpha_{xx}^{(n)}S_{-M_x} - \alpha^{(n)}I + \alpha_{xx}^{(n+M_x)}S_{M_x} + \alpha_{yy}^{(n+M_y)}S_{M_y} + \alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_e^{(n)} = \rho_e^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\alpha_{zz}^{(n)}S_{-M_z} - \alpha_{yy}^{(n)}S_{-M_y} - \alpha_{xx}^{(n)}S_{-M_x} + \alpha^{(n)}I - \alpha_{xx}^{(n+M_x)}S_{M_x} - \alpha_{yy}^{(n+M_y)}S_{M_y} - \alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_e^{(n)} = \rho_e^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\underbrace{\frac{\alpha_{zz}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n)}}S_{-M_z} - \underbrace{\frac{\alpha_{yy}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n)}}S_{-M_y} - \underbrace{\frac{\alpha_{xx}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n)}}S_{-M_x} + I - \underbrace{\frac{\alpha_{xx}^{(n+M_x)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n+M_x)}}S_{M_x} - \underbrace{\frac{\alpha_{yy}^{(n+M_y)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n+M_y)}}S_{M_y} - \underbrace{\frac{\alpha_{zz}^{(n+M_z)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n+M_z)}}S_{M_z}\right)\Phi_e^{(n)} = \frac{\rho_e^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

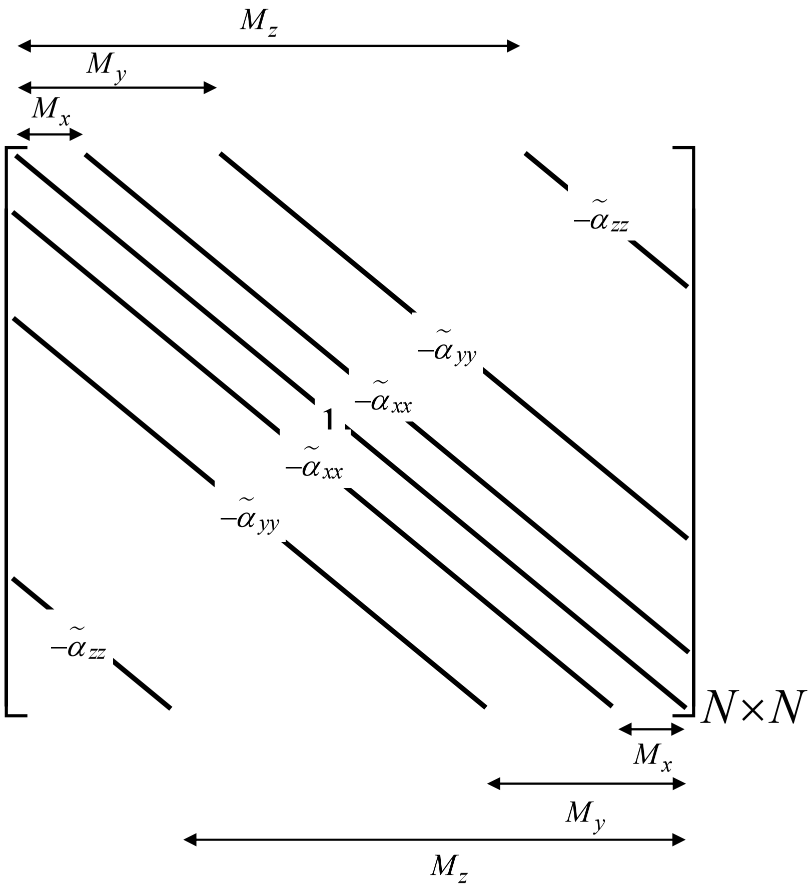
$$\left(-\tilde{\alpha}_{zz}^{(n)}S_{-M_z} - \tilde{\alpha}_{yy}^{(n)}S_{-M_y} - \tilde{\alpha}_{xx}^{(n)}S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)}S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)}S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_e^{(n)} = \frac{\rho_e^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

Discrete Poisson's Grid Equation / Diskrete Poissonsche Gittergleichung

$$\left(-\tilde{\alpha}_{zz}^{(n)} S_{-M_z} - \tilde{\alpha}_{yy}^{(n)} S_{-M_y} - \tilde{\alpha}_{xx}^{(n)} S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)} S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)} S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)} S_{M_z}\right) \Phi_\mathrm{e}^{(n)} = \frac{\rho_\mathrm{e}^{(n)}(t)\Delta x \Delta y \Delta z}{\alpha^{(n)}}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$[\mathbf{A}] =$$



3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

Homogeneous isotropic case for a cubic grid complex /
Homogener isotroper Fall für einen kubischen Gitterkomplex

$$[\widetilde{\boldsymbol{\varepsilon}}]_{3N\times 3N} = \varepsilon_0\varepsilon_\mathrm{r}[\mathbf{I}]_{3N\times 3N}$$

$$[\widetilde{\mathbf{S}}] = (\Delta x)^2[\mathbf{I}]_{3N\times 3N}$$

$$[\mathbf{R}]^{-1} = \frac{1}{\Delta x}[\mathbf{I}]_{3N\times 3N}$$

$$[\widetilde{\mathbf{V}}] = (\Delta x)^3[\mathbf{I}]_{N\times N}$$

$$[\widetilde{\mathbf{div}}]\varepsilon_0\varepsilon_\mathrm{r}[\mathbf{I}](\Delta x)^2[\mathbf{I}]\frac{1}{\Delta x}[\mathbf{I}][\mathbf{grad}]\{\Phi_\mathrm{e}\} = -(\Delta x)^3[\mathbf{I}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_\mathrm{e}\} = -\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}$$

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

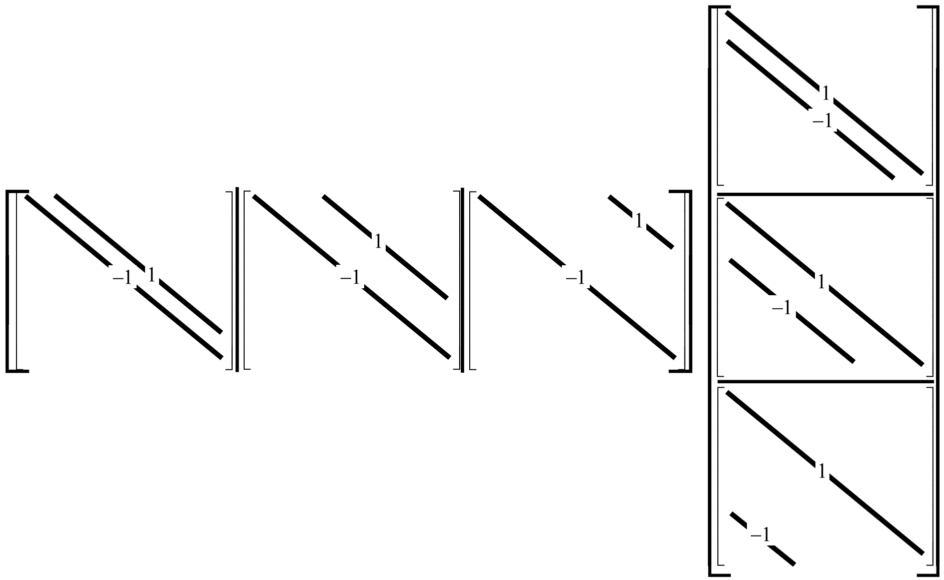
$$-[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_\mathrm{e}\} = \frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}$$

Band Structure of the Divergence Operator on Primary Grid / Bandstruktur des Rotationsoperators in Matrixform

$$[\widetilde{\mathbf{div}}][\mathbf{grad}] = \left[[\mathbf{P}_x],[\mathbf{P}_y],[\mathbf{P}_z]\right]\begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}$$

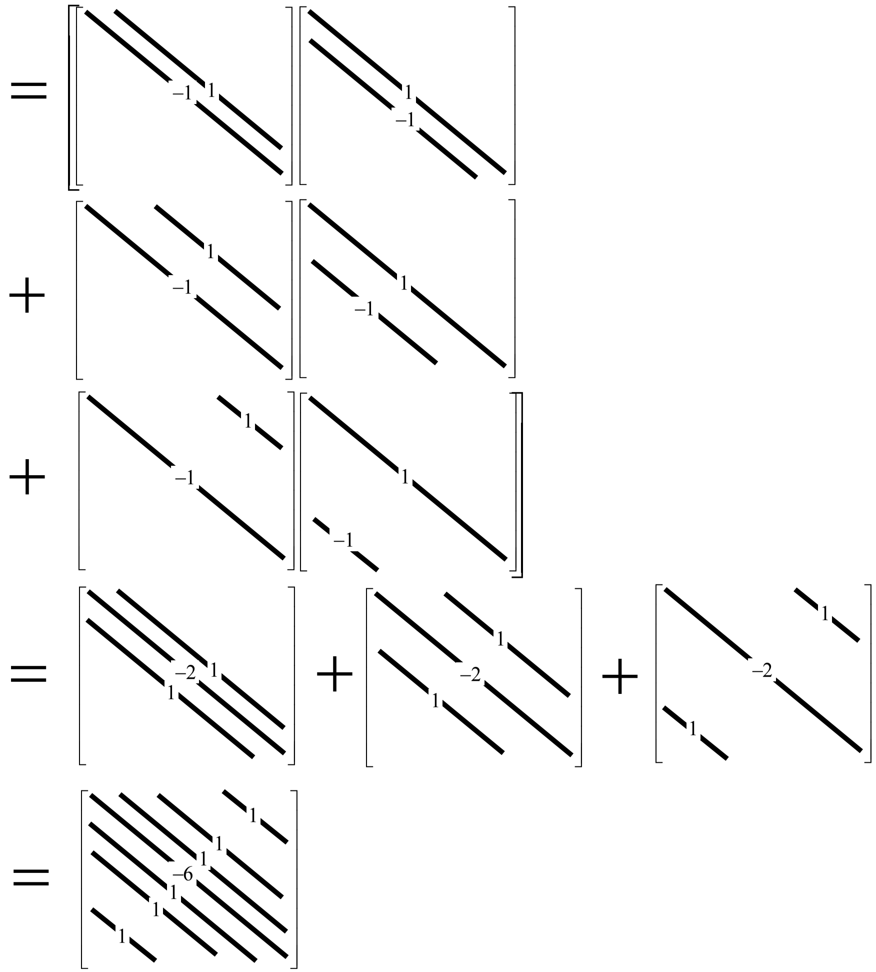
$$= -\left[[\mathbf{P}_x][\mathbf{P}_x]^{\mathrm{T}} + [\mathbf{P}_y][\mathbf{P}_y]^{\mathrm{T}} + [\mathbf{P}_z][\mathbf{P}_z]^{\mathrm{T}}\right]_{N\times N}$$

$$[\widetilde{\mathbf{div}}][\mathbf{grad}] =$$



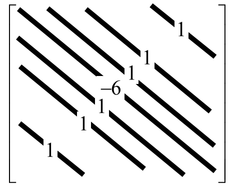
Band Structure of the Divergence Operator on Primary Grid / Bandstruktur des Rotationsoperators in Matrixform

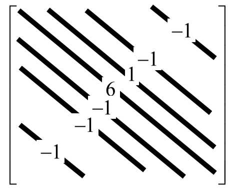
$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$



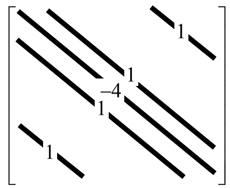
Band Structure of the Divergence Operator on Primary Grid / Bandstruktur des Rotationsoperators in Matrixform

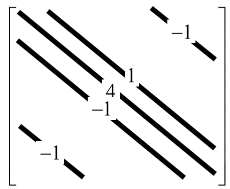
3-D case / 3D-Fall

$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

$$-\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

2-D case in the *xz* plane / 2D-Fall in der *xz*-Ebene

$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

$$-\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

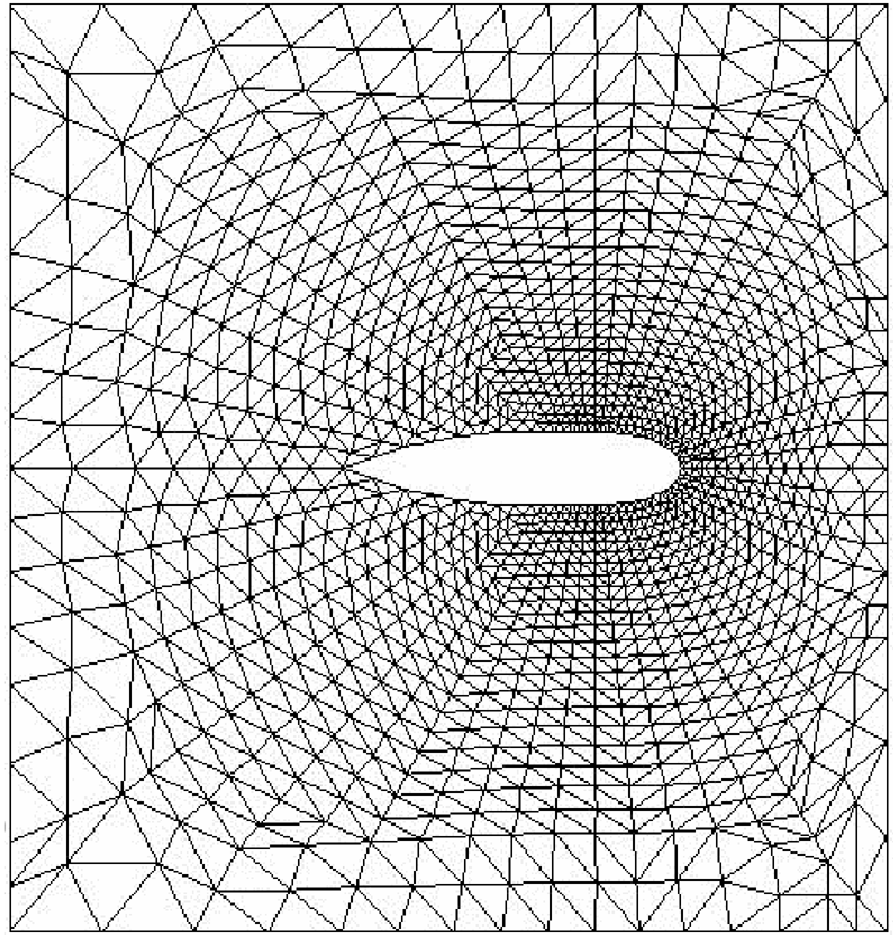
One of the simplest types of iterative methods for solving the linear system /
Einer der einfachsten Typen von iterativen Methoden zur Lösung des linearen Gleichungssystems

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

- Gaussian elimination (Gauss method) / Gaußsches Eliminationsverfahren (Gauß-Methode)
- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)
- Conjugate gradient method (CG method) / Konjugierte Gradientenmethode (KG-Methode)
- Multi grid methods (MG method) / Mehrgitterverfahren (MG-Methode)
- Algebraic multi grid method (AMG method) / Algebraisches Mehrgitterverfahren (AMG-Methode)

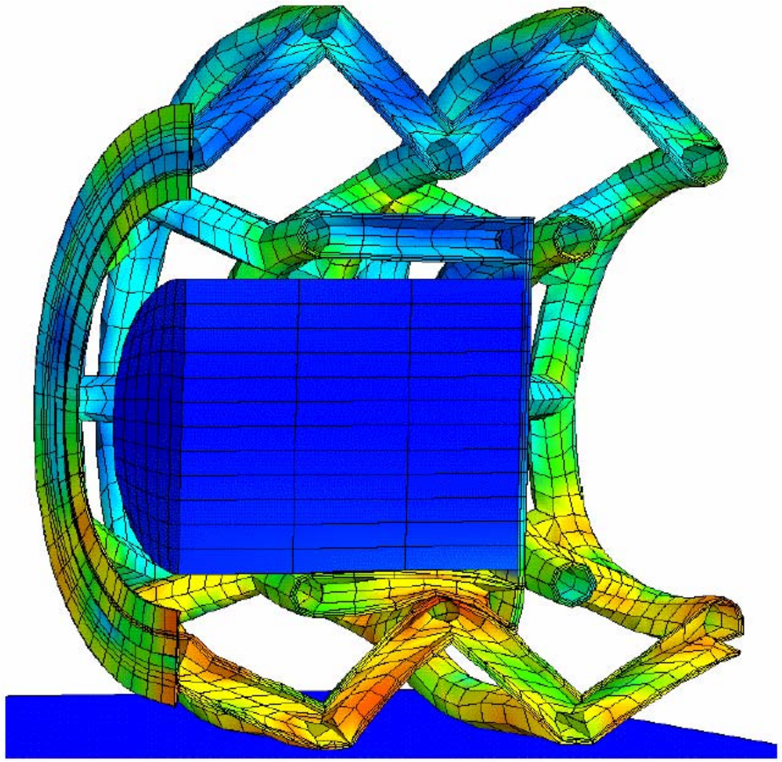
Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

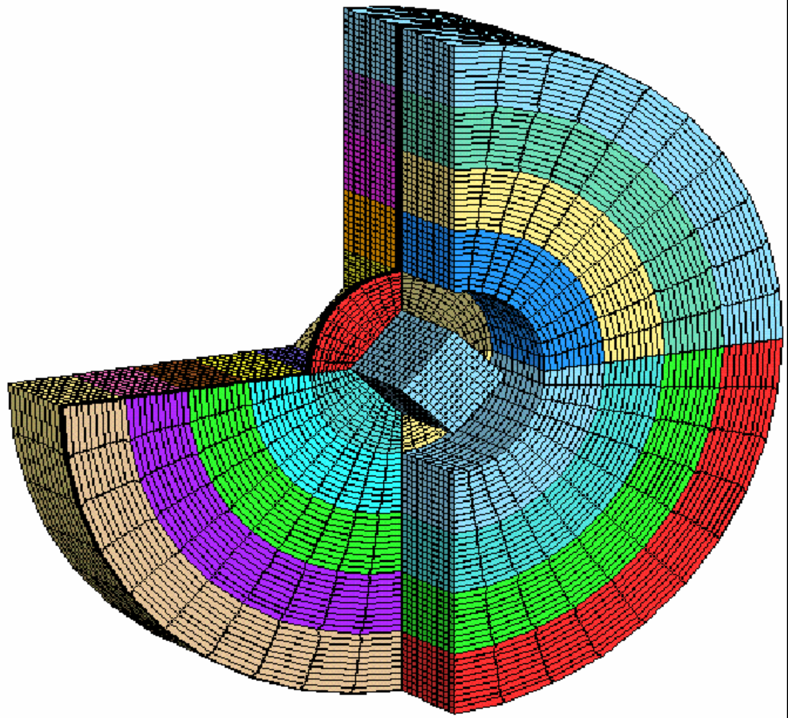
Algebraic multi grid method (AMG method) / Algebraisches Mehrgitterverfahren (AMG-Methode)



Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

Algebraic multi grid method (AMG method) / Algebraisches Mehrgitterverfahren (AMG-Methode)





Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

LU decomposition of matrix $[\mathbf{A}]$ / LU-Zerlegung der Matrix $[\mathbf{A}]$

$$[\mathbf{A}] = [\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]$$

Lower triangular matrix / Untere Dreiecksmatrix

$$[\mathbf{L}] = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ A_{21} & 0 & \cdots & & \vdots \\ A_{31} & A_{32} & \ddots & & 0 \\ \vdots & & & 0 & 0 \\ A_{N1} & A_{N2} & \cdots & A_{N(N-1)} & 0 \end{bmatrix}_{N\times N}$$

Main diagonal matrix / Hauptdiagonalmatrix

$$[\mathbf{D}] = \left[\operatorname{diag}\{A_{11}, A_{22}, \cdots, A_{NN}\}\right]_{N\times N}$$

Upper triangular matrix / Obere Dreiecksmatrix

$$[\mathbf{U}] = \begin{bmatrix} 0 & A_{12} & \cdots & \cdots & A_{1N} \\ 0 & 0 & \cdots & & \vdots \\ 0 & 0 & \ddots & & A_{(N-2)N} \\ \vdots & & & 0 & A_{(N-1)N} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times N}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\} \quad \rightarrow \quad \{[\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]\}\{\mathbf{x}\} = \{\mathbf{b}\} \quad \rightarrow \quad [\mathbf{D}]\{\mathbf{x}\} = -\{[\mathbf{L}] + [\mathbf{U}]\}\{\mathbf{x}\} + \{\mathbf{b}\}$$

Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$\{[\mathbf{L}]+[\mathbf{D}]+[\mathbf{U}]\}\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$[\mathbf{D}]\{\mathbf{x}\} = -\{[\mathbf{L}]+[\mathbf{U}]\}\{\mathbf{x}\} + \{\mathbf{b}\}$$

$$\{\mathbf{x}\} = \underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}}_{=[\mathbf{G}]}\{\mathbf{x}\} + \underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}}$$

$$= [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\} = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\} \quad l = 1, 2, \ldots, L$$

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- ⇨ Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

Jacobi Method / Jacobi-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_\mathrm{J} \{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_\mathrm{J} \quad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1} \{[\mathbf{L}] + [\mathbf{U}]\}$$

$$\{\mathbf{c}\}_\mathrm{J} = [\mathbf{D}]^{-1} \{\mathbf{b}\}$$

$$x_i^{(l+1)} = \sum_{j=1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1, 2, \ldots, L$$

It follows with the LU decomposition of matrix $[\mathbf{A}]$ /
Mit der LU-Zerlegung der Matrix $[\mathbf{A}]$ folgt

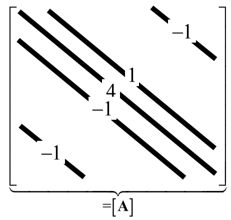
$$x_i^{(l+1)} = \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{(l)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1, 2, \ldots, L$$

Jacobi Method / Jacobi-Methode

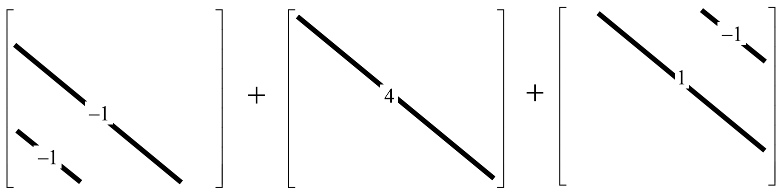
2-D case in the xz plane /
2D-Fall in der xz-Ebene

$$-\widetilde{[\mathbf{div}]}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = \frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$-\widetilde{[\mathbf{div}]}[\mathbf{grad}] =$$

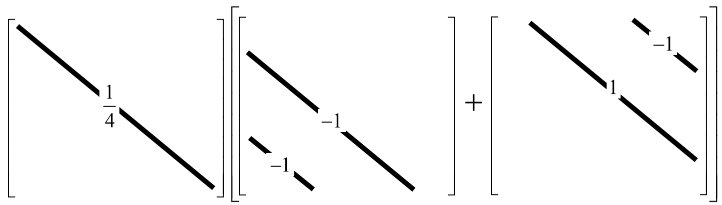


$$[\mathbf{A}] = \{[\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]\} =$$



$$\{\mathbf{x}\} = \underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}}_{=[\mathbf{G}]_\mathrm{J}}\{\mathbf{x}\} + \underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_\mathrm{J}} = [\mathbf{G}]_\mathrm{J}\{\mathbf{x}\} + \{\mathbf{c}\}_\mathrm{J}$$

$$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\} = -$$



Jacobi & Gauss-Seidel Method / Jacobi- & Gauss-Seidel-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{J}} \{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{J}} \quad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_{\mathrm{J}} = -[\mathbf{D}]^{-1} \{[\mathbf{L}] + [\mathbf{U}]\}$$

$$\{\mathbf{c}\}_{\mathrm{J}} = [\mathbf{D}]^{-1} \{\mathbf{b}\}$$

$$x^{(n,l+1)} = \frac{1}{4}\left[x^{(n-M_z,l)} + x^{(n-M_x,l)} + x^{(n+M_x,l)} + x^{(n+M_z,l)} + b^{(n)}\right]$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{GS}} \{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{GS}} \qquad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_{\mathrm{GS}} = -\left([\mathbf{D}] + [\mathbf{L}]\right)^{-1} [\mathbf{U}]$$

$$\{\mathbf{c}\}_{\mathrm{GS}} = \left([\mathbf{D}] + [\mathbf{L}]\right)^{-1} \{\mathbf{b}\}$$

$$x^{(n,l+1)} = \frac{1}{4}\left[x^{(n-M_z,l+1)} + x^{(n-M_x,l+1)} + x^{(n+M_x,l)} + x^{(n+M_z,l)} + b^{(n)}\right]$$

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1,2,\ldots,L$$

$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$\{\mathbf{x}\}^{(l+1)}$: Algebraic field vector at the iteration step l / Algebraischer Feldvektor zum Iterationsschritt l

$\overline{\{\mathbf{x}\}}^{(l+1)}$: Gauss-Seidel value at the iteration step l / Gauß-Seidel-Wert zum Iterationsschritt l

ω : Relaxations Parameter / Relaxationsparameter

$0 < \omega < 1$: Under relaxation method / Unterrelaxationsmethode

$\omega = 1$: Gauss-Seidel method / Gauß-Seidel-Methode

$1 < \omega < 2$: Over relaxation method / Überrelaxationsmethode

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1,2,\ldots,L$$

$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$$x_i^{(l+1)} = (1-\omega)x_i^{(l)} + \omega x_i^{(l+1)}$$

$$= (1-\omega)x_i^{(l)} + \omega\left\{\sum_{j=1}^{i-1} G_{\mathrm{J},ij}\; x_j^{(l+1)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i}\right\} \qquad i = 1,2,\ldots,N$$

$$x_i^{(l+1)} = (1-\omega)x_i^{(l)} + \omega\left\{-\sum_{j=1}^{i-1} D_{ii}^{-1} L_{ij}\; x_j^{(l+1)} - \sum_{j=i+1}^{N} D_{ii}^{-1} U_{ij}\, x_j^{(l)} + c_{\mathrm{J},i}\right\} \qquad i = 1,2,\ldots,N$$

$$\{\mathbf{x}\}^{(l+1)} = (1-\omega)\{\mathbf{x}\}^{(l)} - \omega\left\{[\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+1)} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\}\right\} \qquad i = 1,2,\ldots,N$$

$$\{\mathbf{x}\}^{(l+1)} = \underbrace{\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]}_{=[\mathbf{G}]_{\mathrm{SOR}}}\{\mathbf{x}\}^{(l)} + \underbrace{\omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_{\mathrm{SOR}}} \qquad i = 1,2,\ldots,N$$

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1,2,\ldots,L$$

$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$$\{\mathbf{x}\}^{(l+1)} = \underbrace{\left([\mathbf{D}]+\omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}]-\omega[\mathbf{U}]\right]}_{=[\mathbf{G}]_{\mathrm{SOR}}}\{\mathbf{x}\}^{(l)} + \underbrace{\omega\left([\mathbf{D}]+\omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_{\mathrm{SOR}}} \qquad i = 1,2,\ldots,N$$

$$[\mathbf{G}]_{\mathrm{SOR}} = \left([\mathbf{D}]+\omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}]-\omega[\mathbf{U}]\right]$$

$$\{\mathbf{c}\}_{\mathrm{SOR}} = \omega\left([\mathbf{D}]+\omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{SOR}}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{SOR}} \qquad l = 1,2,\ldots,L$$

Symmetric Successive Overrelaxation Method (SSOR Method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

Forward SOR step / Vorwärts-SOR-Schritt

$$x_i^{\left(l+\frac{1}{2}\right)} = (1-\omega)\, x_i^{(l)} + \omega \left\{ \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{\left(l+\frac{1}{2}\right)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \right\} \qquad i = 1,2,\ldots,N$$

Backward SOR step / Rückwärts-SOR-Schritt

$$x_i^{(l+1)} = (1-\omega)\, x_i^{\left(l+\frac{1}{2}\right)} + \omega \left\{ \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{\left(l+\frac{1}{2}\right)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l+1)} + c_{\mathrm{J},i} \right\} \qquad i = N, N-1,\ldots,1$$

Forward SOR step / Vorwärts-SOR-Schritt

$$\{\mathbf{x}\}^{\left(l+\frac{1}{2}\right)} = (1-\omega)\{\mathbf{x}\}^{(l)} - \omega \left\{ [\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{\left(l+\frac{1}{2}\right)} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\} \right\} \qquad i = 1,2,\ldots,N$$

Backward SOR step / Rückwärts-SOR-Schritt

$$\{\mathbf{x}\}^{(l+1)} = (1-\omega)\{\mathbf{x}\}^{\left(l+\frac{1}{2}\right)} - \omega \left\{ [\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{\left(l+\frac{1}{2}\right)} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\} \right\} \qquad i = N, N-1,\ldots,1$$

Convergence of Point Iterative Methods / Konvergenz von punktiterativen Methoden

$$\rho([\mathbf{G}]) = \max_{n=1,2,\ldots,N} |\nu_n([\mathbf{G}])|$$

$$\rho([\mathbf{G}]) < 1$$

$$0 < \omega_{\mathrm{SOR}} < 2$$

$$0 < \omega_{\mathrm{SSOR}} < 2$$

Error Vector and Error Measure / Fehlervektor und Fehlermaß

$$\omega_{\mathrm{SOR,opt}} = \frac{2}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}} = 1 + \left(\frac{\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}\right)^2$$

$$\rho\left([\mathbf{G}]_{\mathrm{SOR}}\right) = \left(\frac{\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}\right)^2$$

Symmetric positive definite $[\mathbf{A}]$ matrix /
Symmetrische positiv-definite $[\mathbf{A}]$ Matrix

$$\rho\left([\mathbf{L}][\mathbf{U}]\right) \le \frac{1}{4}$$

$$\omega_{\mathrm{SSOR}} = \frac{2}{1+2\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}$$

Approximation for a D-dimensional Dirichlet problem /
Approximation für ein D-dimensionales Dirichlet-Problem

$$\rho\left([\mathbf{G}]_{\mathrm{J}}\right) = \frac{\sum_{d=1}^{D} \cos\left(\frac{\pi}{N_d}\right)}{D}$$

Optimal Relaxation / Optimaler Relaxationsparameter

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\} = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\boldsymbol{\varepsilon}\}^{(l)} = \{\mathbf{x}\}^{(l)} + \{\mathbf{x}\}$$

$$\{\boldsymbol{\varepsilon}\}^{(l+1)} = [\mathbf{G}]\{\boldsymbol{\varepsilon}\}^{(l)}$$

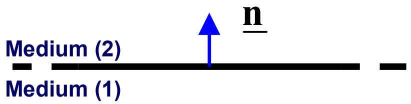
$$\lim_{l\to\infty} \{\boldsymbol{\varepsilon}\}^{(l)} = \{0\}$$

$$\varepsilon_{\mathrm{max}}^{(l+1)} = \max_{n=1,2,\ldots,N} \left| \varepsilon^{(n,l+1)} - \varepsilon^{(n,l)} \right|$$

$$\varepsilon_{\mathrm{rel,max}}^{(l+1)} = \max_{n=1,2,\ldots,N} \left| \frac{\varepsilon^{(n,l+1)} - \varepsilon^{(n,l)}}{\varepsilon^{(n,l+1)}} \right|$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Transition Conditions / Übergangsbedingungen



$\underline{\mathbf{R}} \in S_{\mathrm{I(nterface)}}$

$$E_{tan}^{(2)}(\underline{\mathbf{R}}) - E_{tan}^{(1)}(\underline{\mathbf{R}}) = 0$$

$$D_{n}^{(2)}(\underline{\mathbf{R}}) - D_{n}^{(2)}(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$E_{tan}^{(2)}(\underline{\mathbf{R}}) - E_{tan}^{(1)}(\underline{\mathbf{R}}) = 0$$

$$\Downarrow$$

$$\Phi_{\mathrm{e}}^{(2)}(\underline{\mathbf{R}}) - \Phi_{\mathrm{e}}^{(1)}(\underline{\mathbf{R}}) = \Phi_{\mathrm{e}0} = \mathrm{const.}$$

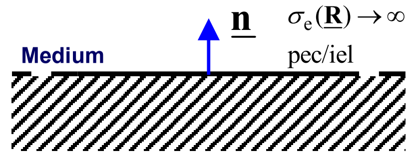
$$D_{n}^{(2)}(\underline{\mathbf{R}}) - D_{n}^{(2)}(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\Downarrow$$

$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}^{(2)}(\underline{\mathbf{R}}) - \frac{\varepsilon_{\mathrm{r}}^{(1)}}{\varepsilon_{\mathrm{r}}^{(2)}}\frac{\partial}{\partial n}\Phi_{\mathrm{e}}^{(1)}(\underline{\mathbf{R}}) = -\frac{1}{\varepsilon_0 \varepsilon_{\mathrm{r}}^{(2)}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Boundary Conditions / Randbedingungen

$\underline{\mathbf{R}} \in S_{\mathrm{B(oundary)}}$



$$E_{tan}(\underline{\mathbf{R}}) = 0 \qquad \text{pec / iel}$$

$$D_{n}(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

$$E_{tan}(\underline{\mathbf{R}}) = 0$$

$$\Downarrow$$

$$\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = \Phi_{\mathrm{e}0} = \mathrm{const.}$$

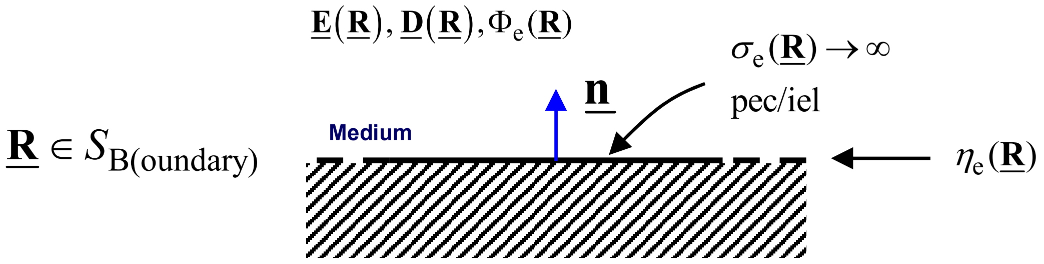
$$D_{n}(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\Downarrow$$

$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{1}{\varepsilon_0 \varepsilon_{\mathrm{r}}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Boundary conditions for the electrostatic potential / Randbedingungen für das elektrostatische Potential



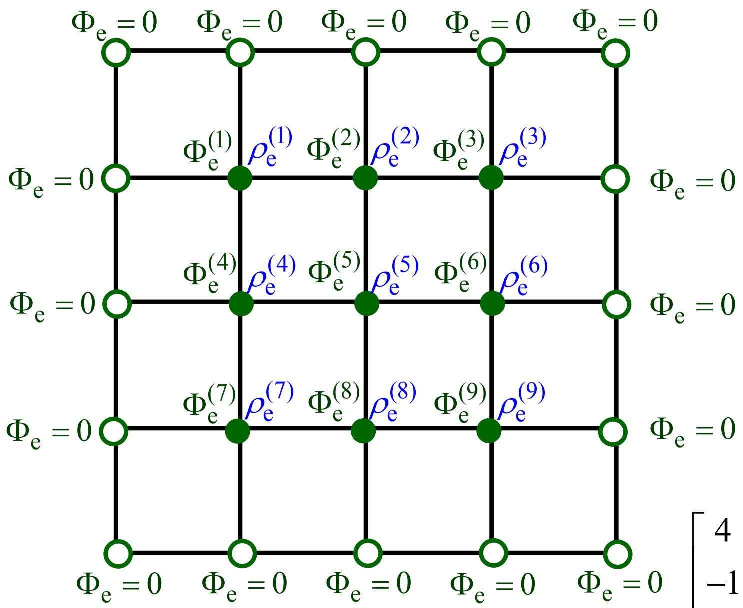
$$\Phi_e(\underline{\mathbf{R}}) = \Phi_{e0} = \text{const.} \quad (\Phi_{e0} = 0 \text{ V})$$

$$\frac{\partial}{\partial n}\Phi_e(\underline{\mathbf{R}}) = -\frac{1}{\varepsilon_0 \varepsilon_r}\eta_e(\underline{\mathbf{R}})$$

Neumann Boundary Condition for Φ_e /
Neumann-Randbedingung für Φ_e

Dirichlet Boundary Condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem



$$\underline{\mathbf{R}} = x\underline{\mathbf{e}}_x + z\underline{\mathbf{e}}_z$$

$$-\Delta\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon_0\varepsilon_\mathrm{r}} \qquad \underline{\mathbf{R}} \in S$$

$$-\oiint_{S=\partial V}\left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\cdot \mathbf{d}\underline{\mathbf{S}} = \frac{1}{\varepsilon_0\varepsilon_\mathrm{r}}\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}})\,\mathrm{d}V$$

$$\underline{\mathbf{R}} \in S$$

Dirichlet boundary condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0 \qquad \underline{\mathbf{R}} \in C_D = \partial S$$

$$\underbrace{-\widetilde{[\mathbf{div}]}[\mathbf{grad}]}_{=[\mathbf{A}]}\underbrace{\{\Phi_\mathrm{e}\}}_{=\{\mathbf{x}\}} = \underbrace{\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}}_{=\{\mathbf{b}\}}$$

Dirichlet boundary condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

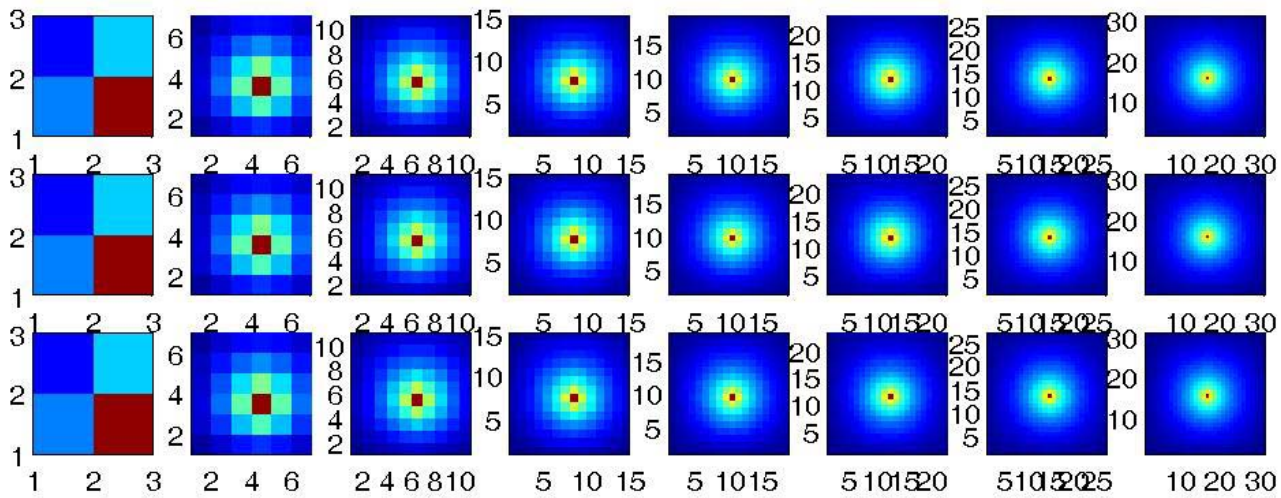
$$x^{(n)} = \Phi_\mathrm{e}^{(n)} = 0 \qquad n \in C_D = \partial S$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} \Phi_\mathrm{e}^{(1)} \\ \Phi_\mathrm{e}^{(2)} \\ \Phi_\mathrm{e}^{(3)} \\ \Phi_\mathrm{e}^{(4)} \\ \Phi_\mathrm{e}^{(5)} \\ \Phi_\mathrm{e}^{(6)} \\ \Phi_\mathrm{e}^{(7)} \\ \Phi_\mathrm{e}^{(8)} \\ \Phi_\mathrm{e}^{(9)} \end{Bmatrix} = \begin{Bmatrix} b^{(1)} \\ b^{(2)} \\ b^{(3)} \\ b^{(4)} \\ b^{(5)} \\ b^{(6)} \\ b^{(7)} \\ b^{(8)} \\ b^{(9)} \end{Bmatrix}$$

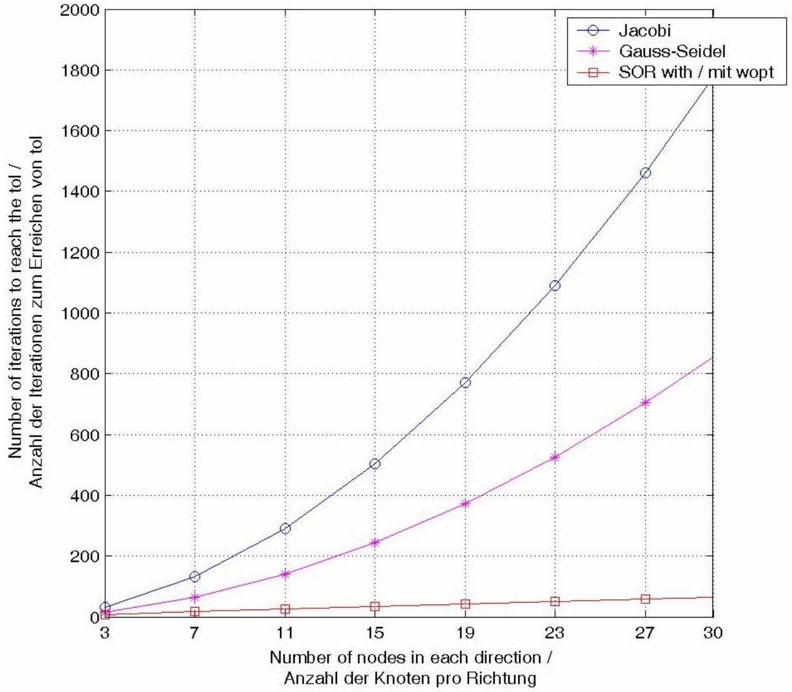
Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem

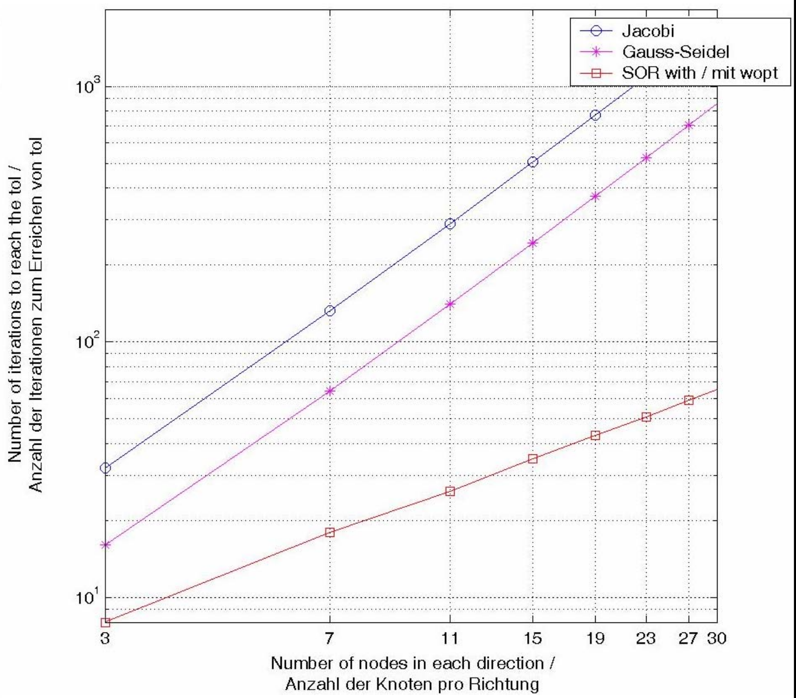
$$\rho_{\mathrm{e}}(x,z) = \xi_{\mathrm{e}0}\,\delta(x-x_{\mathrm{s}})\delta(z-z_{\mathrm{s}}) \qquad \underline{\mathbf{R}} = \underline{\mathbf{R}}_{\mathrm{s}} = x_{\mathrm{s}}\underline{\mathbf{e}}_x + z_{\mathrm{s}}\underline{\mathbf{e}}_z$$

$N=3$ $N=7$ $N=11$ $N=15$ $N=19$ $N=23$ $N=27$ $N=31$



Example: Dirichlet Problem / Beispiel: Dirichlet-Problem





**End of Lecture 8 /
Ende der 8. Vorlesung**