Spatial Econometrics

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1. Introduction

1.1 Spatial Econometrics

Applied work in regional science: Use of spatial data

Regional science provides models of cities and regions which have to be operationalised in empirical analysis. Using spatial data, model estimation, hypothesis testing and prediction has to allow for spatial effects. This requires a special, namely spatial econometric methodology.

Spatial data: Data collected with reference to location
- administrative spatial units (states, districts, counties, etc.)
- functional regions (e.g. labour market regions)
- points in space (e.g. cities, municipalities, plants)

Distinction between mainstream econometrics and spatial econometrics: existence of spatial effects:

1. Spatial dependence,
2. Spatial heterogeneity
1.2 Spatial Dependence

Lack of independence among spatial data:
Observations at location i depend on other observations at locations j ≠ i

Tobler’s first law of geography:
„Everything is related to everything else, but near things are more related than distant things“

Spatial dependence is associated with the notion of relative space (location):
- Neighbouring regions are expected to be more alike than arbitrary regions,
- Spatial dependence is expected to diminish with increasing distance
• Causes for spatial dependence:

1. Measurement errors

   The delineation of spatial units is always somewhat arbitrary. Spatial data are usually collected for administrative units (states, districts, counties, etc.) which do not accurately reflect the scale of the underlying spatial processes generating the sample data. If the correspondence between the spatial scale of a phenomenon under study and the delineation of the spatial units of observation is not strong, measurement errors are to be expected.

   Spatial dependence can be caused by measurement errors that occur by spatial aggregation.
Example:
Let $F_1$ and $F_2$ be functional regions that cover the true scale of a spatial process. When considering a geo-referenced variable $Y$ like unemployment, for the sake of simplicity we assume that $F_1$ and $F_2$ are travel-to-work areas. People who live in $F_1$ (F2) also work there. There are no commuter flows between $F_1$ and $F_2$. $D_1$, $D_2$ and $D_3$ are administrative units (e.g. districts) for which unemployment data are collected.

$D_1$ and the part $D_{2.1}$ of $D_2$ belong to $F_1$, $D_3$ and the other part $D_{2.2}$ of $D_2$ to $F_2$:

\[
F_1 = D_1 + D_{2.1} \\
F_2 = D_3 + D_{2.2}
\]

As data are only available for complete administrative units, in reality, labour market regions $R_1$ and $R_2$ are delineated in the following way:

\[
R_1 = D_1 \\
R_2 = D_2 + D_3
\]
Then we get following aggregate unemployment data for the two labour market regions R1 and R2:

\[ U(R1) = U(D1) \]
\[ U(R2) = U(D2) + U(D3) = U(D2.1) + U(D2.2) + U(D3) \]

Suppose that there is a negative employment shock in the functional region F1 but not in F2. Then U(R1) will tend to increase, because employees living in district D1 are affected. However, as employees living in the part D2.1 of district D2 are affected too, U(R2) is also expected to increase.

This shows how spatial dependence can arise from spatial aggregation. Due to data availability, aggregation cannot be arranged for the „true“ functional regions F1 and F2, but only for the „real“ labour markets R1 and R2. In this case, aggregation is involved with measurement errors that cause the spatial dependence between the observations of the geo-referenced variable U i.e. unemployment.
Spatial error dependence is often interpreted as a **nuisance**, which may be due to
- measurement errors or
- omitted variables.

The latter case applies when the variables are correlated across space without being crucial to the model (→ substantive spatial dependence).

In regression analysis, measurement errors (and omitted variables) are reflected in the error terms $\varepsilon$ of a model. The errors of regions $i$ and $j$ are no more independent, but correlated:

\[(1.1) \quad \text{Cov}(\varepsilon_i, \varepsilon_j) = E(\varepsilon_i \cdot \varepsilon_j) \neq 0 \text{ for } i \neq j.\]

In matrix notation, the assumption of identically independently normally distributed errors,

$$\varepsilon \sim N(o, \sigma^2 \cdot I),$$

has to be replaced by a non-spherical covariance structure:

$$\varepsilon \sim N(o, \Omega),$$

The covariance matrix $\Omega$ is modelled with the aid of spatial weights defined by distance or contiguity measures.
2. Substantive spatial dependence

Substantive spatial dependence is due to the inherent spatial organisation and spatial structure of phenomena. The complex pattern of interaction and dependence of socio-demographic and economic activity on the regional level may be itself a modelling problem. In regional science, location and distance are important forces at work in human geography and market activity. Regional science theory relies *inter alia* on notions of
- spatial interactions,
- diffusion processes,
- spatial spillovers,
- hierarchy of places.

Formally, in case of substantive dependence the geo-referenced variable to be explained, $Y$, is subject to the moment condition:

\[(1.2) \quad \text{Cov}(y_i, y_j) \neq 0 \text{ for } i \neq j.\]

In time-series analysis current and future values of a variable are explained by past realizations of that variable. Thus, *temporal dependence* is termed to *unidirectional*. In contrast, *spatial dependence* is not restricted to one direction. Dependence in space is *multidirectional* by nature. This necessitates the use of a different methodological framework in econometric analysis.
The spatial process underlying more fundamental of spatial dependence can be expressed as following:

\[ y_i = f(y_1, y_2, \ldots, (y_i), \ldots, y_n), \quad i = 1, 2, \ldots, n \]

\((y_i)\) : without the observation \( y_i \)

Geo-referenced variable \( Y \):
Values of \( Y \) are measured for spatial units (regions, districts, etc.)
1.3 Spatial Heterogeneity

Spatial heterogeneity refers to varying economic relationships or disturbances over space. Relations between socioeconomic variables may not be stable over space. A different relationship may hold for every spatial unit. This situation characterizes the case of structural instability. Spatial nonstationarity may as well be expressed in form of heteroscedasticity where the error variance varies across space.

In case of structural instability, response parameters are not homogenous throughout the sample but vary with location. The regression coefficients $\beta_1, \beta_2, \ldots, \beta_k$, of the explanatory variables $X_1, X_2, \ldots, X_k$, are not constant across the spatial units but may differ from region to region. Using $i$ as an index for the spatial units, a cross-sectional linear regression model can then be written in the form

\begin{equation}
(1.4) \quad y_i = \beta_{1i} + \beta_{2i} \cdot x_{2i} + \ldots + \beta_{ki} \cdot x_{ki} + \epsilon_i
\end{equation}

$\epsilon_i$ is a stochastic disturbance term.

More generally, not only parameters but also functional forms can vary over space:

\begin{equation}
(1.5) \quad y_i = f_i(x_i, \beta_i, \epsilon_i)
\end{equation}

where $x_i$ is a $k \times 1$ vector of the $k$ explanatory variables observed in region $i$, $\beta_i$ a $k \times 1$ vector of region-specific parameters and $\epsilon_i$ the disturbance term for region $i$.3
Operationability of specifications:

Note that the regression models (1.4) and (1.5) are not operational as they contain more parameters than observations. It is not possible to estimate $k$ regression coefficients for each of the $n$ regions with $n$ observations of the variables. To ensure identifiability, a number of constraints must be imposed.

Econometric methods in case of **structural instability**:
- (Spatially) varying parameter models (e.g. method of spatial expansion),
- Geographically weighted regression (GWR),
- Random coefficients models,
- Switching regressions (spatial regimes)

**Example**: Spatial regimes

Suppose the spatial units can be classified in two groups: urban and rural regions. There may exist two different relationships between geo-referenced variables: one across all urban regions and another across all rural regions. Then urban regions are homogenous among each other on the one hand and the rural regions are homogenous among each other on the other hand.

For this **two-regimes case** switching regression models can be applied.
Example:
Spatial heterogeneity arises with the distribution of living area of homes. While the distributions of low- and mid-priced homes have roughly similar distributions, a different pattern arises for high-priced homes.
2. Connectivity in Space
2.1 Neighbourhood and Location

Neighbours in space:
System S of n spatial units (i=1,2,...,n)
Variable X observed for each spatial unit
Spatial unit under consideration: i → variable value $x_i$
Set of neighbours: J, j ∈ J
Spatial unit in neighbourhood of i: j ∈ J → variable value $x_j$

Formal definition of the set of neighbours:

\[(2.1) \quad \{ \, j \mid P(x_i) \neq P(x_i \mid x_j) \, \} \]

or

\[(2.2) \quad \{ \, j \mid P(x_i) \neq P(x_i \mid x_j) \, \} \quad \text{and} \quad d_{ij} < c \]

$d_{ij}$: distance between i and j
$c$: cut-off value

Definition (2.2) combines the notion of statistical dependence with the notion of space (distance and relative location).
2.2 Spatial Weight Matrix

Location has to be quantified for analyzing spatial effects i.e. spatial dependence and spatial heterogeneity. **Locational information** can be used from two sources:

1. **Contiguity (neighbourhood)**
   Contiguity (neighbourhood) reflects the relative location of one spatial unit to other regions in space. Neighbourhoodships of spatial units are usually established from a map. Neighbouring units are expected to exhibit a higher degree of spatial dependence than units located far apart. Regarding spatial heterogeneity, relationships may be similar for neighbouring units.

2. **Distance**
   The location in space represented by **latitude and longitude** is one source of information. This information allows to calculate **distance** between points in space. In regional science points in space may represent centres or cities of regions.
   It is expected that the strength of spatial dependence will decline with distance. Observations that are near should exhibit similar relationships, those that are more distant may exhibit dissimilar relationships (spatial heterogeneity).
## Spatial contiguity matrix

The spatial contiguity matrix $W^*$ is a binary $nxn$ matrix which entries $w_{ij}^*$ are 0 or 1. An entry $w_{ij}^*$ is equal to one if regions $i$ and $j$ and neighbours otherwise; the diagonal elements of $W^*$ are set equal to 0:

\[
(2.3) \quad w_{ij}^* = \begin{cases} 
1 & \text{if } i \text{ and } j \text{ are neighbours} \\
0 & \text{otherwise}
\end{cases}
\]

In a **regular grid**, neighbours can be defined in a number of ways. In analogy of the game of chess, rook contiguity, bishop contiguity and queen contiguity are distinguished.

### Rook contiguity:
A spatial unit is a neighbour of another unit if both areas share a common edge (side). In the figure to the right, the units $B_1$, $B_2$, $B_3$ and $B_4$ are neighbours of unit $A$ according to the rook criterion.

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Bishop contiguity:
A spatial unit is a neighbour of another unit if both areas share a common vertex. In the figure to the right, the units $C_1$, $C_2$, $C_3$ and $C_4$ are neighbours of unit $A$ according to the bishop criterion.

Queen contiguity:
A spatial unit is a neighbour of another unit if both areas share a common edge or vertex. In the figure to the right, the units $B_1$, $B_2$, $B_3$ and $B_4$ as well as $C_1$, $C_2$, $C_3$ and $C_4$ are neighbours of unit $A$ according to the queen criterion.
In **irregular grids**, neighbours are usually defined by a common border (not vertex).

Figure: Irregular arrangement of spatial units

Contiguity matrix:

$$W^*_{5 \times 5} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}$$
Standardized contiguity matrix $W$

Row-standardization:

\[ w_{ij} = \frac{w_{ij}^*}{\sum_{j=1}^{n} w_{ij}^*} \]

Standardized contiguity matrix for the irregular grid:

\[
W = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
1/3 & 0 & 1/3 & 1/3 & 0 \\
1/3 & 1/3 & 0 & 1/3 & 0 \\
0 & 1/3 & 1/3 & 0 & 1/3 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Effect of row-standardization (X: geo-referenced variable):

\[
\begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
1/3 & 0 & 1/3 & 1/3 & 0 \\
1/3 & 1/3 & 0 & 1/3 & 0 \\
0 & 1/3 & 1/3 & 0 & 1/3 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= \begin{bmatrix}
(x_2 + x_3)/2 \\
(x_1 + x_3 + x_4)/3 \\
(x_1 + x_2 + x_4)/3 \\
(x_2 + x_3 + x_5)/3 \\
x_4 \\
\end{bmatrix}
= \begin{bmatrix}
\bar{x}_1^N \\
\bar{x}_2^N \\
\bar{x}_3^N \\
\bar{x}_4^N \\
\bar{x}_5^N \\
\end{bmatrix}
\]

The matrix product $W \cdot x$ gives a vector $\bar{x}^N$ that contains the means of the observations in the neighbouring regions (=spatial lag $\rightarrow$ section 3.1).
• Distance-based spatial weight matrix

It is assumed that spatial interaction will decline with increasing distance due to increasing geographical impediments. Nearer regions have a greater potential influence.

1. Power function

\[ w_{ij}^* = d_{ij}^{-\alpha} \]

\( \alpha \) : power parameter

\( \alpha = 1 \) : inverse distance

\( \alpha = 2 \) : quadratic inverse distance (→ gravity model of spatial interaction)

For spatial units outside a critical distance cut-off \( d_{\text{max}} \), the weights may be set equal to 0.

The distances \( d_{ij} \) are usually measured between the centres of the regions. Using the latitude and longitude coordinates the shortest distances between two centres are given by the great circle distances:

Great Circle Distance Formula using radians:

\[ d = r \times \arccos[\sin(\text{lat}_1) \times \sin(\text{lat}_2) + \cos(\text{lat}_1) \times \cos(\text{lat}_2) \times \cos(\text{lon}_2 - \text{lon}_1)] \]

radius of the earth: \( r = 6378.7 \) kilometers
2. Negative exponential function

\[ w_{ij}^* = e^{-\beta \cdot d_{ij}}, \quad 0 < \beta < \infty \]

\( \beta \): distance decay parameter

Percentage of decrease of spatial effects if distance expands by a unit of \( \bar{d} \):

\[ \gamma \bar{d} = 1 - e^{-\beta \cdot \bar{d}} \]

\( \bar{d} \): Average distance between immediate neighbouring regions over the whole cross-section

\( \gamma \): Transformed distance decay parameter

It is assumed that spatial interaction such as commuting, migration or interregional trade is exposed to the frictional effects of geographical distance. With increasing distance from region \( i \) these geographical impediments gain in strength, so that the decline of spatial effects becomes more and more pronounced.

Distance corresponding to a decrease of spatial effects of \( \gamma \cdot 100\% \):

\[ d_{\gamma} = -\frac{\ln (1 - \gamma)}{\beta}, \quad 0 < \gamma < 1 \]
Half-life distance (distance that reduces spatial interaction by 50%):

(2.9) \( d_{0.5} = \ln 2 / \beta \)

Determination of the distance decay parameter \( \beta \) using \( \gamma \):

(2.10) \( \beta = -\frac{\ln (1-\gamma)}{d} \), \( 0 < \gamma < 1 \)

3. General spatial weights

- Cliff-Ord weights (combination of distance measure and relative length of the common border):

(2.11) \( w_{ij}^* = d_{ij}^{-\alpha} \cdot b_{ij}^\delta \)

\( b_{ij} \): Proportion of the common boundary of regions i and j to the entire boundary of region i
\( \alpha \) and \( \beta \): parameters

More general spatial weights may be defined by economic variables such as communication links or trade flows.