Assignment 19 (Maximum Likelihood II)

A simple random sample (x_1, x_2, …, x_n) is drawn from an exponentially distributed population with an unknown parameter $\lambda>0$:

$$f(x \mid \lambda) = \lambda \cdot e^{-\lambda x}, \quad x \geq 0$$

Derive the maximum likelihood (ML) estimator for λ and interpret it!

Likelihood function:

$$L(\lambda|x_1,\ldots,x_n) = \prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n} \lambda \cdot e^{-\lambda x_i} = \lambda^n \cdot \prod_{i=1}^{n} \cdot e^{-\lambda x_i}$$

Log likelihood function:

$$\ln L(\lambda|x_1,\ldots,x_n) = n \ln \lambda + \sum_{i=1}^{n} -\lambda x_i = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$$

Necessary condition for a maximum:

$$\frac{d \ln L(\lambda|x_1,\ldots,x_n)}{d\lambda} = n \cdot \frac{1}{\lambda} - \sum_{i=1}^{n} x_i \overset{!}{=} 0$$

ML estimator:

$$\hat{\lambda} = n \cdot \frac{1}{\sum_{i=1}^{n} x_i} = \frac{1}{\overline{x}}$$

The ML estimator $\hat{\lambda}$ of the unknown parameter λ of an exponentially distributed population is the reciprocal sample mean.

Sufficient condition for a maximum:

$$\frac{d^2 \ln L(\lambda|x_1,\ldots,x_n)}{d\lambda^2} = -n \frac{1}{\lambda^2} < 0$$