3. Significance tests and confidence intervals

3.1 Significance tests of the individual regression coefficients

If the multiple regression model conforms with the underlying economic theory, one would expect the exogenous variables x_j to influence the endogenous variable y in particular directions. In the econometric model, the estimated regression coefficients should therefore display the theoretically expected signs.

In addition it needs to be examined whether the influencing factors do in fact matter for the explanation of the endogenous variable. Because, if a regression coefficient has the expected sign but only randomly deviates from 0, the explanatory variable will have no systematic influence on the endogenous variable. Whether an independent variable exhibits a systematic influence on the dependent variable, can be checked with a **significance test:**

If the **null hypothesis** $H_0 : \beta_j = 0$ is being contrasted to

the **alternative hypothesis** $H_1 : \beta_j \neq 0$

one speaks of a two-sided significance test. Only if the null hypothesis can be refuted, has the empirical relevance of the exogenous variable for the explanation of the endogenous variable been shown.

Under the null hypothesis, due to the normal distribution of the error term u, the estimated vector of regression coefficients, $\hat{\boldsymbol{\beta}}$, is normally distributed with the expectation vector $\boldsymbol{\beta}$ and the covariance matrix $\sigma^2(\mathbf{X'X})^{-1}$:

(3.1) $$\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta}, \sigma^2(\mathbf{X'X})^{-1})$$

The variances of the regression coefficients $\hat{\beta}_j$ are therefore given by

$$\mathrm{Var}(\hat{\beta}_j) = \sigma^2 \cdot x^{jj}$$

where the quantities x^{jj} indicate the main diagonal elements of the inverse $(\mathbf{X'X})^{-1}$. Estimators of the variances $\mathrm{Var}(\hat{\beta}_j)$ are derived by replacing the unknown disturbance variance σ^2 with its unbiased estimator

(3.2) $$\hat{\sigma}^2 = \frac{1}{n-k}\hat{\mathbf{u}}'\hat{\mathbf{u}} = \frac{1}{n-k}\sum_{t=1}^{n}\hat{u}_t^2$$

(3.3) $$\hat{\mathrm{Var}}(\hat{\beta}_j) = \hat{\sigma}^2 \cdot x^{jj} = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-k} \cdot x^{jj}$$

In the unbiased estimator $\hat{\sigma}^2$ the sum of the squared residuals is divided not by the number of residuals n, but by the number of the degrees of freedom n-k.

In the special case of the simple regression model:

(3.4) $$\hat{\mathrm{Var}}(\hat{\beta}_1) = \hat{\sigma}^2 \frac{\sum x_t^2}{n\sum x_t^2 - (\sum x_t)^2}$$

and

(3.5) $$\hat{\mathrm{Var}}(\hat{\beta}_2) = \hat{\sigma}^2 \frac{n}{n\sum x_t^2 - (\sum x_t)^2}$$

Example:

The variance $\hat{\sigma}^2$ of the disturbance variables shall now be estimated for the macroeconomic consumption function

$$\hat{C}_t = -38{,}522 + 0{,}9335 \cdot Y_t^V$$

To this end a work table is created which contains the required figures and computations for the calculation of the estimator:

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_t^2}{n-k} = \frac{\sum (C_t - \hat{C}_t)^2}{n-k}$$

$\hat{C}_t$ is the regression value of private consumption in period t,
for example consumption in 1994 is given by

$$\hat{C}_1 = -38{,}522 + 0{,}9335 \cdot 1159{,}39 = 1043{,}81$$

In the same way, the further regression values can be calculated, which are then included in the work table:

t	Y_t^v	C_t	$\hat{C}_t$	$C_t - \hat{C}_t$	$(C_t - \hat{C}_t)^2$
1	1159,39	1032,75	1043,81	11,06	122,30
2	1190,61	1066,47	1072,95	6,48	42,04
3	1212,71	1088,64	1093,59	4,95	24,46
4	1231,47	1110,82	1111,10	0,28	0,08
5	1251,21	1130,14	1129,53	-0,61	0,38
6	1278,66	1161,86	1155,15	-6,71	45,00
7	1300,70	1195,04	1175,73	-19,31	372,99
8	1347,78	1233,43	1219,68	-13,75	189,12
9	1363,26	1240,58	1234,13	-6,45	41,62
10	1394,62	1264,51	1263,41	-1,11	1,22
11	1419,00	1283,61	1286,17	2,56	6,53
12	1448,13	1306,98	1313,36	6,38	40,68
13	1480,52	1339,54	1343,60	4,06	16,45
14	1502,37	1356,73	1363,99	7,26	52,75
15	1541,25	1389,62	1400,29	10,67	113,83
16	1533,14	1391,55	1392,72	1,17	1,36
17	1578,89	1433,16	1435,43	2,27	5,14
18	1630,14	1487,66	1483,27	-4,39	19,26
19	1666,05	1521,59	1516,80	-4,79	22,99
Σ					1118,20

The sum of deviation squares is thus 1,118.20. With n = 19 and k = 2 one gets the unbiased estimator for the disturbance term σ^2 as

$$\hat{\sigma}^2 = \frac{1118,20}{19-2} = 65,7765$$

The standard deviation $\hat{\sigma}$ of the residuals depicts the standard error of the regression (SER):

$$\hat{\sigma} = \sqrt{65,7765} = 8,1103$$

□

Test statistic and critical value

The test statistic for the significance test

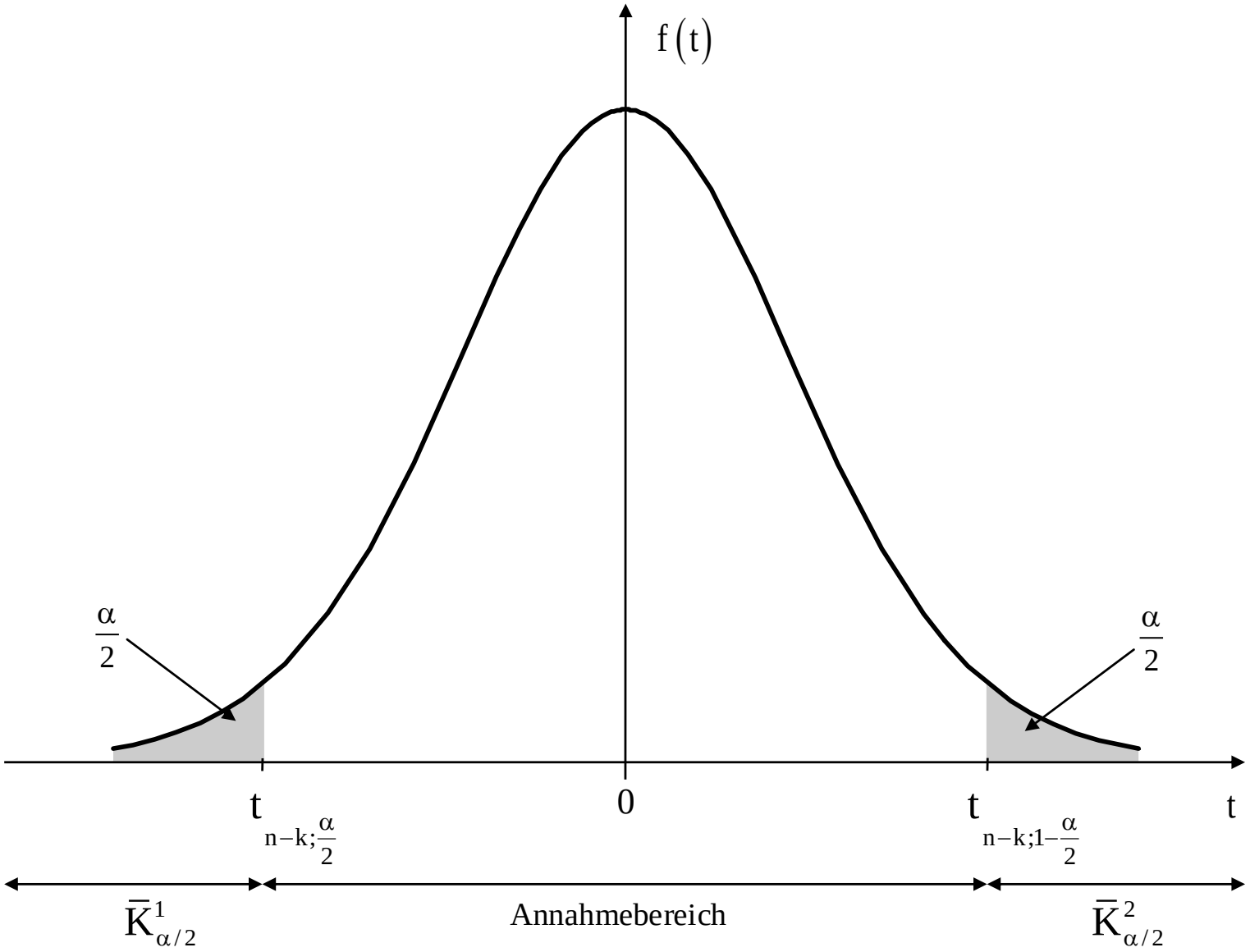
(3.6) $$t = \frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{x^{jj}}}$$

is t-distributed with n–k degrees of freedom. With a two-sided test the null hypothesis H_0 will be rejected at significance level α if the absolute value of the test statistic t exceeds the **critical value**, that is the (1-α/2)–quantile $t_{n-k;\,1-\alpha/2,}$ of the t-distribution with n-k degrees of freedom:

$$|t| > t_{n-k;1-\alpha/2} \rightarrow H_0 \text{ reject and discard}$$

In this case, the test statistic falls within the rejection region $\overline{K}_\alpha = \overline{K}^1_{\alpha/2} \cup \overline{K}^2_{\alpha/2}$ (see Fig.). At a significance level of e.g. 5% (α = 0.05) the probability of such a large deviation from the value $\beta_j = 0$, alleged by the null hypothesis, is less than 5%. While small deviations (with t values situated in the acceptance region K_α) are taken to be randomly determined, such large differences are interpreted as being substantial or **significant.**

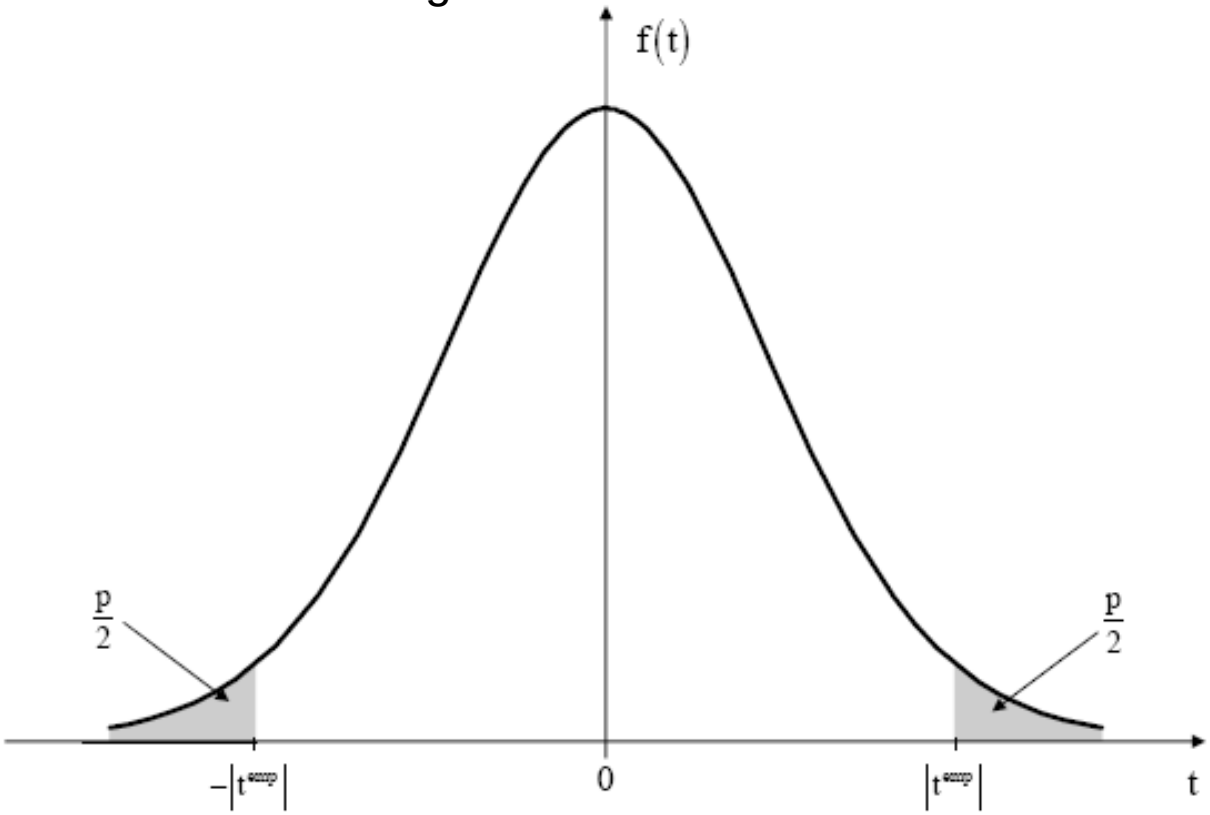
Figure: Acceptance and rejection regions for the two-sided significance test



Significance test and p-value

In statistical packages such as EViews, Stata, and R, the table displaying the regression output reveals under "Prob." or "Sign." so-called **p-values** for the regression coefficients. As the figure shows, the **p-value** indicates the twofold probability that the absolute empirical t-value $| t_{emp}|$ in the t-distribution with n-k degrees of freedom is exceeded.

Figure: p-value at two-sided significance test



In the statistical packages, the results of a two-sided significance test are reported in general. The p-value then indicates the probability P((T>$|t_{emp}|$) and (T < -$|t_{emp}|$)). Due to the symmetry of the t distribution

$$p = P(T < -|t_{emp}|) + P(T > |t_{emp}|) = 2 \cdot P(T > |t_{emp}|) = 2 \cdot P(T < -|t_{emp}|)$$

applies, that is, the p-value is twice the shaded right (= left) surface in the above figure.

While α indicates the **predetermined level of significance** (**nominal level of significance)**, the p-value corresponds to the **actual significance level,** i.e. the actual error probability of the test. Insofar the p-value conveys more information than the comparison of the empirical t-value with the critical value $t_{n-1;\, 1\alpha/\, 2}$. Using the p-value, the decision rule of the test is given as:

$p < \alpha \Rightarrow$ reject the null hypothesis

$p \geq \alpha \Rightarrow$ maintain the null hypothesis

Test decisions for one-sided significance tests:

- Right-sided significance test

$t > t_{n-k;1-\alpha} \rightarrow H_0$ reject

or

$p / 2 < \alpha$ and $t > 0 \rightarrow H_0$ reject

- Left-sided significance test

$t < t_{n-k;\alpha} = -t_{n-k;1-\alpha} \rightarrow H_0$ reject

or

$p / 2 < \alpha$ and $t < 0 \rightarrow H_0$ reject

Example:

For the macroeconomic consumption function over the period 1994 to 2012 we have ascertained the OLS-estimators $\hat{\beta}_1$ = -38.522 and $\hat{\beta}_2$ = 0.9335. Assuming normally distributed disturbance variables, these are also the ML-estimators. For a significance test of the regression coefficients we also need their variances. Given the estimator $\hat{\sigma}^2$ = 65.7767 for the variance of the error term and n = 19, $\sum x_t = 26529{,}9 \; and \; \sum x_t^2 = 37472832$ one gets:

$$\hat{Var}(\hat{\beta}_1) = \hat{\sigma}^2 \cdot \frac{\sum x_t^2}{n\sum x_t^2 - (\sum x_t)^2}$$

$$= 65{,}7767 \cdot \frac{37472832}{19 \cdot 37472832 - 26529{,}9^2}$$

$$= 65{,}7767 \cdot 4{,}5989 = 302{,}5$$

and

$$\hat{Var}(\hat{\beta}_2) = \hat{\sigma}^2 \cdot \frac{n}{n\sum x_t^2 - (\sum x_t)^2}$$

$$= 65{,}7767 \cdot \frac{19}{19 \cdot 37472832 - 26529{,}9^2}$$

$$= 65{,}7767 \cdot 2{,}3318 \cdot 10^{-6}$$

$$= 0{,}00015338$$

With this one can obtain the t-values for the estimated regression coefficients

$$t(\hat{\beta}_1) = \frac{\hat{\beta}_1}{\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_1)}} = \frac{-38{,}522}{\sqrt{302{,}5}} = -2{,}2149$$

and

$$t(\hat{\beta}_2) = \frac{\hat{\beta}_2}{\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_2)}} = \frac{0{,}9335}{\sqrt{0{,}00015338}} = 75{,}3755$$

With a two-sided test at the 5% significance level, the critical value of the t-distribution with 17 degrees of freedom $t_{17;\,0.975}$ equals 2,110. Because

$$\left|t\left(\hat{\beta}_1\right)\right| = 2{,}2149 > t_{17;0{,}975} = 2{,}110$$

and

$$\left|t\left(\hat{\beta}_2\right)\right| = 73{,}3755 > t_{17;0{,}975} = 2{,}110$$

the two estimated regression coefficients $\hat{\beta}_1$ and $\hat{\beta}_2$ are significantly different from zero. The regression coefficient $\hat{\beta}_2$ is even to be interpreted as highly significant as its t-value exceeds critical levels of significance of a test at the 1% level ($t_{17;\,0.995}$= 2.898).

For certain issues it is not only of interest to clarify if an exogenous variable has a substantial effect on a dependent variable. One would furthermore like to know whether the empirical data are compatible with a hypothesis

$$H_0 : \beta_j = c, \quad c \neq 0$$

To test this null hypothesis the already discussed t–test can be applied in an otherwise unchanged form by using the modified test statistic

(3.7)
$$t = \frac{\hat{\beta}_j - c}{\hat{\sigma}\sqrt{x^{jj}}}$$

Example:

With the money demand function, it is not the sole interest to examine whether the explanatory variables income and rate of interest do indeed have a statistically validated effect on money demand. In addition, for monetary policy it is important to establish to what extent ecomomies of scale do matter for holding money. To posit the existence of scale economies in terms of a rational holding of money, the income elasticity of money demand would have to be significantly smaller than 1. Conversely, diseconomies of scale occur if the income elasticity is greater than 1. A significance test of the hypothesis that the income elasticity equals 1 therefore reveals information about the extent to which the advantages or disadvantages of scale, indicated by the estimates of the regression coefficients, can in fact be interpreted as substantiated.

First, however, the significance of the factors of influence has to be tested. For this we need the variances of the regression coefficients , $\hat{\beta}_1, \hat{\beta}_2 \text{ und } \hat{\beta}_3$
With a squared sum of the residuals of

$$\sum \hat{u}_t^2 = \hat{u}'\hat{u} = 0,054376$$

and with n = 19 and k = 3, the unbiased estimator $\hat{\sigma}^2$ of the variance of the error term σ^2 is obtained from

$$\hat{\sigma}^2 = \frac{1}{n-k}\hat{u}'\hat{u} = \frac{0,054376}{19-3} = 0,003398$$

Using the diagonal elements

x^{11}= 451.821, x^{22}= 7.4157 and x^{33}= 0.2598

the unbiased estimators for the variances of the estimated regression coefficients can be calculated:

$$\hat{\mathrm{Var}}(\hat{\beta}_1) = \hat{\sigma}^2 \cdot x^{11} = 0,003398 \cdot 451,821 = 1,5355$$

$$\hat{\mathrm{Var}}(\hat{\beta}_2) = \hat{\sigma}^2 \cdot x^{22} = 0,003398 \cdot 7,4157 = 0,0252$$

and

$$\hat{\mathrm{Var}}(\hat{\beta}_3) = \hat{\sigma}^2 \cdot x^{33} = 0,003398 \cdot 0,2598 = 0,00088296$$

The test values for the significance tests of the estimated regression coefficients being 0 are thus given by:

$$t = \frac{\hat{\beta}_1}{\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_1)}} = \frac{-15,0174}{\sqrt{1,5355}} = -12,1191$$

$$t = \frac{\hat{\beta}_2}{\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_2)}} = \frac{2,8112}{\sqrt{0,0252}} = 17,7084$$

and

$$t = \frac{\hat{\beta}_3}{\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_3)}} = \frac{-0,1891}{\sqrt{0,00088296}} = -6,3641$$

All regression coefficients are statistically validated at a significance level of 5% because their t-values are larger in absolute terms than the critical value $t_{16;\,0.975}$= 2.120 of a two-sided test.

In a test of the income elasticity being 1,

$$H_0 : \beta_2 = 1,$$

the test statistic is

$$t = \frac{\hat{\beta}_2 - 1}{\sqrt{\hat{\mathrm{Var}}\left(\hat{\beta}_2\right)}} = \frac{2,8112 - 1}{\sqrt{0,0252}} = 11,4092$$

The null hypothesis is on the basis of a two-sided significance test ($t_{0.975;\,16}$= 2.120) and even more so on the basis of a one-sided (right side) significance test ($t_{0.95,\,16}$= 1.746) to be rejected. Concerning the money supply M1, within the observation period 1994-2012, "diseconomies of scale" in holding money can then in fact be assumed. □

3.2 Test of linear restrictions

In the tests considered so far, each regression coefficient has been looked at separately. However, econometric studies serve equally much in the detection of possible common effects of factors of influence. For example, when estimating the Cobb-Douglas-production function

(3.8) $$y_t = \beta_1 \cdot L_t^{\beta_2} \cdot K_t^{\beta_3} \cdot e^{u_t}$$

in which output y depends on the production factors labour L and capital K, it is of interest to know wether or not constant returns to scale do prevail. Mathematically one would then speak of a linear homogeneous (i.e. homogeneous of degree 1) production function. In this case, a concurrent multiplication of the two production factors by a factor λ would lead to the output $\lambda \cdot y_t$. The OLS–estimators of the production elasticities $\hat{\beta}_2$ und $\hat{\beta}_3$, resulting from the linearized production function

(3.9) $\ln y_t = \ln ß_1 + ß_2 \cdot \ln L_t + ß_3 \cdot \ln K_t + u_t$

would then appear to be in accord with the hypothesis

$$H_0: ß_2 + ß_3 = 1$$

This would involve a test of a linear combination of regression coefficients:

(3.10) $$r_1\beta_1 + r_2\beta_2 + ... + r_k\beta_k = \mathbf{r'\beta}$$

with

$$\mathbf{r}' = (r_1\ r_2\ ...\ r_k).$$

Specifically, it is tested whether the linear combination (3.10) takes on a specific value c. Consequently, we speak of a **test of a linear restriction.** The null hypothesis of the test is given as

(3.11) H_0: **r'ß** = C.

From an OLS-estimation of an econometric model, we get $\mathbf{r'\hat{\beta}}$ as an estimator for the linear combination **r'ß.** Under the condition that regression coefficients $\hat{\beta}_j$ are normally distributed, $\mathbf{r'\hat{\beta}}$ will also be normally distributed, because of the reproduction property of the normal distribution with respect to any linear combination of normally distributed random variables. Under the null hypothesis (3.11) the expected value of the linear combination $\mathbf{r'\hat{\beta}}$ is equal to c. Generalizing the variance of the random variables a* X (a const.),

$$\text{Var}(a \cdot X) = a^2 \cdot \text{Var}(X) = a * \text{Var}(X) * a,$$

the variance of the linear combination $\mathbf{r}'\hat{\boldsymbol{\beta}}$ is given by

(3.12a) $$\mathrm{Cov}(\mathbf{r}'\hat{\boldsymbol{\beta}}) = \mathbf{r}' \cdot \mathrm{Cov}(\hat{\boldsymbol{\beta}}) \cdot \mathbf{r}$$

With

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = \sigma^2 \cdot (\mathbf{X}'\mathbf{X})^{-1}$$

one gets

(3.12b) $$\mathrm{Cov}(\mathbf{r}'\hat{\boldsymbol{\beta}}) = \sigma^2 \cdot \mathbf{r}' \cdot (\mathbf{X}'\mathbf{X})^{-1}\mathbf{r} \ .$$

Similar to the estimates of the regression coefficients, the linear combination is under the condition of a known variance σ^2 of the error term and following a standardization standard normal distributed. If however σ^2 with $\hat{\sigma}^2$ is estimated in an unbiased manner, we get the test statistic

(3.13) $$t = \frac{\mathbf{r}'\hat{\boldsymbol{\beta}} - c}{\hat{\sigma}\sqrt{\mathbf{r}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{r}}}$$

which is t-distributed with n–k degrees of freedom.

In some econometrics packages, the t-test of linear restrictions is not directly available, but instead we find an equivalent F–test. The test statistic for the **F-test on a linear restriction**

(3.14) $$F = \frac{(\mathbf{r}'\hat{\boldsymbol{\beta}} - c)^2}{\hat{\sigma}^2 \cdot \mathbf{r}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{r}}$$

is F-distributed with 1 and n-k degrees of freedom. When testing on a linear restriction, the F-value is exactly equal to the square of the t-value:

$$F = t^2 \text{ (in case of a linear restriction)}$$

The F-test can consistently be expanded to the simultaneous testing of multiple linear restrictions. It is a special case of the Wald-test, which can also be used more generally in the case of non-linear restrictions.

Example

With the data on gross domestic product (Y), the number of employees (L), and capital stock (real assets) (K), we estimate the linearized production function (3.9) with the OLS method by using EViews:

Dependent Variable: LOG(BIP)
Method: Least Squares
Sample: 1991 2004 Included obs.: 14

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-3.359728	0.338956	-9.911979	0.0000
LOG(ET)	0.618047	0.111450	5.545485	0.0002
LOG(KAPITAL)	0.616240	0.020519	30.03262	0.0000

R-squared	0.993282	Mean dependent var	4.545379
Adjusted R-squared	0.992060	S.D. dependent var	0.065752
S.E. of regression	0.005859	Akaike info criterion	-7.254299
Sum squared resid	0.000378	Schwarz criterion	-7.117359
Log likelihood	53.78010	F-statistic	813.1610
Durbin-Watson stat	2.019197	Prob(F-statistic)	0.000000

Using the OLS-estimator

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_1 = e^{-3,3597} = 0,0347 \\ \hat{\beta}_2 = 0,6180 \\ \hat{\beta}_3 = 0,6162 \end{bmatrix}$$

it is possible to determine the required value of the linear combination $\mathbf{r}'\hat{\boldsymbol{\beta}}$ which is needed for the testing of the null hypothesis (restriction)

H_0: $\mathbf{r}'\boldsymbol{ß} = (0\ 1\ 1)\cdot (ß_1\ ß_2\ ß_3)' = 1$

or

H_0: $ß_2 + ß_3 = 1$

with the required value given by:

$$\mathbf{r}'\hat{\boldsymbol{\beta}} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \cdot \begin{bmatrix} 0,0347 \\ 0,6180 \\ 0,6162 \end{bmatrix} = 1,234286$$

Using the R package, the product matrix $\mathbf{X}'\mathbf{X}$ can be computed

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 14 & 51,0018 & 128,4403 \\ 510018 & 1858028 & 467,9183 \\ 128,4403 & 467,9183 & 1178,469 \end{bmatrix}$$

and then inverted:

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 3346{,}994 & -976{,}7859 & 23{,}05285 \\ -976{,}7859 & 361{,}8519 & -37{,}21647 \\ 23{,}05285 & -37{,}21647 & 12{,}26536 \end{bmatrix}$$

The radicand of the square root term in the denominator of the test statistic (3.13) then takes the value:

$$\mathbf{r}'\cdot(\mathbf{X}'\mathbf{X})^{-1}\cdot\mathbf{r} = (0 \;\; 1 \;\; 1)\cdot\begin{bmatrix} 3346{,}994 & -976{,}7859 & 23{,}05285 \\ -976{,}7859 & 361{,}8519 & -37{,}21647 \\ 23{,}05285 & -37{,}21647 & 12{,}26536 \end{bmatrix},\cdot\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 299{,}6843$$

The estimated standard deviation $\hat{\sigma}$ corresponds to the standard error of the regression (SER) and can therefore e.g. be taken from the R output (SER = $\hat{\sigma}$ = 0.005859).

Inserting the calculated values in the test statistic (3.13), we obtain

$$t = \frac{\mathbf{r}'\hat{\boldsymbol{\beta}} - c}{\hat{\sigma}\sqrt{\mathbf{r}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{r}}} = \frac{1{,}234286 - 1}{0{,}005859 \cdot \sqrt{299.6843}} = \frac{0{,}234286}{0{,}005859 \cdot 17{,}31139} = \frac{0{,}234286}{0{,}101427} = 2{,}310$$

In a test at a significance level of 5% (α= 0.05) the critical value (n = 14, k = 3) is

t (n-k; 1-α/ 2) = T (11; 0.975) = 2.201.

to be compared to:

| t = 2.310 | > t (11; 0.975) = 2.201 → H_0 to be rejected.

The null hypothesis of constant returns to scale thus can not be maintained in this case. The econometric estimation of the overall economic production function indicates increasing economies of scale, a finding that is statistically secured.

The F-statistic, which in our analysis takes the value 5.336, matches - except for rounding deviations - exactly the square of the estimated t-statistic

$$F = \frac{(\mathbf{r}'\hat{\boldsymbol{\beta}} - c)^2}{\hat{\sigma}^2 \cdot \mathbf{r}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{r}} = \frac{(1{,}234286 - 1)^2}{0{,}005859^2 \cdot 299{,}6843} = 5{,}33558 \ ,$$

If we compare the empirical F value with the critical value

F (1; n-k; 1-α) = F (1; 11; 0,95) = 4.84,

it becomes evident that in keeping with the t-test the null hypothesis of constant returns to scale must be rejected at the significance level of 5%:

F = 5,336> F (1; 11; 0.95) = 4.84 → H_0 to be rejected