5. Heteroscedasticity

5.1 Form and consequencies

So far it has been assumed that homoscedasticity of the disturbance term applies in the multiple regression model, i.e. its variance is the same in all periods and/or for all statistical units. Together with the assumption of a lack of autocorrelation this means that the variance-covariance-matrix is of a scalar type:

(2.7) $$\mathrm{Cov}(\mathbf{u}) = \mathrm{E}(\mathbf{uu}') = \sigma^2 \underset{\text{nxn}}{\mathbf{I}}$$

For the time being we retain the assumption of an absence of autocorrelation, but allow for different variances of the disturbance term u. If the exogenous variables e.g. follow an ascending trend, then for time-dependent data the variance of the disturbance term may be rising over time. In the case of cross-sectional data, consumption spending of high income classes usually varies more strongly than consumption spending of lower income classes. In both cases, the variance of the disturbance term u is no longer constant, but varies within the time or the cross-sectional units. In that case one speaks of **heteroscedasticity**. In the presence of a heteroscedastic disturbance term, its variance-covariance-matrix is given not by (2.7) but instead by (5.1):

(5.1) $$\mathrm{Cov}(\mathbf{u}) = \mathrm{E}(\mathbf{u}\mathbf{u}') = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

The variance-covariance-matrix Cov (**u**) is then an n×n diagonal matrix whose main diagonal elements display the variances of the disturbance terms of the individual observations. In the case of heteroscedasticity generally $\sigma^2_t \neq \sigma^2_s$ for $t \neq s$ applies, so that the variances of the disturbance terms of the different observations differ from each other.

Thus, in the variance-covariance matrix (5.1) n parameters are unknown. Therefore, together with the k unknown regression coefficients, k+n parameters would need to be estimated in a general model, which, however, with n observations would not be possible due to a lack of degrees of freedom. In an econometric estimation of the multiple regression model, which takes into account the presence of heteroscedasticity, assumptions about the type of heteroscedasticity therefore have to be made.

What impact does heteroscedasticity then have on the ordinary least squares estimation (OLS-estimation)? Let us first for a moment assume that the existence of heteroscedasticity is ignored and estimate a simple regression model with the OLS method

(2.21) $$y_t = \beta_1 + \beta_2 x_t + u_t$$

We limit ourselves to the effects on the OLS-estimator $\hat{\beta}_2$ of the slope coefficient β_2, which according to (2.25) takes the form

(5.2) $$\hat{\beta}_2 = \frac{s_{xy}}{s_x^2} = \frac{\sum(x_t - \bar{x})(y_t - \bar{y})}{\sum(x_t - \bar{x})^2} = \frac{\sum(x_t - \bar{x})y_t - \sum(x_t - \bar{x})\bar{y}}{\sum(x_t - \bar{x})^2}$$

Equation (5.2) can be simplified by using (2.21)

(5.3) $$\hat{\beta}_2 = \frac{\sum(x_t - \bar{x})y_t}{\sum(x_t - \bar{x})^2} = \beta_2 + \frac{\sum(x_t - \bar{x})\cdot u_t}{\sum(x_t - \bar{x})^2}$$

With respect to the expected value it follows from this

$$E(\hat{\beta}_2) = E\left[\beta_2 + \frac{\sum(x_t - \bar{x})\cdot u_t}{\sum(x_t - \bar{x})^2}\right] = \beta_2 + \frac{\sum(x_t - \bar{x})\cdot E(u_t)}{\sum(x_t - \bar{x})^2}$$

and with the assumption (2.2), $E(u_t) = 0$

$$E(\hat{\beta}_2) = \beta_2 \quad .$$

The OLS estimator $\hat{\boldsymbol{\beta}}$ remains unbiased even in case of heteroscedasticity.

By contrast, the variances of the regression coefficients are biased with OLS estimation. We show this with respect to the slope measure $\hat{\beta}_2$. Using (5.3) one gets

$$\mathrm{Var}\left(\hat{\beta}_2\right) = \mathrm{Var}\left[\beta_2 + \frac{\sum\left(x_t - \bar{x}\right)\cdot u_t}{\sum\left(x_t - \bar{x}\right)^2}\right] = \mathrm{Var}\left[\frac{\sum\left(x_t - \bar{x}\right)\cdot u_t}{\sum\left(x_t - \bar{x}\right)^2}\right]$$

since β_2, being a constant, has a variance of 0. While in the case of a homoscedastic disturbance term u_t one can extract the entire fraction because of $\mathrm{Var}\left(u_t\right) = \sigma^2$,

$$\text{(5.4)} \qquad \mathrm{Var}\left(\hat{\beta}_2\right) = \frac{\sum\left(x_t - \bar{x}\right)^2 \cdot \mathrm{Var}\left(u_t\right)}{\left[\sum\left(x_t - \bar{x}\right)^2\right]^2} = \frac{1}{\sum\left(x_t - \bar{x}\right)^2}\cdot\sigma^2,$$

this cannot be done in the case of a heteroscedastic disturbance term because of $\mathrm{Var}\left(u_t\right) = \sigma_t^2$. Instead one gets

$$\text{(5.5)} \qquad \mathrm{Var}\left(\hat{\beta}_2\right) = \frac{\sum\left(x_t - \bar{x}\right)^2 \cdot \mathrm{Var}\left(u_t\right)}{\left[\sum\left(x_t - \bar{x}\right)^2\right]^2} = \frac{\sum\left(x_t - \bar{x}\right)^2 \cdot \sigma_t^2}{\left[\sum\left(x_t - \bar{x}\right)^2\right]^2} \ .$$

Even though in the case of heteroscedasticity the variance (5.5) is valid, in an OLS estimation the estimation of the variance of $\hat{\beta}_2$ on the basis of expression (5.4) will be biased. Thereby at the same time the t–statistic looses its validity, since the denominator contains the square root of $Var\left(\hat{\beta}_2\right)$. Consequently with heteroscedastic disturbance variables it is no longer possible on the basis of the usual ordinary least square estimation (OLS estimation) to derive any valid statements about the significance of the estimated regression coefficients.

5.2 Tests for heteroscedasticity

5.2.1 Goldfeld-Quandt-Test

With the Goldfeld-Quandt test we introduce a rather traditional approach to the testing of the null hypothesis of the existence of homoscedastic disturbance terms, which, however, is quite popular in practice. For the disturbance terms the already discussed assumptions hold

$$E(u_t) = 0, \quad t = 1,\ldots,n$$

$$Cov(u_t, u_s) = E(u_t u_s) = 0, \quad t \neq s, \quad t = 1,\ldots,n$$

i.e. the disturbance terms have an expected value of 0 and are uncorrelated. Yet, the hypothesis of variance homogeneity is being questioned. In the Goldfeld-Quandt test normally distributed disturbance terms are assumed.

The underlying idea of the test is that the variance of the disturbance terms develops in dependence of the regressors. If the variance of the disturbance term, for Example, varies with increasing values of the j-th exogenous variable in the form

(5.6) $$\sigma_t^2 = \sigma^2 \cdot x_{jt}^2, \quad t = 1,2,...,n$$

then the variable x_j serves as a scaling variable ω_t . In (5.6) , is $\omega_t = x_{jt}^2$.

The Goldfield-Quandt–Test checks for the existence of a **groupwise heteroscedasticity**.

Grouping of the n observations so that their new order corresponds with the ascending order of the values of a regressor x_j , to which the heteroscedasticity is related back to:

Group 1	Group 2
low regressor values	high regressor values
disturbance variance σ_1^2	disturbance variance σ_2^2
sample size (n-c) / 2	sample size (n-c) / 2

[In the middle of the ordered observations, c observations are omitted to increase the discriminatory power of the test.

Rule of thumb: c = 4 with n = 30, c = 10 with n = 60]

Null hypothesis: H_0: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Alternative hypothesis: H_1: $\sigma_1^2 < \sigma_2^2$ or H_1: $\sigma_1^2 > \sigma_2^2$

Implementation of the Goldfeld-Quandt-Test

In each of the two samples a regression

(5.7) $$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \mathbf{u}_i\,, \quad i = 1,2$$

is carried out based on the OLS method, where the index i indicates the respective sample. From the estimated regressions, the OLS residuals can then be calculated

(5.8) $$\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}_i \quad i = 1,2$$

Teststatístik (for $\sigma_1^2 < \sigma_2^2$) :

(5.9a) $$GQ = \hat{\mathbf{u}}_2'\,\hat{\mathbf{u}}_2 \big/ \hat{\mathbf{u}}_1'\hat{\mathbf{u}}_1$$ or (5.9b) $$GQ = \hat{\sigma}_2^2 \big/ \hat{\sigma}_1^2$$

with the residual variances

(5.10) $$\hat{\sigma}_i^2 = \hat{\mathbf{u}}_i'\,\hat{\mathbf{u}}_i \big/ \left[(n\text{-}c)/2 - k\right], \quad i = 1,2$$

In case of $\sigma_1^2 > \sigma_2^2$ instead of GQ the reciprocal value of the test statistic 1/GQ is used.

Being a quotient of two independent χ^2-distributed random variables, the test statistic GQ is F-distributed with (n-c)/2-k numerator and denominator degrees of freedom.

Critical value: $F_{(n-c)/2 - k;\ (n-c)/2-k;\ 1-\alpha}$

Test decision:

$GQ > F_{(n-c)/2 - k;\ (n-c)/2-k;\ 1-\alpha}$ → H_0 to be rejected

Acceptance of the null hypothesis: Acceptance of homoscedasticity
Rejection of the null hypothesis: Acceptance of heteroscedasticity

Example:

In applying the Goldfeld-Quandt test on the money demand function (2.28) we set c = 1, so that the middle observation is excluded. Due to the high correlation between money supply and gross domestic product (r = 0,971 for the real value terms on a logarithmic scale) it seems reasonable to arrange the observations in ascending order using the regressor ln y. The two samples of the "smaller" and "larger“ values are then distinguished as follows:

1. Sample		2. Sample	
$\ln y_t$	year	$\ln y_t$	year
7,48560384	1994	7,69425618	2004
7,52212978	1995	7,70724250	2005
7,53636394	1996	7,74668969	2006
7,55621885	1997	7,77254217	2009
7,58054668	1998	7,79502906	2007
7,60100245	1999	7,81351071	2008
7,62437482	2000	7,82252485	2010
7,65059698	2001	7,86041651	2011
7,66490959	2002	7,88001038	2012

The first sample thus covers the period from 1994 to 2002; the second sample contains the observations from 2004 to 2012. In 2003, the regressor ln y takes the value of 7,672060, which lies exactly between the two groups and is therefore excluded.

OLS estimation of the money demand function:

1. Sample: $\hat{\ln m}_t = -18,259 + 3,215 \cdot \ln y_t - 0,042 \cdot \ln r_t$,

$R_1^2 = 0,984, \quad \hat{u}_1' \hat{u}_1 = 0,005178,$

2. Sample: $\hat{\ln m}_t = -17,283 + 3,102 \cdot \ln y_t - 0,203 \cdot \ln r_t$,

$R_2^2 = 0,964, \quad \hat{u}_2' \hat{u}_2 = 0,026826$

Test statístic (for $\sigma_1^2 < \sigma_2^2$) : $GQ = \hat{u}_2' \hat{u}_2 / \hat{u}_1' \hat{u}_1 = 0,026826/0,005178 = 5,1808$

Critical value (α= 0,05, n = 19, k = 3, c = 1): $F_{6;\,6;\,0,95} = 4,28$

Test decision:
GQ = 5,1808 > $F_{6;\,6;\,0,95}$ = 4.28 → H_0 to be rejected
GQ = 5,1808 < $F_{6;\,6;\,0,99}$ = 8.47 → H_0 to be accepted

5.2.2 Breusch-Pagan-Test

The Breusch-Pagan test is inasmuch a more general heteroscedasticity test as the Goldfeld-Quandt test as it does not rest on groupwise heteroscedasticity. It assumes that the variances of the disturbance terms u_t of a multiple regression model

(5.11) $$y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t$$

under the alternative hypothesis H_1 can be expressed as a function of the observable variables z_2, z_3...., z_p in the form

(5.12a) $$\sigma_t^2 = h\left(\alpha_1 + \alpha_2 \cdot z_{2t} + \ldots + \alpha_p \cdot z_{pt}\right)$$

or with

$$\boldsymbol{\alpha}' = \left(\alpha_1 \;\; \alpha_2 \;\; \cdots \;\; \alpha_p\right) \quad \text{and} \quad \mathbf{z}_t' = \left(z_{1t} = 1 \;\; z_{2t} \;\; \cdots \;\; z_{pt}\right)$$

(5.12b) $$\sigma_t^2 = h\left(\mathbf{z}_t'\boldsymbol{\alpha}\right)$$

The function h does not need to be further specified in this connection. With the approach, different instances of possible heteroscedasticity such as e.g.

$$\sigma_t^2 = \exp\left(z_t'\alpha\right) \quad or \quad \sigma_t^2 = z_t'\alpha$$

are covered. In this context, the variance of the disturbance terms can vary over the whole period. It is however stipulated that this variation can be completely traced back to the variables z_2, z_3, ... , z_p .

For p=2 the variance σ_t^2 of the disturbance variables u_t is solely dependent on one variable $z = z_2$:

$$\sigma_t^2 = h(\alpha_1 + \alpha_2 z_t)$$

Specifically, the disturbance variance σ_t^2 is in the case of linearity given by:

$$\sigma_t^2 = \alpha_1 + \alpha_2 z_t$$

As a z-variable, particular exogenous variables such as x_2 might be considered, which can also be used in logarithmic ($\ln x_2$) or squared (x_2^2) form.

Null hypothesis H_0: $\alpha_2 = \alpha_3 = ... = \alpha_p = 0$
Alternative hypothesis H_1: at least one $\alpha_j \neq 0$, j = 2,3, ... , p

The null hypothesis depicts the case of **homoscedasticity** because from this $\sigma_t^2 = \sigma^2$ follows for all t. Under the alternative hypothesis, at least two variances are unequal, $\sigma_t^2 \neq \sigma_s^2$ for $t \neq s$, which implies **heteroscedasticity**.

Implementation of the Breusch-Pagan-Test

The regression model (5.11) is first OLS-estimated, from which the residuals $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ arise. Using the maximum likelihood estimator (ML estimator) of the disturbance variance

(5.13) $$\hat{\sigma}^2_{ML} = \sum_{t=1}^{n} \hat{u}_t^2 \Big/ n$$

the terms $\hat{u}_t^2 \Big/ \hat{\sigma}^2_{ML}$ are constructed. The terms $\hat{u}_t^2 \Big/ \hat{\sigma}^2_{ML}$ serve as the dependent variable of the auxiliary regression

(5.14) $$\hat{u}_t^2 \Big/ \hat{\sigma}^2_{ML} = \mathbf{z}_t' \boldsymbol{\alpha} + v_t$$

with v_t denoting the disturbance term.

Dispersion decomposition of the auxiliary regression: $SS_T = SS_e + SS_R$

SS_T: total sum of squares of deviations of the dependent variables
SS_e: through the auxiliary regression explained sum of squares of deviations
SS_R: Sum of squared residuals

Calculating SS_e from: $SS_e = SS_T - SS_R$

Test statistic:

(5.15a) $BP_1 = \frac{1}{2} SS_E$ or (5.15b) $BP_2 = n \cdot R^2$

(The test statistics BP_1 and BP_2 are asymptotically equivalent. R^2 is the coefficient of determination of the auxiliary regression (5.14).)

For a normally distributed disturbance term u_t the test statistics BP_1 respectively BP_2 are under the null hypothesis asymptotically (i.e. with large samples) χ^2-distributed with p-1 degrees of freedom.

Critical value: $\chi^2_{p-1;\,1-\alpha}$

Test decision:

BP_1 (BP_2)> $\chi^2_{p-1;\,1-\alpha}$ → H_0 to be rejected

Acceptance of the null hypothesis: acceptance of homoscedasticity
Rejection of the null hypothesis: acceptance of heteroscedasticity

Example:

We deploy the Breusch-Pagan test to test for heteroscedasticity of the disturbance term of the money demand function (2.28). It is in this connection assumed that the variance of the disturbance term under the alternative hypothesis of heteroscedasticity, σ_t^2, is solely a function of (logarithmic) real GDP ln y:

$$\sigma_t^2 = h(\alpha_0 + \alpha_1 \cdot \ln y_t)$$

The logarithmic form of y is here chosen in keeping with the demand for money function.

To determine the ML estimator $\hat{\sigma}^2_{ML}$ for the variance σ^2 of the disturbance terms u_t of the econometric money demand model (2.28), we revert to the sum of squared residuals determined in section 2.1

$$\sum \hat{u}_t^2 = \hat{u}'\hat{u} = 0,054376$$

from which results

$$\hat{\sigma}^2_{ML} = \frac{\hat{u}'\hat{u}}{n} = \frac{0,054376}{19} = 0,002862$$

On this basis the expressions $\hat{u}_t^2 / \hat{\sigma}^2_{ML}$ can be calculated which in turn are regressed on ln y.

From the OLS estimation

$$\left(\hat{u}_t^2 / \hat{\sigma}_{ML}^2\right) = \underset{(-0,348)}{-8,156} + \underset{(0,391)}{1,192} \cdot \ln(y_t). \qquad R^2 = 0,061$$

the sum of squared residuals SS_R= 40,50927 can be determined. With SS_T= 43,14451 one obtains the explained deviation sum of squares from the auxiliary regression as

$$SS_e = SS_T - SS_R = 43,14451 - 40,50927 = 2,63524.$$

Test statistics: $BP_1 = \frac{1}{2} SS_E = 1,318$ and $BP_2 = n \cdot R^2 = 19 \cdot 0,061 = 1,159$

Critical value (α= 0,05; p = 2): $\chi^2_{1;\,0,95} = 3,841$

Test decision:

$BP_1 = 1,318 < \chi^2_{1;\,0,95} = 3,841 \rightarrow H_0$ maintained

$BP_2 = 1,159 < \chi^2_{1;\,0,95} = 3,841 \rightarrow H_0$ maintained

Both test statistics BP_1 and BP_2 indicate that the null hypothesis of homoscedasticity cannot be rejected at a significance level of 5%.

5.2.3 White-Test

The **White-Test** is based on a comparison of the estimated variance-covariance-matrices of the Oestimator for the parameter vector **β.** The covariance-matrix of the classical model, which is consistent only in the case of homosceda-sticity, is compared with an estimation which is consistent both with hetero– as well as with homoscedastic disturbance terms.

The test is implemented by squaring the residuals of the original model and subsequent, in compliance with

$$\hat{u}_t^2 = \alpha_0 + \sum_{i=2}^{k} \alpha_i x_{it} + \sum_{i=2}^{k} \sum_{j \geq i}^{k} \alpha_{ij} x_{it} x_{jt} + v_t \qquad (5.16)$$

by running a regression on the k-1 original regressors, their squares, and their crossproducts. The term v_t is the disturbance variable in the auxiliary regression (5.16).

Test statistic: (5.17) $W = n \cdot R^2$

The test statistic is under the null hypothesis of homoscedasticity χ^2-distributed with r = (k - 1)(k + 2)/2 degrees of freedom.

Critical value: $\chi^2_{r;\, 1-\alpha}$

In the case of a large number of regressors, the White test can be conducted disregarding the quadratic terms and cross products. In this case the test statistic has r = k-1 degrees of freedom.

Example:

With the White test the residuals of the money demand function (2.28) is to be examined for heteroscedasticity. For this purpose, the squared residuals are regressed on the original regressors, their squares, and their cross products. Estimated is thus this auxiliary regression

$$\hat{u}_t^2 = \alpha_0 + \alpha_1 \cdot \ln y_t + \alpha_2 \cdot \ln r_t + \alpha_3 \cdot (\ln y_t)^2 + \alpha_4 \cdot (\ln r_t)^2 + \alpha_5 \cdot \ln y_t \cdot \ln r_t + v_t$$

from which a coefficient of determination R^2 of 0,238 appears.

Test statistics (n = 19): $W = n \cdot R^2 = 19 \cdot 0{,}238 = 4{,}522$

Number of degrees of freedom (k = 3): $r = (k-1)(k+2)/2 = (3-1)(3+2)/2 = 5$

Critical value ($\alpha = 0{,}05$; r = 5): $\chi^2_{5;0,95} = 11{,}070$

Test decision:
$W = 4{,}522 < \chi^2_{5;0,95} = 11{,}070 \rightarrow H_0$ maintained, i.e. the null hypothesis of homoscedasticity cannot be rejected at a significance level of 5%.

5.3 Model estimation in the presence of heteroscedasticity

5.3.1 Generalized Least Squares (GLS)

In an application of **generalized least squares (GLS)** the assumption of a scalar covariance-matrix of the disturbance terms u_t, Cov (**u**) = $\sigma^2 \cdot \mathbf{I}$ is replaced by the more general assumption

(5.18) $$\mathrm{Cov}(\mathbf{u}) = \sigma^2 \boldsymbol{\Omega}$$

With **heteroscedastic** but uncorrelated **disturbance terms** u, the **covariance matrix** Cov(**u**) takes the form:

(5.19) $$\mathrm{Cov}(\mathbf{u}) = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{pmatrix}$$

(5.19) is however typically represented in the form (5.18). In general, heteroske-dasticity can be modeled with a scaling variable ω_t, usually perceived as a function of one (or more generally: multiple) exogenous variable z_t. A simple **model for determining heteroscedastic variances** σ_t^2 is then given by

(5.20a) $$\sigma_t^2 = \sigma^2 \cdot \omega_t = \sigma^2 z_t^{\delta}, t = 1, \ldots, n$$

In (5.20a) merely the parameters σ^2 and δ are unknown. While σ^2 now is a scale factor, the parameter δ needs to be chosen so that the disturbance variances σ_t^2 are positive.

Let us, however, first take a look at the basic estimation of a regression model with heteroscedastic disturbance terms. As will be seen, in this case the OLS method is to be replaced by the GLS method. To make the basic structure of the generalized least-squares estimator in the context of the linear regression model more transparent, it is for a start assumed that the n×n-Matrix **Ω** is known.

If **Ω** is symmetric and positive definite, it also applies to its inverse $\mathbf{\Omega}^{-1}$ which can then be factored as follows:

(5.21) $$\mathbf{\Omega}^{-1} = \mathbf{T}'\mathbf{T}$$

[A n×n–Matrix **A** is positive definite if its quadratic form is positive, i.e. for all n×1–vectors $\mathbf{x} \neq \mathbf{0}$ the expression $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ must apply.]

T is therein a regular matrix of the order n. If now the econometric model

(2.5) $$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

is multiplied from the left with the matrix **T,** wich is initially assumed to be known, it follows:

(5.22) $$\mathbf{T}\mathbf{y} = \mathbf{T}\mathbf{X}\boldsymbol{\beta} + \mathbf{T}\mathbf{u}$$

Equation (5.22) is a transformed regression model of the type (2.5)

(5.23) $$\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \mathbf{u}^*$$

with

(5.24) $$\mathbf{Ty} = \mathbf{y}^*, \quad \mathbf{TX} = \mathbf{X}^* \quad \text{und} \quad \mathbf{Tu} = \mathbf{u}^*$$

The key point now is that the transformed disturbance variable u* fulfils the standard assumptions:

(5.25) $$\mathrm{E}\left(\mathbf{u}^*\right) = \mathbf{0}$$ and (5.26) $$\mathrm{Cov}(\mathbf{u}^*) = \mathrm{E}\left(\mathbf{u}^*\mathbf{u}^{*\prime}\right) = \sigma^2\mathbf{I}$$

which does not hold for the original heteroscedastic disturbance variable **u.** For this reason, the unknown vector of regression coefficients, **β**, can be estimated from the transformed model in the usual way by the ordinary least squares method. Since the estimate is, however, not derived from the original regression model (2.5), but from the transformed model (5.23), the estimation is denoted **generalized least-squares estimation (GLS estimation).**

Proof of (5.25):

$$\mathrm{E}\left(\mathbf{u}^{*}\right)=\mathrm{E}(\mathbf{T}\mathbf{u})=\mathbf{T}\mathrm{E}(\mathbf{u})=\mathbf{0}$$ □

Proof of (5.26):

$$\mathrm{Cov}\left(\mathbf{u}^{*}\right)=\mathrm{E}(\mathbf{u}^{*}\mathbf{u}^{*\prime})=\mathrm{E}\left[\mathbf{T}\mathbf{u}(\mathbf{T}\mathbf{u})'\right]=\mathrm{E}(\mathbf{T}\mathbf{u}\mathbf{u}'\mathbf{T}')=\mathbf{T}\mathrm{E}(\mathbf{u}\mathbf{u}')\mathbf{T}'$$

$$\mathrm{Cov}(\mathbf{u}^{*})=\mathbf{T}\sigma^{2}\boldsymbol{\Omega}\mathbf{T}'=\sigma^{2}\mathbf{T}\Omega\mathbf{T}'$$

Owing to $\boldsymbol{\Omega}=(\mathbf{T}'\mathbf{T})^{-1}=\mathbf{T}^{-1}(\mathbf{T}')^{-1}$ it follows

[The calculation of the inverse is done according to the rule $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$.]

$$\mathrm{Cov}(\mathbf{u}^{*})=\sigma^{2}\mathbf{T}\mathbf{T}^{-1}(\mathbf{T}')^{-1}\mathbf{T}'$$

Because of

$$\mathbf{T}\mathbf{T}^{-1}=\mathbf{I} \quad \text{und} \quad (\mathbf{T}')^{-1}\mathbf{T}'=\mathbf{I}$$

we finally get

$$\mathrm{Cov}(\mathbf{u}^{*})=\sigma^{2}\mathbf{I}$$ □

For this reason, the parameter vector **β** of the transformed model (5.23) can be estimated with the OLS method. This estimator, given by

(5.27a) $$\hat{\boldsymbol{\beta}}_{GLS} = \left(\mathbf{X}^{*'}\mathbf{X}^{*}\right)^{-1}\mathbf{X}^{*'}\mathbf{y}^{*},$$

is called **GLS–estimator** (Generalized-Least-Squares-Estimator).

In determining the GLS-estimator $\hat{\boldsymbol{\beta}}_{GLS}$, no need whatsoever exists to revert to the transformation matrix **T**. Rather, it can be directly determined from

(5.27b) $$\hat{\boldsymbol{\beta}}_{GLS} = \left(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y}$$

wherein only the inverse of the covariance matrix **Ω** except for the scalar σ^2 enters.
To carry out significance tests and for the determination of confidence intervals one still needs the covariance matrix of $\hat{\boldsymbol{\beta}}_{GLS}$, given by

(5.28a) $$\mathrm{Cov}\left(\hat{\boldsymbol{\beta}}_{GLS}\right) = \sigma^2\left(\mathbf{X}^{*'}\mathbf{X}^{*}\right)^{-1}$$

or with **X*** = **TX**

(5.28b) $$\mathrm{Cov}\left(\hat{\boldsymbol{\beta}}_{GLS}\right) = \sigma^2\left(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}\right)^{-1}$$

The unknown disturbance variance σ^2 can in this connection be estimated by

(5.29a) $$\hat{\sigma}^2_{GLS} = \frac{\hat{\mathbf{u}}^{*\prime}\hat{\mathbf{u}}^{*}}{n - k}$$

where $\hat{\mathbf{u}}^{*}$ are the GLS residuals:

(5.30) $$\hat{\mathbf{u}}^{*} = \mathbf{y}^{*} - \mathbf{X}^{*}\hat{\boldsymbol{\beta}}_{GLS}$$

From the original data, the GLS estimator (5.29a) of the disturbance variance σ^2 can be determined by using

(5.29b) $$\hat{\sigma}^2_{GLS} = \frac{\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS}\right)^{\prime}\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS}\right)}{n - k}$$

As can be seen from the formulae (5.27b), (5.28b), and (5.29b), knowledge of the transformation matrix **T** is not necessarily required in order to carry out the GLS estimation. Only the covariance matrix **Ω** must be known or estimated. With its determination, based on model (5.20a) concerning the determination of the heteroscedastic variances σ_t^2, we will turn to in the following section.

5.3.2 Weighted Least Squares method (WLS)

The **weighted least-squares method** (**WLS method**), being a special case of the GLS-method, is in the presence of heteroscedastic disturbance terms based on the model

(5.20b) $$\sigma_t^2 = \sigma^2 z_t^{\delta}, t = 1,...,n$$

for the variances σ_t^2. The validity of this approach can be empirically examined with the Breusch-Pagan test of heteroscedasticity. In the simplest case the parameter δ, which has to be an even number, is set equal to 2:

(5.20c) $$\sigma_t^2 = \sigma^2 z_t^2, t = 1,...n,$$

Under the assumption (5.20b), the covariance matrix of the disturbance variables Cov(**u**) is given by:

$$\mathrm{Cov}(\mathbf{u}) = \sigma^2 \mathbf{\Omega} = \begin{pmatrix} \sigma^2 z_1^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 z_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma^2 z_n^2 \end{pmatrix} = \sigma^2 \cdot \begin{pmatrix} z_1^2 & 0 & \cdots & 0 \\ 0 & z_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n^2 \end{pmatrix}$$

From

$$\mathbf{\Omega} = \begin{pmatrix} z_1^2 & 0 & \cdots & 0 \\ 0 & z_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n^2 \end{pmatrix}$$

the transformation matrix **T** can in an easy way be determined.
Since the inverse of **Ω**

$$\mathbf{\Omega}^{-1} = \begin{pmatrix} 1/z_1^2 & 0 & \cdots & 0 \\ 0 & 1/z_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/z_n^2 \end{pmatrix}$$

due to (5.21) is given by the product matrix **T'T**, it follows immediately

$$\mathbf{T} = \begin{pmatrix} 1/z_1 & 0 & \cdots & 0 \\ 0 & 1/z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/z_n \end{pmatrix}$$

If the z values are given by the observations of a single exogenous variable, e.g. x_2, the transformation matrix **T** becomes specified as

$$\mathbf{T} = \begin{pmatrix} 1/x_{21} & 0 & \cdots & 0 \\ 0 & 1/x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/x_{2n} \end{pmatrix}$$

One then obtains a GLS estimator for **β** by multiplying the variables of the model equation

(2.1a) $$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \ldots + \beta_k x_{kt} + u_t$$

with the inverse observation values $1/x_{2t}$ and by estimating the transformed model

(5.31) $$\underset{\uparrow\, y_t^*}{\frac{y_t}{x_{2t}}} = \beta_1 \underset{\uparrow\, x_{1t}^*}{\frac{1}{x_{2t}}} + \underset{\uparrow\, x_{2t}^* = 1}{\beta_2} + \beta_3 \underset{\uparrow\, x_{3t}^*}{\frac{x_{3t}}{x_{2t}}} + \ldots + \beta_k \underset{\uparrow\, x_{kt}^*}{\frac{x_{kt}}{x_{2t}}} + \underset{\uparrow\, u_t^*}{\frac{u_t}{x_{2t}}}$$

with the OLS method.

[The transformed disturbance variable u * is homoscedastic, i.e. its variance varies no longer within the observation units, but is instead constant. It is

$$\mathrm{Var}\left(u_t^*\right) = \mathrm{Var}\left(\frac{u_t}{x_{2t}}\right) = \frac{1}{x_{2t}^2}\mathrm{Var}(u_t) = \frac{1}{x_{2t}^2}\sigma^2 z_t^2 \ .$$

so that with $z_t = x_{2t}$ one obtains

$$\mathrm{Var}\left(u_t^*\right) = \sigma^2 \ ]$$

The heteroscedasticity is with this procedure therefore eliminated by a weighting of the model variables. The weights are in this connection generally given by the factor1/z_t. The estimation of the vector of the regression coefficients, **β**, of the multiple regression model (2.1a) with the OLS method using the transformed variables

$$y_t^* = \frac{y_t}{z_t}, \quad x_{1t}^* = \frac{x_{1t}}{z_t}, \quad x_{2t}^* = \frac{x_{2t}}{z_t}, \quad \ldots, \quad x_{kt}^* = \frac{x_{kt}}{z_t}, \quad u_t^* = \frac{u_t}{z_t}$$

is for this reason also called **WLS estimation** (w**eighted-least squares)**.

Example:

Although the null hypothesis of homoscedastic disturbance terms of the money demand function (2.28) could not be rejected, shall now the application of the WLS method as a special case of the GLS method be charted in an exemplary fashion for the case of heteroscedasticity. The starting point is in this connection the OLS estimation of the money demand function:

$$\underset{\wedge}{\ln m_t} = \underset{(-12{,}119)}{-15{,}017} + \underset{(17{,}708)}{2{,}811} \cdot \ln y_t \underset{(-6{,}364)}{-0{,}189} \cdot \ln r_t\,, \quad R^2 = 0{,}984$$

To model heteroscedasticity we use the approach (5.20b), where heteroscedasticity is attributed to an exogenous variable. If the regressor ln y is chosen for this purpose, then can the disturbance variance σ_t^2 be specified in the form

$$\sigma_t^2 = \sigma^2 \cdot (\ln y_t)^2$$

The regression model of the weighted method of least squares (WLS method) as a special case of the GLS method then becomes:

(5.32)
$$\left(\frac{\underset{\wedge}{\ln m_t}}{\ln y_t} \right) = \beta_1 \cdot \frac{1}{\ln y_t} + \beta_2 + \beta_3 \cdot \frac{\ln r_t}{\ln y_t} + u_t^*$$

with $u_t^* = u_t / \ln y_t$. OLS estimation of the econometric model (5.32) leads to the GLS estimators

$$\hat{\beta}_{1,GLS} = \quad -15,025 \quad (t = -12,114)$$

$$\hat{\beta}_{2,GLS} = \quad 2,812 \quad (t = 17,679)$$

und

$$\hat{\beta}_{3,GLS} = \quad -0,188 \quad (t = -6,261),$$

whose t-statistics can be used for the purpose of valid significance tests and confidence intervals.

One notes that the coefficient of determination of the OLS estimation $R_{OLS}^2 = 0,984$ is not directly comparable with e.g. the coefficient of determination $R_{WLS}^2 = 0,973$ which is displayed in program packages such as EViews or SPSS, because the latter relates to the transformed variables.

In order to obtain a coefficient of determination comparable to R_{OLS}^2, the WLS-regression values for the original (not for the transformed!) observations are first of all to be to determined:

$$\ln \hat{m}_{t,WLS} = -15,02488 + 2,81211 \cdot \ln y_t - 0,18835 \cdot \ln r_t$$

Thus one gets, for example for period 1, the WLS regression value

$$\ln \hat{m}_{1,GLS} = -15,02488 + 2,81211 \cdot \ln y_1 - 0,18835 \cdot \ln r_1$$

$$= 5,71135.$$

If one carries out the corresponding calculations for the support period of the regression, then one obtains the time series of the regression values $\hat{\ln m}_{t,WLS}$:

t	1	2	3	4	5	6
$\hat{\ln m}_{t,WLS}$	5,71135	5,84489	5,94334	5,99917	6,05650	6,14945
t	7	8	9	10	11	12
$\hat{\ln m}_{t,WLS}$	6,14098	6,21915	6,30482	6,39293	6,47248	6,50900
t	13	14	15	16	17	18
$\hat{\ln m}_{t,WLS}$	6,55275	6,62531	6,67728	6,79804	7,01496	7,03007
t	19					
$\hat{\ln m}_{t,WLS}$	7,26515					

With the variance of the logarithmic money supply $s^2_{\ln m} = 0,1860949$ and the variance explained by the regression $s^2_{\hat{\ln m},WLS} = 0,1828249$, one obtains the coefficient of determination of the WLS estimation for the variables at their original measurment level:

$$R^2_{WLS} = \frac{s^2_{\hat{\ln m},WLS}}{s^2_{\ln m}} = \frac{0,1828249}{0,1860949} = 0,982$$

□

5.3.3 Heteroscedasticity-consistent (HC) standard errors

If the heteroscedasticity can actually be attributed to an exogenous variable on the basis of the approach (5.20a), then the vector of the regression coefficients, **β,** can in fact be efficiently estimated with the **generalized method of least squares (GLS)** or **weighted method of least squares (WLS)**. What is the situation if there is uncertainty about the form of heteroscedasticity?

Since the regression coefficients β_j in the case of heteroscedasticity continue to be unbiased if estimated with the OLS method, but this does not apply to their standard deviations, $\sigma_{\hat{\beta}_j}$, the point of departure in econometric practice is commonly taken solely with respect to the latter. In particular, often one confines oneself to estimating the standard errors of the regression coefficients, $\sigma_{\hat{\beta}_j}$, in such a way that the significance tests of an OLS estimation of the regression coefficients β_j can once again be carried out in a valid fashion.

To conceive an approach for the determination of heteroscedasticity-consistent (HC) standard errors, it is necessary to start from the general form of the variance-covariance matrix of the estimated regression coefficients (see intermediate steps relating to the derivation of $\mathrm{Cov}(\hat{\boldsymbol{\beta}})$ by the significance tests of an OLS-estimation),

.

(5:33) $$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

which in the case of heteroscedastic disturbance terms is given by

(5.33a) $$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\,\mathrm{diag}(\sigma_1^2,\sigma_2^2,...,\sigma_n^2)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

with

(5.1) $$\mathrm{Cov}(\mathbf{u}) = \sigma^2\cdot\boldsymbol{\Omega} = \mathrm{diag}(\sigma_1^2,\sigma_2^2,...,\sigma_n^2) = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Using the matrix

(5:34) $$\mathbf{S}_0 = \frac{1}{n}\mathbf{X}'\,\mathrm{diag}(\sigma_1^2,\sigma_2^2,...,\sigma_n^2)\mathbf{X} = \frac{1}{n}\sum_{t=1}^{n}\sigma_t^2\cdot\mathbf{x}_t\mathbf{x}_t'$$

the variance-covariance matrix $\mathrm{Cov}(\hat{\boldsymbol{\beta}})$ takes the form

(5.33b) $$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = n(\mathbf{X}'\mathbf{X})^{-1}\mathbf{S}_0(\mathbf{X}'\mathbf{X})^{-1}$$

White (1980) has shown that the estimator

(5.35) $$\hat{\mathbf{S}}_0 = \frac{1}{n}\mathbf{X}'\,\mathrm{diag}(\hat{u}_1^2,\hat{u}_2^2,...,\hat{u}_n^2)\mathbf{X} = \frac{1}{n}\sum_{t=1}^{n}\hat{u}_t^2\cdot\mathbf{x}_t\mathbf{x}_t'$$

as a function of sample data for $n\to\infty$ converges to the same (positive definite) matrix as $\mathbf{S}_0$, which is the basis of his HC–estimator

(5.36A) $$\hat{Cov}(\hat{\boldsymbol{\beta}}) = n(\mathbf{X}'\mathbf{X})^{-1}\hat{\mathbf{S}}_0(\mathbf{X}'\mathbf{X})^{-1}$$

White's heteroscedasticity-consistent (HC) estimator for $Cov(\hat{\beta})$ can be calculated equivalently with the use of the formula .

(5.36b) $$\hat{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\,\mathrm{diag}(\hat{u}_1^2, \hat{u}_2^2, ..., \hat{u}_n^2)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

Following White, in the case of a simple regression (k = 2) the HC estimator for the variance of the regression coefficients $\hat{\beta}_2$ (= slope measure) is given by

(5.37) $$\hat{Var}(\hat{\beta}_2) = \frac{\sum (x_t - \bar{x})^2 \cdot \hat{u}_t^2}{\left[\sum (x_t - \bar{x})^2\right]^2}$$

Equation (5.37) can be interpreted as sample equivalent to equation (5.5),

(5.5) $$Var(\hat{\beta}_2) = \frac{\sum (x_t - \bar{x})^2 \cdot \sigma_t^2}{\left[\sum (x_t - \bar{x})^2\right]^2}$$

insofar as the unknown disturbance variances σ_t^2 have been replaced by the squared residuals $\hat{u}_t^2$.

White‘s HC estimator possesses due to its consistency one for large samples desirable property, which however may not necessarily prevail for small samples. For this reason, various modifications have been suggested to help improve its performance in small samples.

The starting point for the modifications is always **White‘s HC estimator** (5.36a/b), which is therefore referred to as the **HC0 estimator**. A simple adjustment consists in replacing the sample size n in (5.36a) with the number of degrees of freedom n-k. The **HC1 estimator** is obtained after multiplication of HC0 with the adjusting factor n/(n-k):

(5.38) $$\mathrm{HC1} = \frac{n}{n-k} \cdot \mathrm{HC0} \quad ,$$

With the **HC2 estimator**, the squared residuals $\hat{u}_t^2$ of the White estimator (5.36b) is replaced by the modified squared residuals $\hat{u}_t^2 /(1-h_{tt})$:

(5.39) $$\mathrm{HC2} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\,\mathrm{diag}[\hat{u}_1^2 /(1-h_{11}),\ldots,\hat{u}_n^2 /(1-h_{nn})]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \quad .$$

Here, the terms h_{tt} are the main diagonal elements of the hat matrix **H** (projection matrix, which transforms the vector of the dependent variable **y** into the vector of the regression values $\hat{\mathbf{y}}$),

(5.40) $$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \,,$$

with which the relationship

(5.41) $\hat{\mathbf{u}} = (\mathbf{I} - \mathbf{H}) \cdot \mathbf{u}$

between the OLS residuals $\hat{u}_t$ and the unknown disturbance terms u_t can be established. In light of the necessity to replace the disturbance terms u_t with the residuals, $\hat{u}_t$, the HC2 estimator (5.39) exploits the relationship between these two terms.

With $\mathbf{x}_t'$ being the t-th row of the observation matrix **X**, the diagonal elements h_{tt} can be directly determined from

(5.42) $$h_{tt} = \mathbf{x}_t'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_t$$

A further modification in the construction of a HC estimator for the variance-covariance-matrix (5.33a /b) is obtained by relating the residual squares $\hat{u}_t^2$ to the squared expressions $(1 - h_{tt})^2$. Herewith is the **HC3 estimator** defined as

(5.42) $$HC3 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\,\mathrm{diag}[\hat{u}_1^2/(1-h_{11})^2,\ldots,\hat{u}_n^2/(1-h_{nn})^2]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

which in simulation studies has been shown to be advantageous in small samples.

[HC3 estimator: Default HC estimator in the functions HCCM and vcovHC of the R-packages car and sandwich.]

Example:

As an example, the significance tests of the OLS estimated regression coefficients for the money demand function are performed by using the heteroscedasticity-consistent (HC) standard errors.

To begin with, that various HC estimators of the variance-covariance-matrix $\widehat{Cov}(\hat{\boldsymbol{\beta}})$ are compared. [With the program R, HC estimators can be calculated via the functions hccm (package car) and vcovHC (package sandwich)]. The original **White-corrected variance-covariance matrix (HC0 estimator)**, due to the absence of an adjustment for finite samples, exhibits the smallest variance values, which are indicated on the main diagonal:

HC0 (White):

	(Intercept)	lny	lnr
(Intercept)	0.407079263	-0.0519598602	-0.0066556362
lny	-0.051959860	0.0066412016	0.0008046838
Lnr	-0.006655636	0.0008046838	0.0004015473

The estimated variances increase successively with the correction for finite samples using the HC estimators HC1, HC2, and HC3:

HC1:

	(Intercept)	lny	lnr
(Intercept)	0.483406625	-0.061702334	-0.0079035680
lny	-0.061702334	0.007886427	0.0009555620
Lnr	-0.007903568	0.000955562	0.0004768374

HC2:

	(Intercept)	lny	lnr
(Intercept)	0.530998314	-0.067824698	-0.0087957622
lny	-0.067824698	0.008673837	0.0010702662
Lnr	-0.008795762	0.001070266	0.0004960037

HC3:

	(Intercept)	lny	lnr
(Intercept)	0.70461568	-0.09005106	-0.0119252672
lny	-0.09005106	0.01152118	0.0014599601
lnr	-0.01192527	0.00145996	0.0006278258

Since the by default calculated variances of the OLS estimated regression coefficients are greater than the preferred HC3 estimators, it is sufficient in the context of the significance tests to compare these two estimators.

To this end, the estimated variances of the regression coefficients $\hat{\beta}_j$, displayed on the main diagonal, are extracted:

$$\hat{\mathrm{Var}}(\hat{\beta}_1)_{HC3} = 0.70461568, \quad \hat{\mathrm{Var}}(\hat{\beta}_2)_{HC3} = 0.01152118, \quad \hat{\mathrm{Var}}(\hat{\beta}_3)_{HC3} = 0.00062782$$

From this, the heteroscedasticity-consistent (HC3) standard errors emerge

$$\hat{\sigma}_{1,HC3} = \sqrt{\hat{\mathrm{Var}}(\hat{\beta}_1)_{HC3}} = 0.83941389,$$

$$\hat{\sigma}_{2,HC3} = \sqrt{\hat{\mathrm{Var}}(\hat{\beta}_2)_{HC3}} = 0.10733676,$$

$$\hat{\sigma}_{3,HC3} = \sqrt{\hat{\mathrm{Var}}(\hat{\beta}_3)_{HC3}} = 0.02505645,$$

which in an OLS estimation in the presence of heteroscedasticity are to be used instead of the uncorrected standard errors for significance tests of the estimated regression coefficients.

Variable	Standard errors for homosce-dastic disturbance variables	Heteroscedastic–consistent standard errors (HC3)
C	1.23616	0.83941
LNY	0.15837	0.10734
LNR	0.02964	0.02506

While the standard deviation of the regression coefficient for the interest rate (LNR) with the assumption of homoscedastic disturbance variables exceeds the chosen HC estimator by almost 20%, the potential overestimation of the standard error for income (LNY) amounts to almost 50%

Variable	t-values for homosceda-stic disturbance variables	t-values for heteroscedastic–consistent standard errors (HC3)
C	-12.145	-17.885
LNY	17.748	26.186
LNR	-6.382	-7.550

The t-values increase when using the HC3 standard error, in part considerably. However, owing to the already acquired high significance of the estimated regression coefficients in the case of the uncorrected tests, no qualitative changes in the test result show up in this example.