APPENDIX 2.1
Change of variables in density functions (use of the Jacobian)

The basic idea may be simply illustrated for the univariate case. Suppose u is a random variable with density function $f(u)$. Let a new variable y be defined by the relation $y = g(u)$. The y variable must then also have a density function, which will obviously depend on both $f(u)$ and $g(u)$. Suppose that the relation between y and u is monotonically increasing, as in Fig. A-2.1. Whenever u lies in the interval Δu, y will be in the corresponding interval Δy. Thus

$$\boxed{\Pr(y \text{ lies in } \Delta y) = \Pr(u \text{ lies in } \Delta u)}$$

or

$$\boxed{f(y')\Delta y = f(u')\Delta u}$$

where u' and y' denote appropriate values of u and y in the intervals Δu and Δy, and $f(y)$ indicates the postulated density function for y. Taking limits as Δu goes to zero gives

$$f(y') = f(u')\frac{\Delta u}{\Delta y} \xrightarrow{\lim_{\Delta u \to 0}} \boxed{f(y) = f(u)\frac{du}{dy}}$$

If y were a decreasing function of u, the derivative in this last expression would be negative, thus implying an impossible negative value for the density function. Therefore, the *absolute* value of the derivative must be taken and the result reformulated to read

Jacobian

$$\boxed{f(y) = f(u)\left|\frac{du}{dy}\right|}$$

If $y = g(u)$ were not a monotonic function this last result would require amendment, Zusatz/Abänderung but we are only concerned with monotonic transformations. The relevant relationship

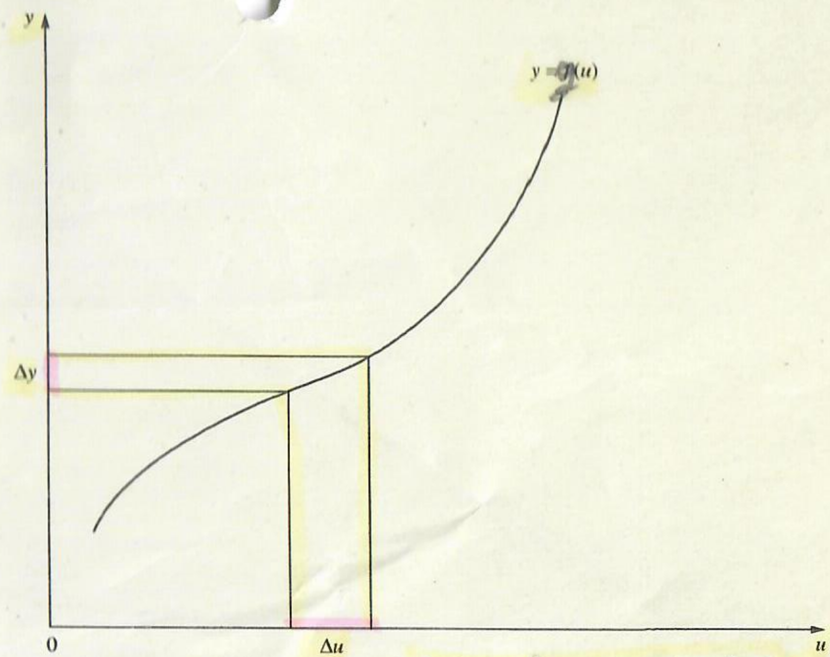

FIGURE A-~~2~~.1

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$
$$u_i = Y_i - \beta_1 - \beta_2 X_i$$
$$\frac{du_i}{dY_i} = 1$$

our cor- in the text is

$$\cancel{Y_t = \alpha + \beta Y_{t-1} + u_t} \tag{2.19}$$

which gives

$$\boxed{\frac{du_t}{dY_t} = 1 \qquad \text{for all } t} \Rightarrow f(y) = f(u)$$

which gives the joint density shown in Eq. (2.37).

APPENDIX 2.2
Maximum likelihood estimators for the AR(1) model

The log-likelihood is given in Eq. (2.39) as

$$l = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(Y_t - \alpha - \beta Y_{t-1})^2$$

The partial derivatives with respect to the three parameters are

$$\frac{\partial l}{\partial \alpha} = \frac{1}{\sigma^2}\sum_{t=1}^{n}(Y_t - \alpha - \beta Y_{t-1})$$