Lecture Notes on General Relativity

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This is a summary of the main ideas, equations and formulae appearing in the course on General Relativity. The purpose of these notes is to supplement the course, not to replace a text-book. You should turn to other sources for other explanations, more on the historical and experimental side, and exercises to test your understanding of these matters. Nevertheless, I hope that these notes are reasonably self-contained and comprehensible.

I make no claim to originality in these notes. In particular, the presentation of much of the introductory material follows quite closely the treatment in Weinberg's 'Gravitation and Cosmology', but I have also used a number of other sources.

Sections marked with a * contain supplementary material that is not strictly necessary for an understanding of the subsequent sections. I am grateful for feedback of any kind: complaints, constructive criticism, and corrections.

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PART I: TOWARDS THE EINSTEIN EQUATIONS

1 FROM THE EINSTEIN EQUIVALENCE PRINCIPLE TO GEODESICS

INTRODUCTION

The year 1905 was Einstein's magical year. In that year, he published three articles, on light quanta, on the foundations of the theory of Special Relativity, and on Brownian motion, each one separately worthy of a Nobel prize. Immediately after his work on Special Relativity, Einstein started thinking about gravity and how to give it a relativistically invariant formulation. He kept on working on this problem during the next ten years, doing little else. This work, after many trials and errors, culminated in his masterpiece, the *General Theory of Relativity*, presented in 1915/1916. It is clearly one of the gratest scientific achievements of all time, a beautiful theory derived from pure thought and physical intuition, capable of explaining, still today, 80 years later, virtually every aspect of gravitational physics ever observed.

Einstein's key insight was that gravity is not a physical external force like the other forces of nature but rather a manifestation of the *curvature of space-time* itself. This realization, in its simplicity and beauty, has had a profound impact on theoretical physics as a whole, and Einstein's vision of a geometrization of all of physics is still with us today.

Of course, we do not have ten years to reach these insights but nevertheless the first half of this course will be dedicated to explaining this and to developing the machinery (of tensor calculus and Riemannian geometry) required to describe physics in a curved space time, i.e. in a gravitational field.

In the second half of this course, we will then turn to various applications of General Relativity. Foremost among them is the description of the classical predictions of General Relativity and their experimental verification. Other subjects we will cover include the strange world of Black Holes, Cosmology, gravitational waves, and some intriguing theories of gravity in higher dimensions known as Kaluza-Klein theories.

General Relativity may apear to you to be a difficult subject at first, since it requires a certain amount of new mathematics and takes place in an unfamiliar arena. However, this course is meant to be essentially self-contained, requiring only a basic familiarity with Special Relativity, vector calculus and coordinate transformations. That means that I will attempt to explain every single other thing that is required to understand Einstein's theory of gravity.

CAVEATS AND OMISSIONS

Invariably, any set of (introductory) lecture notes has its shortcomings, due to lack of space and time, the requirements of the audience and the expertise (or lack thereof) of the lecturer. These lecture notes are, of course, no exception. I believe/hope that the strengths of these lecture notes are that

- they are elementary, requiring nothing beyond special relativity and calculus,

- they are essentially self-contained,

- they provide a balanced overview of the subject, the second half of the course dealing with a larger variety of different subjects than is usually covered in a 20 lecture introductory course.

In my opinion, among the weaknesses of this course or these lecture notes are the following:

- The history of the development of general relativity is an important and complex subject, crucial for a thorough appreciation of general relativity. My remarks on this subject are scarce and possibly even misleading at times and should not be taken as gospel.

- Exercises are an essential part of the course, but so far I have not included them in the lecture notes themselves.

- In the first half of the course, on tensor calculus, no mention is made of manifolds and bundles as this would require some background in topology I did not want to assume.

- Moreover, practically no mention is made of the manifestly coordinate independent calculus of differential forms. Given a little bit more time, it would be possible to cover the (extremely useful) vielbein and differential form formulations of general relativity, and a supplement to these lecture notes on this subject is in preparation.

- The discussion of the causal structure of the Schwarzschild metric and its Kruskal extension stops short of introducing Penrose diagrams. These are useful and important and, once again, given a bit more time, this is a subject that could and ought to be covered as well.

- Cosmology is a very active, exciting, and rapidly developing field. Unfortunately, not being an expert on the subject, my treatment is rather old-fashioned and certainly not Y2K compatible. I would be grateful for suggestions how to improve this section.

- Something crucial is mising form the section on gravitational waves, namely a derivation of the famous quadrupole radiation formula. If I can come up with,

or somebody shares with me, a simple five-line derivation of this formula, I will immediately include it here.

- There are numerous other important topics not treated in these notes, foremost among them perhaps a discussion of the canonical ADM formalism, a discussion of notions of energy in general relativity, the post-Newtonian approximation, other exact solutions, and aspects of black hole thermodynamics.

Including all these topics would require at least one more one-semester course and would turn these lecture notes into a (rather voluminous) book. The former is not possible, given the constraints of the Diploma Course, and the latter is not my intention, so I can only hope that these lecture notes provide the necessary background for studying these more advanced topics.

Motivation: The Einstein Equivalence Principle

Let us now, very briefly and in a streamlined way, try to retrace Einstein's thoughts which, as we will see, will lead us rather quickly to the geometric picture of gravity sketched above.

First of all, let us ask the question why we should not be happy with the classical Newtonian description of gravity. Well, for one, this theory is not Lorentz invariant, postulating an action at a distance and an instantaneous propagation of the gravitational field to every point in space. This is something that Einstein had just successfully exorcised from other aspects of physics, and clearly Newtonian gravity had to be revised as well.

It is then immediately clear that what would have to replace Newton's theory is something rather more complicated. The reason for this is that, according to Special Relativity, mass is just another form of energy. But then, since gravity couples to masses, in a relativistically invariant theory, gravity will also couple to energy. In particular, therefore, gravity would have to couple to gravitational energy, i.e. to itself. As a consequence, the new gravitational field equations will, unlike Newton's, have to be non-linear: the field of the sum of two masses cannot equal the sum of the gravitational fields of the two masses because it should also take into account the gravitational energy of the two-body system.

But now, having realized that Newton's theory cannot be the final word on the issue, how does one go about finding a better theory? Einstein approached this by thinking about three issues,

1. the equivalence principle of Special Relativity;

2. the relation between inertial and gravitational mass;

3. Special Relativity and accelerations.

As regards the first issue, let me just recall that Special Relativity postulates a preferred class of inertial frames, namely those travelling at constant velocity to each other. But this raises the questions (I will just raise and not attempt to answer) what is special about constant velocities and, more fundamentally, velocities constant with respect to what? Some absolute space? The background of the stars? ...?

Regarding the second issue, recall that in Newtonian theory, classical mechanics, there are two a priori independent concepts of mass: inertial mass m_i, which accounts for the resistance against acceleration, and gravitational mass m_g which is the mass gravity couples to. Now it is an important empirical fact that the inertial mass of a body is equal to its gravitational mass. This is usually paraphrased as 'all bodies fall at the same rate in a gravitational field'. This realization, at least with this clarity, is usually attributed to Galileo (it is probably not true, though, that Galileo dropped objects from the leaning tower of Pisa to test this - he used an inclined plane, a water clock and a pendulum).

These experiments were later on improved, in various forms, by Huygens, Newton, Bessel and others and reached unprecedented accuracy with the work of Baron von Eötvös (1889-...), who was able to show that inertial and gravitational mass of different materials (like wood and platinum) agree to one part in 10^9. In the 1950/60's, this was still further improved by R. Dicke to something like one part in 10^{11}. More recently, rumours of a 'fifth force', based on a reanalysis of Eötvös' data (but buried in the meantime) motivated experiments with even higher accuracy and no difference between m_i and m_g was found.

Now Newton's theory is in principle perfectly consistent with $m_i \neq m_g$, and Einstein was very impressed with their observed equality. This should, he reasoned, not be a mere coincidence but is probably trying to tell us something rather deep about the nature of gravity. With his unequalled talent for discovering profound truths in simple observations, he concluded that the equality of inertial and gravitational mass suggests a close relation between inertia and gravity itself, suggests, in fact, that locally effects of gravity and acceleration are indistinguishable,

$$\text{locally: GRAVITY = INERTIA = ACCELERATION}$$

He substantiated this with some classical thought experiments, *Gedankenexperimente*, as he called them, which have come to be known as the *elevator thought experiments*.

Consider somebody in a small sealed box (elevator) somewhere in outer space. In the absence of any forces, this person will float. Likewise, two stones he has just dropped (see Figure 1) will float with him.
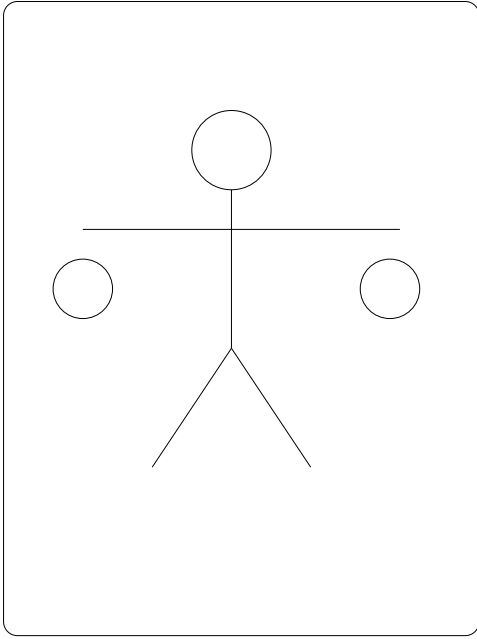
Figure 1: An experimenter and his two stones freely floating somewhere in outer space, i.e. in the absence of forces.

Now assume (Figure 2) that somebody on the outside suddenly pulls the box up with a constant acceleration. Then of course, our friend will be pressed to the bottom of the elevator with a constant force and he will also see his stones drop to the floor.

Now consider (Figure 3) this same box brought into a constant gravitational field. Then again, he will be pressed to the bottom of the elevator with a constant force and he will see his stones drop to the floor. With no experiment inside the elevator can he decide if this is actually due to a gravitational field or due to the fact that somebody is pulling the elevator upwards.

Thus our first lesson is that, indeed, locally the effects of acceleration and gravity are indistinguishable. Now consider somebody cutting the cable of the elevator (Figure 4). Then the elevator will fall freely downwards but, as in Figure 1, our experimenter and his stones will float as in the absence of gravity.

Thus lesson number two is that, locally the effect of gravity can be eliminated by going to a freely falling reference frame (or coordinate system). This should not come as a surprise. In the Newtonian theory, if the free fall in a constant gravitational field is described by the equation

$$\ddot{x} = g \ \ (+ \ \text{other forces}) \ , \tag{1.1}$$

then in the accelerated coordinate system

$$\xi(x,t) = x - gt^2/2 \tag{1.2}$$

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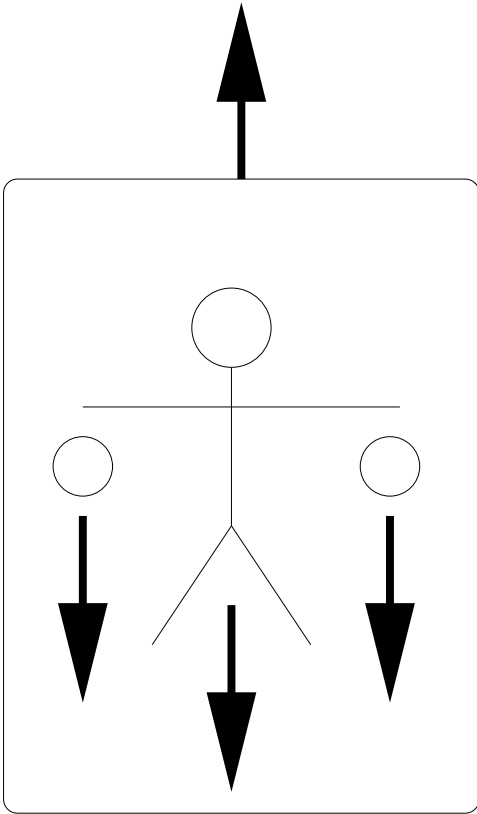
Figure 2: Constant acceleration upwards mimics the effect of a gravitational field: experimenter and stones drop to the bottom of the box.
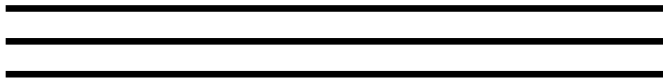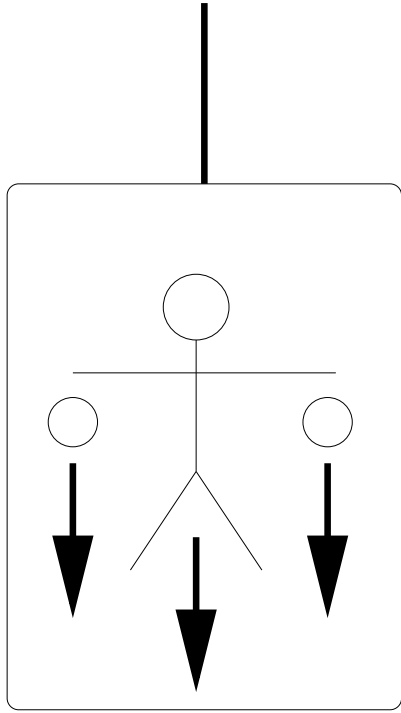
Figure 3: The effect of a constant gravitational field: indistinguishable for our experimenter from that of a constant acceleration in Figure 2.
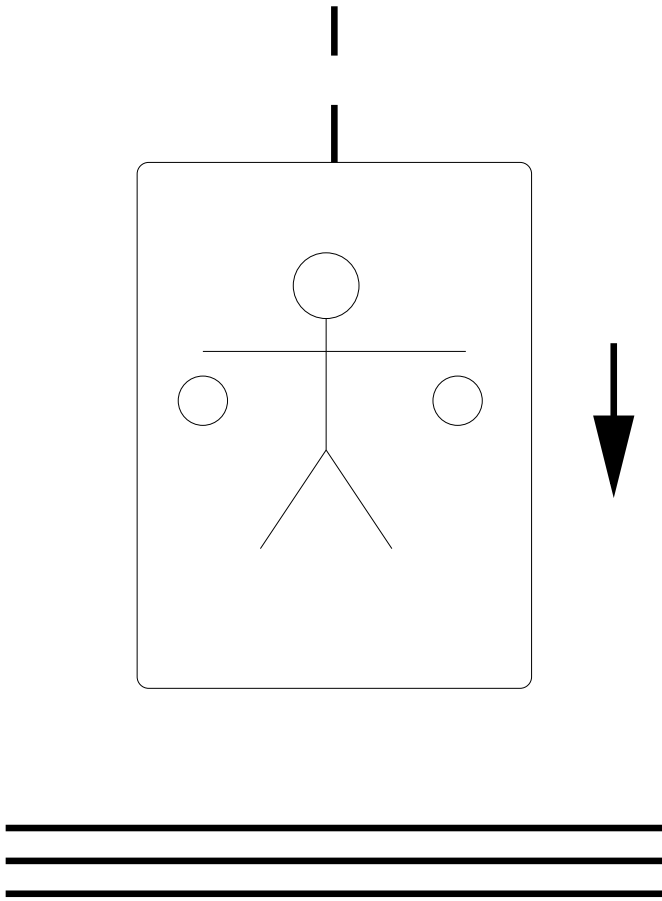
Figure 4: Free fall in a gravitational field has the same effect as no gravitational field (Figure 1): experimenter and stones float.
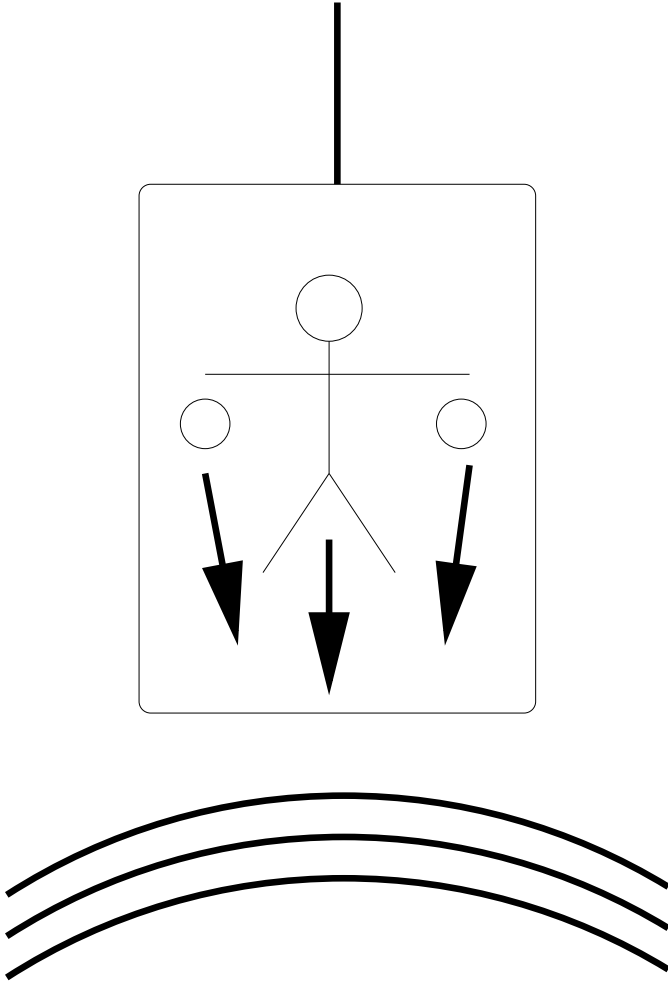
Figure 5: The experimenter and his stones in a non-uniform gravitational field: the stones will approach each other slightly as they fall to the bottom of the elevator.

the same physics is described by the equation

$$\ddot{\xi} = 0 \;\; (+ \;\; \text{other forces}) \;\; , \tag{1.3}$$

and the effect of gravity has been eliminated by going to the freely falling coordinate system ξ.

In the above discussion, I have put the emphasis on constant accelerations and on 'locally'. To see the significance of this, consider our experimenter with his elevator in the gravitational field of the earth (Figure 5). This gravitational field is not constant but spherically symmetric, pointing towards the center of the earth. Therefore the stones will slightly approach each other as they fall towards the bottom of the elevator, in the direction of the center of the gravitational field.

Thus, if somebody cuts the cable now and the elevator is again in free fall (Figure 6), our experimenter will float again, so will the stones, but our experimenter will also notice
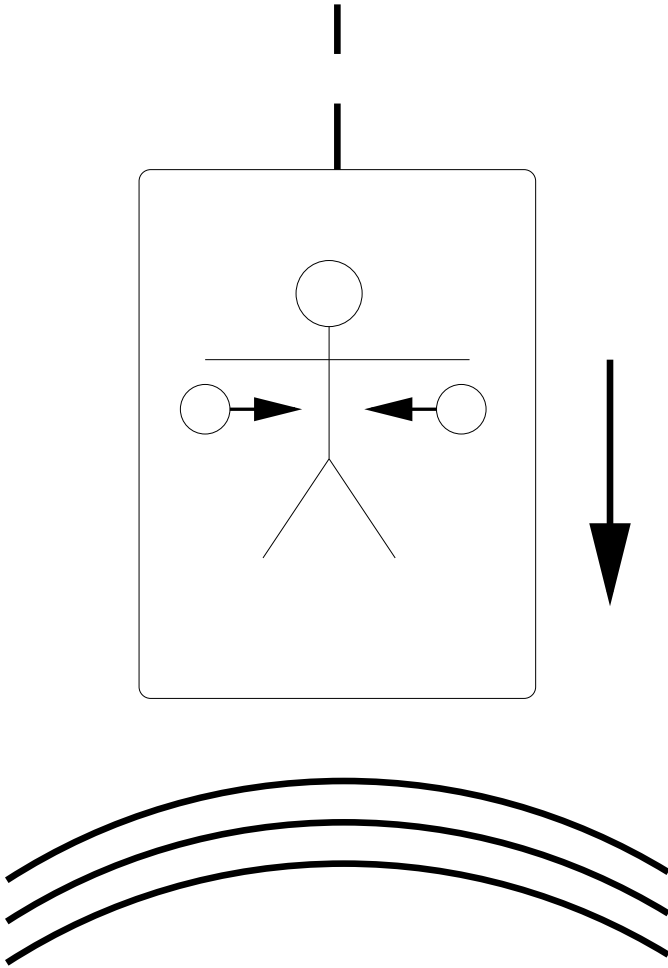
Figure 6: Experimentator and stones freely falling in a non-uniform gravitational field. The experimenter floats, so do the stones, but they move closer together, indicating the presence of some external force.

that the stones move closer together for some reason. He will have to conclude that there is some force responsible for this.

This is lesson number three: in a non-uniform gravitational field the effects of gravity cannot be eliminated by going to a freely falling coordinate system. This is only possible locally, on such scales on which the gravitational field is essentially constant.

Einstein formalized the outcome of these thought experiments in what is now known as the *Einstein Equivalence Principle*:

> At every space-time point in an arbitrary gravitational field it is possible to choose a *locally inertial* (or *freely falling*) *coordinate system* such that, within a sufficiently small region of this point the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence

of gravitation.

<center>GEODESICS</center>

Thus, conversely, we can learn about the effects of gravitation by transforming the laws of nature (equations of motion) from an inertial Cartesian coordinate system to other (accelerated, curvilinear) coordinates. We do this for the motion of a massive free particle, described in an inertial coordinate system $\{\xi^A\}$ by

$$\frac{d^2}{d\tau^2}\xi^A(\tau) = 0 \ , \tag{1.4}$$

where τ is proper time, defined by

$$d\tau^2 = -\eta_{AB}d\xi^A d\xi^B \ . \tag{1.5}$$

Let us now see what this equation looks like when written in some other (non-inertial, accelerating) coordinate system $x^\mu(\xi^A)$. Following the usual rules for changes of variables,

$$\frac{d}{d\tau}\xi^A = \frac{\partial \xi^A}{\partial x^\mu}\frac{dx^\mu}{d\tau} \ , \tag{1.6}$$

etc., the equation of motion becomes the *geodesic equation*

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ , \tag{1.7}$$

while proper time is calculated from

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \ . \tag{1.8}$$

Here, the *metric tensor* $g_{\mu\nu}$ and the *affine connection* or *Christoffel symbol* $\Gamma^\mu_{\ \nu\lambda}$ are given by

$$g_{\mu\nu} = \eta_{AB}\frac{\partial \xi^A}{\partial x^\mu}\frac{\partial \xi^B}{\partial x^\nu} \tag{1.9}$$

$$\Gamma^\mu_{\ \nu\lambda} = \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} \ . \tag{1.10}$$

Thus, the affine connection represents the gravitational force. It can be expressed in terms of first derivatives of the metric tensor which thus plays the role of a gravitational potential:

$$\Gamma^\mu_{\ \nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda}$$
$$\Gamma_{\rho\nu\lambda} = \tfrac{1}{2}(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}) \ . \tag{1.11}$$

Here $g^{\mu\rho}$ is the *inverse metric*, i.e. $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_{\ \nu}$.

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Hence, given the metric one can directly calculate the Christoffel symbol without having to know (or determine) an inertial coordinate system first. This identifies the metric tensor as the fundamental dynamical variable of gravity.

For massless particles, some other parameter instead of the proper time τ (e.g. $\sigma = \xi^0$) has to be used because $d\tau^2 = 0$ but the equations remain the same, i.e. one has

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\ \nu\lambda} \frac{dx^\nu}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0 \quad , \tag{1.12}$$

with

$$0 = -g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \quad . \tag{1.13}$$

The latter equation, rather than telling how to calculate proper time (as for massive particles) sets the initial conditions approporiate to a massless particle.

METRICS AND COORDINATE TRANSFORMATIONS

Above we saw that the motion of free particles in Minkowski space in curvilinear coordinates is described in terms of a modified metric, $g_{\mu\nu}$, and a force term $\Gamma^\mu_{\ \nu\lambda}$ representing the 'pseudo-force' on the particle. It thus follows from the Einstein Equivalence Principle that an appropriate description of true gravitational fields is in terms of a metric tensor $g_{\mu\nu}(x)$ (and its associated Christoffel symbol) which can only locally be related to the Minkowski metric via a suitable coordinate transformation (to locally inertial coordinates). Thus our starting point will now be a space-time equipped with some metric $g_{\mu\nu}(x)$.

A space-time equipped with a metric tensor $g_{\mu\nu}$ is called a metric space-time or (pseudo-)Riemannian space-time. It encodes the information how to measure (spatial and temporal) distances via

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad . \tag{1.14}$$

Such distances should not depend on which coordinate system is used. Hence, changing coordinates from the $\{x^\mu\}$ to new coordinates $\{y^{\mu'}(x^\mu)\}$ and demanding that

$$g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu'\nu'}(y)dy^{\mu'} dy^{\nu'} \quad , \tag{1.15}$$

one finds that under a coordinate transformation the metric transforms as

$$g_{\mu'\nu'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \quad . \tag{1.16}$$

Objects which transform in such a nice and simple way under coordinate transformations are known as *tensors* - the metric is an example of a covariant symmetric rank two tensor. We will study these in much more detail later.

One point to note about this transformation behaviour is that if in one coordinate system the metric tensor has one negative and three positive eigenvalues (as in a locally inertial

coordinate system), then the same will be true in any other coordinate system (even though the eigenvalues themselves will in general be different) - this statement should be familiar from linear algebra as Sylvester's law of inertia. This explains the qualifier 'pseudo': a pseudo-Riemannian space-time is a space-time equipped with a metric tensor with one negative and three positive eigenvalues while a Riemannian space is a space equipped with a positive definite metric. Space-like distances correspond to $ds^2 > 0$, time-like distances to $d\tau^2 = -ds^2 > 0$, and null or light-like distances to $ds^2 = d\tau^2 = 0$.

By drawing the coordinate grid determined by the metric tensor, one can convince onseself that in general a metric space or space-time need not or cannot be flat. Example: the coordinate grid of the metric $d\theta^2 + \sin^2\theta d\phi^2$ cannot be drawn in flat space but can be drawn on the surface of a two-sphere because the infinitesimal parallelograms described by ds^2 degenerate to triangles at $\theta = 0, \pi$.

At this point the question naturally arises how one can tell whether a given (perhaps complicated looking) metric is just the flat metric written in other coordinates or whether it describes a genuinely curved space-time. We will see later that there is an object, the *Riemann curvature tensor*, constructed from the second derivatives of the metric, which has the property that all of its components vanish if and only if the metric is a coordinate transform of the flat space Minkowski metric. Thus, given a metric, by calculating its curvature tensor one can decide if the metric is just the flat metric in disguise or not.

CHRISTOFFEL SYMBOLS, GEODESICS AND COORDINATE TRANSFORMATIONS

Knowing how the metric transforms under coordinate transformations, we can now also determine how the affine connection transforms. A straightforward calculation gives

$$\Gamma^{\mu'}_{\nu'\lambda'} = \Gamma^{\mu}_{\nu\lambda}\frac{\partial y^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial y^{\nu'}}\frac{\partial x^{\lambda}}{\partial y^{\lambda'}} + \frac{\partial y^{\mu'}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial y^{\nu'}\partial y^{\lambda'}} \ . \tag{1.17}$$

Thus, $\Gamma^{\mu}_{\nu\lambda}$ is *not* a tensor, but the second term is there precisely to compensate for the fact that $\ddot{x}^{\mu}$ is also not a tensor - the combined geodesic equation transforms in a nice way under coordinate transformations. Indeed one has

$$\frac{d^2 y^{\mu'}}{d\tau^2} + \Gamma^{\mu'}_{\nu'\lambda'}\frac{dy^{\nu'}}{d\tau}\frac{dy^{\lambda'}}{d\tau} = \frac{\partial y^{\mu'}}{\partial x^{\mu}}\left[\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}\right] \tag{1.18}$$

Therefore, as $(\partial y^{\mu'}/\partial x^{\mu})$ is an invertible matrix, the geodesic equation is true in one coordinate system (y) if it is true in another coordinate system (x). This is the prototype of the kinds of physical laws we are looking for - those which are valid in any coordinate system. The reason why this is true for the geodesic equation is that it transforms in such a simple way under coordinate transformations - as a contravariant one-tensor (equivalently: as a vector).

2 The Physics and Geometry of Geodesics

The Geodesic Equation from a Variational Principle

We obtained the geodesic equation by transforming the equation for a straight line into an arbitrary coordinate system. It is thus likely that in general a geodesic extremizes the proper distance (or proper time) between two space-time points. This is indeed the case. The action is simply $\int d\tau$ with $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$. Actually, for a particle of rest mass m one should really use the action principle $m \int d\tau$. But of course m drops out of the variational equations (as it should by the equivalence principle) and we will therefore ignore m in the following.

In order to perform the variation, it is useful to introduce an auxiliary parameter σ in the initial stages of the calculation and to write

$$\int d\tau = \int (d\tau/d\sigma)d\sigma = \int (-g_{\mu\nu}\tfrac{dx^\mu}{d\sigma}\tfrac{dx^\nu}{d\sigma})^{1/2}d\sigma \ . \tag{2.1}$$

By the standard variational procedure one then finds

$$\delta \int d\tau = \int d\tau g_{\mu\nu}(\ddot{x}^\mu + \Gamma^\mu_{\rho\lambda}\dot{x}^\rho\dot{x}^\lambda)\delta x^\nu \ . \tag{2.2}$$

There is a small problem with this action principle for massless particles (null geodesics). For this reason and many other practical purposes it is much more convenient to use the Lagrangian

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{2.3}$$

and the action $S = \int \mathcal{L}d\tau$ which also gives rise to the geodesic equation.

The Newtonian Limit

We saw that the 10 components of the metric $g_{\mu\nu}$ play the role of potentials for the gravitational force. We now want to find the relation of these potentials to the Newtonian potential. For that we consider a particle moving *slowly* in a *weak stationary* gravitational field (because it is only under these conditions that we know and trust the validity of Newton's equations).

Split the coordinates $x^\mu = (t, x^i)$. Using $dx^i/d\tau \ll dt/d\tau$ (slow), $g_{\mu\nu,0} = 0$ (stationary), $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ (weak), the geodesic equation can be shown to reduce to

$$\frac{d^2 x^i}{dt^2} = \tfrac{1}{2}h_{00,i} \ . \tag{2.4}$$

Indeed, the condition of slow motion implies that the geodesic equation can be approximated by

$$\ddot{x}^\mu + \Gamma^\mu_{00}\dot{t}^2 = 0 \ . \tag{2.5}$$

Stationarity tells us that

$$\Gamma^{\mu}_{\ 00} = -\tfrac{1}{2} g^{\mu\nu} \partial_{\nu} g_{00} = -\tfrac{1}{2} g^{\mu i} \partial_i g_{00} \ .\tag{2.6}$$

From the weak field condition we learn that

$$\Gamma^{\mu}_{\ 00} = -\tfrac{1}{2} \eta^{\mu i} \partial_i h_{00} \ ,\tag{2.7}$$

so that

$$\Gamma^{0}_{\ 00} = 0 \ , \qquad \Gamma^{i}_{\ 00} = -\tfrac{1}{2} \partial^i h_{00} \ .\tag{2.8}$$

Thus the geodesic equation splits into

$$\begin{aligned}
\ddot{t} &= 0 \\
\ddot{x}^i &= \tfrac{1}{2} \partial^i h_{00} \dot{t}^2 \ .
\end{aligned}\tag{2.9}$$

As the first of these just says that $\dot{t}$ is constant, we can use this in the second equation to convert the τ-derivatives into derivatives with respect to the coordinate time t. Hence we obtain (2.4). Comparing this with

$$\frac{d^2 x^i}{dt^2} = -\phi_{,i}\tag{2.10}$$

where ϕ is the Newtonian potential, e.g.

$$\phi = -\frac{GM}{r} \ ,\tag{2.11}$$

leads to $h_{00} = -2\phi$ (the constant of integration is fixed by demanding that the metric approach the flat metric at infinity) or

$$g_{00} = -(1 + 2\phi) \ .\tag{2.12}$$

Restoring the appropriate units (in particular a factor of c^2), one finds that $\phi \sim 10^{-9}$ on the surface of the earth, 10^{-6} on the surface of the sun, so that the distortion in the space-time geometry produced by gravitation is in general quite small (justifying our approximations).

<center>THE GRAVITATIONAL REDSHIFT</center>

The gravitational redshift (i.e. the fact that photons lose or gain energy when rising or falling in a gravitational field) is a consequence of the Einstein Equivalence Principle (and therefore also provides an experimental test of the Einstein Equivalence Principle).

It is clear from the expression $d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu$ that e.g. the rate of clocks is affected by the gravitational field. However, as everything is affected in the same way by gravity it is impossible to measure this effect locally. In order to find an observable effect, one needs to compare data from two different points in a gravitational potential.

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The situation we could consider is that of two observers A and B moving on worldlines (paths) γ_A and γ_B, A sending light signals to B. In general the frequency, measured in the observers rest-frame at A (or in a locally inertial coordinate system there) will differ from the frequency measured by B upon receiving the signal.

In order to seperate out Doppler-like effects due to relative accelerations, we consider two observers A and B at rest radially to each other, at radii r_A and r_B, in a stationary spherically symmetric gravitational field. This means that the metric depends only on a radial coordinate r and we can choose it to be of the form

$$g_{00}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 \ , \tag{2.13}$$

where $d\Omega^2$ is the standard volume element on the two-sphere (see section 11 for a more detailed justification of this ansatz for the metric).

Observer A sends out light of a given frequency, say n pulses per proper time unit $\Delta\tau_A$. Observer B receives these n pulses in his proper time $\Delta\tau_B$. Thus the relation between the frequency ν_A emitted at A and the frequency ν_B observed at B is

$$\frac{\nu_A}{\nu_B} = \frac{\Delta\tau_B}{\Delta\tau_A} \ . \tag{2.14}$$

The geometry of the situation dictates that the coordinate time intervals recorded at A and B are equal, $\Delta t_A = \Delta t_B$ as nothing in the metric actually depends on t. In equations, this can be seen as follows. First of all, the equation for a radial light ray is

$$-g_{00}(r)dt^2 = g_{rr}(r)dr^2 \ . \tag{2.15}$$

From this we can calculate the coordinate time for the light ray to go from A to B. Say that the first light pulse is emitted at point A at time $t(A)_1$ and received at B at coordinate time $t(B)_1$. Then

$$t(B)_1 - t(A)_1 = \int_{r_A}^{r_B} dr(-g_{rr}(r)/g_{00}(r))^{1/2} \tag{2.16}$$

But the right hand side obviously does not depend on t, so we also have

$$t(B)_2 - t(A)_2 = \int_{r_A}^{r_B} dr(-g_{rr}(r)/g_{00}(r))^{1/2} \tag{2.17}$$

where t_2 denotes the coordinate time fo the arrival of the n-th pulse. Therefore,

$$t(B)_1 - t(A)_1 = t(B)_2 - t(A)_2 \ , \tag{2.18}$$

or

$$t(A)_2 - t(A)_1 = t(B)_2 - t(B)_1 \ , \tag{2.19}$$

as claimed. Thus the coordinate time intervals recorded at A and B between the first and last pulse are equal. However, to convert this to proper time, we have to multiply the coordinate time intervals by an r-dependent function,

$$\Delta\tau_{A,B} = (-g_{\mu\nu}(r_{A,B})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt})^{1/2}\Delta t_{A,B} \ , \tag{2.20}$$

22

and therefore the proper time intervals will not be equal. For observers at rest, $dx^i/dt = 0$, one has

$$\Delta\tau_{A,B} = (-g_{00}(r_{A,B}))^{1/2}\Delta t_{A,B} \quad . \tag{2.21}$$

Therefore

$$\frac{\nu_A}{\nu_B} = (g_{00}(r_B)/g_{00}(r_A))^{1/2} \quad . \tag{2.22}$$

Using the Newtonian approximation, this becomes

$$\frac{\nu_A}{\nu_B} \sim 1 + \phi(r_B) - \phi(r_A) \quad , \tag{2.23}$$

or

$$\frac{\nu_A - \nu_B}{\nu_B} = \frac{GM(r_B - r_A)}{r_A r_B} \tag{2.24}$$

Note that, for example, for $r_B > r_A$ one has $\nu_B < \nu_A$ so that, as expected, a photon loses energy when rising in a gravitational field.

While difficult to observe directly (by looking at light form the sun), this prediction has been verified with one percent accuracy in the laboratory by Pound and Snider (using the Mössbauer effect).

This result can also be deduced from energy conservation. A local inertial observer at the emitter A will see a change in the internal mass of the emitter $\Delta m_A = -\hbar\nu_A$ when a photon of frequency of ν_A is emitted. Likewise, the absorber at point B will experience an increase in inertial mass by $\Delta m_B = \hbar\nu_B$. But the total internal plus gravitational potential energy must be conserved. Thus

$$0 = \Delta m_A(1 + \phi(r_A)) + \Delta m_B(1 + \phi(r_B)) \quad . \tag{2.25}$$

Thus

$$\frac{\nu_A}{\nu_B} = \frac{1 + \phi(r_B)}{1 + \phi(r_A)} \sim 1 + \phi(r_B) - \phi(r_A) \quad , \tag{2.26}$$

as before. This derivation shows that gravitational red-shift experiments test the Einstein Equivalence Principle in its strong form, in which the term 'laws of nature' is not restricted to mechanics (inertial = gravitational mass), but also includes quantum mechanics in the sense that it tests if in an inertial frame the relation between photon energy and frequency is unaffected by the presence of a gravitational field.

Locally Inertial and Riemann Normal Coordinates

Central to our initial discussion of gravity was the Einstein Equivalence Principle which postulates the existence of locally inertial (or freely falling) coordinate systems in which locally at (or around) a point the effects of gravity are absent. Now that we have decided that the arena of gravity is a general metric space-time, we should establish that such coordinate systems indeed exist. Looking at the geodesic equation, it it is clear that

'absence of gravitational effects' is tantamount to the existence of a coordinate system $\{\xi^A\}$ in which at a given point p the metric is the Minkowski metric, $g_{AB}(p) = \eta_{AB}$ and the Christoffel symbol is zero, $\Gamma^A_{BC}(p) = 0$. Owing to the identity

$$g_{\mu\nu,\lambda} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda} \; , \tag{2.27}$$

the latter condition is equivalent to $g_{AB,C}(p) = 0$. I will sketch three arguments establishing the existence of such coordinate systems, each one having its own virtues and providing its own insights into the issue.

1. Direct Construction

   We know that given a coordinate system $\{\xi^A\}$ that is inertial at a point p, the metric and Christoffel symbols at p in a new coordinate system $\{x^\mu\}$ are determined by (1.9,1.10). Conversely, we will now see that knowledge of the metric and Christoffel symbols at a point p is sufficient to construct a locally inertial coordinate system at p.

   Equation (1.10) provides a second order differential equation in some coordinate system $\{x^\mu\}$ for the inertial coordinate system $\{\xi^A\}$, namely

   $$\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} = \Gamma^\mu_{\;\nu\lambda} \frac{\partial \xi^A}{\partial x^\mu} \; . \tag{2.28}$$

   By a general theorem, a local solution around p with given initial conditions $\xi^A(p)$ and $(\partial \xi^A / \partial x^\mu)(p)$ is guaranteed to exist. In terms of a Taylor series expansion around p one has

   $$\xi^A(x) = \xi^A(p) + \frac{\partial \xi^A}{\partial x^\mu}|_p(x^\mu - p^\mu) + \frac{1}{2}\frac{\partial \xi^A}{\partial x^\mu}|_p \Gamma^\mu_{\;\nu\lambda}(p)(x^\nu - p^\nu)(x^\lambda - p^\lambda) + \dots \tag{2.29}$$

   It follows from (1.9) that the metric at p in the new coordinate system is indeed the Minkowski metric,

   $$g_{AB}(p) = g_{\mu\nu}(p)\frac{\partial x^\mu}{\partial \xi^A}|_p \frac{\partial x^\nu}{\partial \xi^B}|_p \tag{2.30}$$

   for an appropriate choice of the initial condition $(\partial \xi^A / \partial x^\mu)(p)$ (a symmetric matrix at a point can always be diagonalized by a similarity transformation). With a little bit more work it can also be shown that in these coordinates $g_{AB,C}(p) = 0$. Thus this is indeed an inertial coordinate system at p. As the matrix $(\partial \xi^A / \partial x^\mu)(p)$ which transforms the metric at p into the standard Minkowski form is only unique up to Lorentz transformations, overall (counting also the initial condition $\xi^A(p)$) a locally inertial coordinate system is unique only up to Poincaré transformations - an unsurprising result.

2. Geodesic (or Riemann Normal) Coordinates

A slightly more insightful way of constructing a locally inertial coordinate system, rather than by directly solving the relevant differential equation, makes use of geodesics at p. Recall that in Minkowski space the metric takes the simplest possible form in coordinates whose coordinate lines are geodesics. One might thus suspect that in a general metric space-time the metric will also (locally) look particularly simple when expressed in terms of such *geodesic coordinates*. Since locally around p we can solve the geodesic equation with four linearly independent initial conditions, we can assume the existence of a coordinate system $\{\xi^A\}$ in which the coordinate lines are geodesics $\xi^A(\tau)$. But this means that $\ddot{\xi}^A = 0$. Hence the geodesic equation reduces to

$$\Gamma^A_{BC}\dot{\xi}^B\dot{\xi}^C = 0 \ . \tag{2.31}$$

As at p the $\dot{\xi}^A$ were chosen to be linearly independent, this implies $\Gamma^A_{BC}(p) = 0$, as desired. It is easy to see that the coordinates ξ^A can also be chosen in such a way that $g_{AB}(p) = \eta_{AB}$.

3. A Numerological Argument

This is my favourite argument because it requires no calculations. Assuming that the local existence of solutions to differential equations is guaranteed by some mathematical theorems, it is frequently sufficient to check that one has enough degrees of freedom to satisfy the desired initial conditions (one may also need to check integrability conditions). In the present context, this argument is useful because it also reveals some information about the 'true' curvature hidden in the second derivatives of the metric. It works as follows:

Consider a Taylor expansion of the metric around p in the sought-for new coordinates. Then the metric at p will transform with the matrix $(\partial x^\mu/\partial \xi^A)(p)$. This matrix has $(4 \times 4) = 16$ independent components, precisely enough to impose the 10 conditions $g_{AB}(p) = \eta_{AB}$ up to Lorentz transformations.

The derivative of the metric at p, $g_{AB,C}(p)$, will appear in conjunction with the second derivative $\partial^2 x^\mu/\partial \xi^A \partial \xi^B$. The $4 \times (4 \times 5)/2 = 40$ coefficients are precisely sufficient to impose the 40 conditions $g_{AB,C}(p) = 0$.

Now let us look at the second derivatives of the metric. $g_{AB,CD}$ has $(10 \times 10) = 100$ independent components, while the third derivative of $x^\mu(\xi)$ at p, $\partial^3 x^\mu/\partial \xi^A \partial \xi^B \partial \xi^C$ has $4 \times (4 \times 5 \times 6)/(2 \times 3) = 80$ components. Thus 20 linear combinations of the second derivatives of the metric at p cannot in general be set to zero by a coordinate transformation. Thus these encode the information about the real curvature at p. This agrees nicely with the fact that the Riemann curvature tensor we will construct later turns out to have precisely 20 independent components.

MORE ON GEODESICS AND THE VARIATIONAL PRINCIPLE

Recall from above that the geodesic equation for a metric $g_{\mu\nu}$ can be derived from the Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. This has several immediate consequences which are useful for the determination of Christoffel symbols and geodesics in practice.

The geodesic equation is just the Euler-Lagrange equation

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \ . \tag{2.32}$$

Just as in classical mechanics, a coordinate the Lagrangian does not depend on explicitly (a cyclic coordinate) leads to a conserved quantity. In the present context this means that if, say, $\partial\mathcal{L}/\partial x^1 = 0$, then the momentum $p_1 = \partial\mathcal{L}/\partial\dot{x}^1$ is conserved along geodesics.

For example, on the two-sphere the Lagrangian reads

$$\mathcal{L} = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 \ . \tag{2.33}$$

The angle ϕ is a cyclic variable and the angular momentum

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 2\sin^2\theta\dot{\phi} \tag{2.34}$$

is a conserved quantity.

By the way, another conserved quantity that one has for any geodesic is the object $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, i.e.

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \Rightarrow \frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) = 0 \ . \tag{2.35}$$

This quantity can then always be chosen to be (-1) for time-like geodesics and 0 for null geodesics.

Another immediate consequence is the following: consider a space or space-time with coordinates $\{y, x^\mu\}$ and a metric of the form $ds^2 = dy^2 + g_{\mu\nu}(x, y)dx^\mu dx^\nu$. Then the coordinate lines of y are geodesics. Indeed, since the Lagrangian is

$$\mathcal{L} = \dot{y}^2 + g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \ , \tag{2.36}$$

the Euler-Lagrange equations are equivalent to

$$\begin{aligned}
\ddot{y} - \tfrac{1}{2}g_{\mu\nu,y}\dot{x}^\mu\dot{x}^\nu &= 0 \\
\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda + 2\Gamma^\mu_{\nu y}\dot{x}^\nu\dot{y} &= 0 \ .
\end{aligned} \tag{2.37}$$

Therefore $\dot{x}^\mu = 0$, $\ddot{y} = 0$ is a solution of the geodesic equation, and it describes motion along the coordinate lines of y.

In the case of the two-sphere, with its metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$, this translates into the familiar statement that the great circles, the coordinate lines of $y = \theta$, are geodesics.

Finally, the Euler-Lagrange form of the geodesic equations frequently provides the most direct way of calculating Christoffel symbols - by comparing the Euler-Lagrange equations with the expected form of the geodesic equation in terms of Christoffel symbols. For example, once again in the case of the two-sphere, for the θ-equation one has

$$
\begin{aligned}
\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot\theta} &= 2\ddot\theta \\
\frac{\partial\mathcal{L}}{\partial\theta} &= 2\sin\theta\cos\theta\dot\phi^2 \ .
\end{aligned}
\tag{2.38}
$$

Comparing the variational equation

$$
\ddot\theta - \sin\theta\cos\theta\dot\phi^2 = 0
\tag{2.39}
$$

with the geodesic equation

$$
\ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta^2 + 2\Gamma^\theta_{\theta\phi}\dot\theta\dot\phi + \Gamma^\theta_{\phi\phi}\dot\phi^2 = 0 \ ,
\tag{2.40}
$$

one can immediately read off that

$$
\begin{aligned}
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\theta_{\theta\theta} &= \Gamma^\theta_{\theta\phi} = 0 \ .
\end{aligned}
\tag{2.41}
$$

* AFFINE AND NON-AFFINE PARAMETRIZATIONS

The geodesic equation for time-like geodesics (massive particles) is

$$
\ddot x^\mu + \Gamma^\mu_{\nu\lambda}\dot x^\nu\dot x^\lambda = 0 \ .
\tag{2.42}
$$

where a dot denotes the derivative with respect to proper time τ. This equation is not parametrization invariant. Indeed, consider a change of parametrization $\tau \to \sigma = f(\tau)$. Then

$$
\frac{dx^\mu}{d\tau} = \frac{df}{d\tau}\frac{dx^\mu}{d\sigma} \ ,
\tag{2.43}
$$

and therefore the geodesic equation written in terms of σ reads

$$
\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = -\frac{\ddot f}{\dot f^2}\frac{dx^\mu}{d\sigma} \ .
\tag{2.44}
$$

Thus the geodesic equation retains its form only under affine changes of the proper time parameter τ, $f(\tau) = a\tau + b$, and parameters $\sigma = f(\tau)$ related to τ by such an affine transformation are known as affine parameters. Note that the variational principle based on $\mathcal{L}$ always yields the geodesic equation in affine form whereas this is not necessarily the case for the action $S = \int d\tau = \int (d\tau/d\sigma)d\sigma$.

FROM THE EINSTEIN EQUIVALENCE PRINCIPLE TO THE PRINCIPLE OF GENERAL COVARIANCE

The Einstein Equivalence Principle tells us that the laws of nature (including the effects of gravity) should be such that in an inertial frame they reduce to the laws of Special Relativity (SR). As we have seen, this can be implemented by transforming the laws of SR to arbitrary coordinate systems and declaring that these be valid for arbitrary coordinates and metrics.

However, this is a tedious method in general (e.g. to obtain the correct form of the Maxwell equations in the presence of gravity). We will thus replace the Einstein Equivalence Principle by the closely related *Principle of General Covariance* PGC:

> A physical equation holds in an arbitrary gravitational field if
>
> 1. the equation holds in the absence of gravity, i.e. when $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma^{\mu}_{\nu\lambda} = 0$, and
>
> 2. the equation is *generally covariant*, i.e. preserves its form under a general coordinate transformation.

It should be noted here that general covariance alone is an empty statement. Any equation can be made generally covariant simply by writing it in an arbitrary coordinate system. The significance of the PGC lies in the statement about gravity, namely that by virtue of its general covariance an equation will be true in a gravitational field if it is true in the absence of gravitation.

TENSORS

In order to construct generally covariant equations, we need objects that transform in a simple way under coordinate transformations. The prime examples of such objects are *tensors*.

<u>Scalars</u>

The simplest example of a tensor is a function (or scalar) f which under a coordinate transformation $x^{\mu} \to y^{\mu'}(x^{\mu})$ simply transforms as

$$f'(y(x)) = f(x) \ , \tag{3.1}$$

or, simply, $f' = f$.

<u>Vectors</u>

The next simplest case are vectors $V^\mu(x)$ transforming as

$$V'^{\mu'}(y(x)) = \frac{\partial y^{\mu'}}{\partial x^\mu} V^\mu(x) \quad . \tag{3.2}$$

A prime example is the tangent vector $\dot{x}^\mu$ to a curve, for which this transformation behaviour

$$\dot{x}^\mu \to \dot{y}^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^\mu} \dot{x}^\mu \tag{3.3}$$

is just the familiar one.

It is extremely useful to think of vectors as first order differential operators, via the correspondence

$$V^\mu \Leftrightarrow V := V^\mu \partial_\mu \quad . \tag{3.4}$$

One of the advantages of this point of view is that V is completely invariant under coordinate transformations as the *components* V^μ of V transform inversely to the *basis vectors* ∂_μ. For more on this see the (optional) section on the coordinate-independent interpretation of tensors below.

Covectors

A covector is an object $U_\mu(x)$ which under a coordinate transformation transforms inversely to a vector, i.e. as

$$U'_{\mu'}(y(x)) = \frac{\partial x^\mu}{\partial y^{\mu'}} U_\mu(x) \quad . \tag{3.5}$$

A familiar example of a covector is the derivative $U_\mu = \partial_\mu f$ of a function which of course transforms as

$$\partial_{\mu'} f'(y(x)) = \frac{\partial x^\mu}{\partial y^{\mu'}} \partial_\mu f(x) \quad . \tag{3.6}$$

Covariant 2-Tensors

Clearly, given the above objects, we can construct more general objects which transform in a nice way under coordinate transformations by taking products of them. Tensors in general are objects which transform like (but need not be equal to) products of vectors and covectors.

In particular, a covariant 2-tensor, or (0,2)-tensor, is an object $A_{\mu\nu}$ that transforms under coordinate transformations like the product of two covectors, i.e.

$$A'_{\mu'\nu'}(y(x)) = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} A_{\mu\nu}(x) \quad . \tag{3.7}$$

I will from now on use a shorthand notation in which I drop the prime on the transformed object and also omit the argument. In this notation, the above equation would then become

$$A_{\mu'\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} A_{\mu\nu} \quad . \tag{3.8}$$

We already know one example of such a tensor, namely the metric tensor $g_{\mu\nu}$ (which happens to be a symmetric tensor).

<u>Contravariant 2-Tensors</u>

Likewise we define a contravariant 2-tensor (or a (2,0)-tensor) to be an object $B^{\mu\nu}$ that transforms like the product of two vectors,

$$B^{\mu'\nu'} = \frac{\partial y^{\mu'}}{\partial x^{\mu}} \frac{\partial y^{\nu'}}{\partial x^{\nu}} B^{\mu\nu} \quad . \tag{3.9}$$

An example is the inverse metric tensor $g^{\mu\nu}$.

<u>(p, q)-Tensors</u>

It should now be clear how to define a general (p, q)-tensor - as an object $T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q}$ with p contravariant and q covariant indices which under a coordinate transformation transforms like a product of p vectors and q covectors,

$$T^{\mu'_1 \ldots \mu'_p}_{\nu'_1 \ldots \nu'_q} = \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\mu'_p}}{\partial x^{\mu_p}} \frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}} \cdots \frac{\partial x^{\nu_q}}{\partial y^{\nu'_q}} T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q} \quad . \tag{3.10}$$

Note that, in particular, a tensor is zero (at a point) in one coordinate system if and only if the tensor is zero (at the same point) in another coordinate system.

Thus, any law of nature (field equation, equation of motion) expressed in terms of tensors, say in the form $T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q} = 0$, preserves its form under coordinate trasformations and is therefore automatically generally covariant,

$$T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q} = 0 \Leftrightarrow T^{\mu'_1 \ldots \mu'_p}_{\nu'_1 \ldots \nu'_q} = 0 \tag{3.11}$$

An important special example of a tensor is the Kronecker tensor δ^{μ}_{ν}. Together with scalars and products of scalars and Kronecker tensors it is the only tensor whose components are the same in all coordinate systems.

One comment on terminology: it is sometimes useful to distinguish vectors from vector fields and, likewise, tensors from tensor fields. A vector is then just a vector $V^{\mu}(x)$ at some point x of space-time whereas a vector field is something that assigns a vector to each point of space-time and, likewise, for tensors and tensor fields.

Important examples of non-tensors are the Christoffel symbols. Another important example is the the ordinary partial derivative of a (p, q)-tensor, $\partial_{\lambda} T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q}$ which is *not* a $(p, q+1)$-tensor unless $p = q = 0$. This failure of the partial derivative to map tensors to tensors will motivate us below to introduce a *covariant derivative* which generalizes the usual notion of a partial derivative and has the added virtue of mapping tensors to tensors.

<div align="center">TENSOR ALGEBRA</div>

Tensors can be added, multiplied and contracted in certain obvious ways. The basic algebraic operations are the following:

1. Linear Combinations

   Given two (p,q)-tensors $A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q}$ and $B^{\mu_1...\mu_p}{}_{\nu_1...\nu_q}$, their sum

   $$C^{\mu_1...\mu_p}{}_{\nu_1...\nu_q} = A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q} + B^{\mu_1...\mu_p}{}_{\nu_1...\nu_q} \tag{3.12}$$

   is also a (p,q)-tensor.

2. Direct Products

   Given a (p,q)-tensor $A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q}$ and a (p',q')-tensor $B^{\lambda_1...\lambda_{p'}}{}_{\rho_1...\rho_{q'}}$, their direct product

   $$A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q} B^{\lambda_1...\lambda_{p'}}{}_{\rho_1...\rho_{q'}} \tag{3.13}$$

   is a $(p+p', q+q')$-tensor,

3. Contractions

   Given a (p,q)-tensor with p and q non-zero, one can associate to it a $(p-1, q-1)$-tensor via contraction of one covariant and one contravariant index,

   $$A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q} \rightarrow B^{\mu_1...\mu_{p-1}}{}_{\nu_1...\nu_{q-1}} = A^{\mu_1...\mu_{p-1}\lambda}{}_{\nu_1...\nu_{q-1}\lambda} \;. \tag{3.14}$$

   This is indeed a $(p-1, q-1)$-tensor, i.e. transforms like one. Consider, for example, a (1,2)-tensor $A^{\mu}{}_{\nu\lambda}$ and its contraction $B_\nu = A^{\mu}{}_{\nu\mu}$. Under a coordinate transformation B_ν transforms as a covector:

   $$\begin{aligned} B_{\nu'} &= A^{\mu'}{}_{\nu'\mu'} \\ &= \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial x^\lambda}{\partial y^{\mu'}} A^{\mu}{}_{\nu\lambda} \\ &= \frac{\partial x^\nu}{\partial y^{\nu'}} \delta^\lambda{}_\mu A^{\mu}{}_{\nu\lambda} \\ &= \frac{\partial x^\nu}{\partial y^{\nu'}} A^{\mu}{}_{\nu\mu} = \frac{\partial x^\nu}{\partial y^{\nu'}} B_\nu \;. \end{aligned} \tag{3.15}$$

   A particular example of a contraction is the scalar product between a vector and a covector which is a scalar.

   Note that contraction over different pairs of indices will in general give rise to different tensors. E.g. $A^{\mu}{}_{\nu\mu}$ and $A^{\mu}{}_{\mu\nu}$ will in general be different.

4. Raising and Lowering of Indices

   These operations can of course be combined in various ways. A particular important operation is, given a metric tensor, the raising and lowering of indices with the metric. From the above we know that given a (p,q)-tensor $A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q}$, the product plus contraction with the metric tensor $g_{\mu_1\nu} A^{\mu_1...\mu_p}{}_{\nu_1...\nu_q}$ is a $(p-1, q+1)$-tensor.

It will be denoted by the same symbol, but with *one index lowered by the metric*, i.e. we write

$$g_{\mu_1\nu}A^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q} \equiv A_\nu{}^{\mu_2\dots\mu_p}{}_{\nu_1\dots\nu_q} \quad . \tag{3.16}$$

Note that there are p different ways of lowering the indices, and they will in general give rise to different tensors. It is therefore important to keep track of this in the notation. Thus, in the above, had we contracted over the second index instead of the first, we should write

$$g_{\mu_2\nu}A^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q} \equiv A^{\mu_1}{}_\nu{}^{\mu_3\dots\mu_p}{}_{\nu_1\dots\nu_q} \quad . \tag{3.17}$$

Finally note that this notation is consistent with denoting the inverse metric by raised indices because

$$g^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}g_{\lambda\sigma} \quad . \tag{3.18}$$

and raising one index of the metric gives the Kronecker tensor,

$$g^{\mu\lambda}g_{\lambda\nu} \equiv g^\mu{}_\nu = \delta^\mu{}_\nu \quad . \tag{3.19}$$

An observation we will frequently make use of to recognize when some object is a tensor is the following (occasionally known as the *quotient theorem* or quotient lemma):

Assume that you are given some object $A^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}$. Then if for every covector U_μ the contracted object $U_{\mu_1}A^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}$ transforms like a $(p-1,q)$-tensor, $A^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}$ is a (p,q)-tensor. Likewise for contractions with vectors or other tensors so that if e.g. in an equation of the form

$$A_{\mu\nu} = B_{\mu\nu\lambda\rho}C^{\lambda\rho} \tag{3.20}$$

you know that A transforms as a tensor for every tensor C, then B itself has to be a tensor.

<div align="center">TENSOR DENSITIES</div>

While tensors are the objects which, in a sense, transform in the nicest and simplest possible way under coordinate transformations, they are not the only relevant objects. An important class of non-tensors are so-called *tensor densities*. The prime example of a tensor density is the determinant $g := -\det g_{\mu\nu}$ of the metric tensor (the minus sign is there only to make g positive in signature $(-++ +)$). It follows from the standard tensorial transformation law of the metric that under a coordinate transformation $x^\mu \to y^{\mu'}(x^\mu)$ this determinant transforms as

$$g' = \det\left(\frac{\partial x}{\partial y}\right)^2 g = \det\left(\frac{\partial y}{\partial x}\right)^{-2} g \quad . \tag{3.21}$$

An object which transforms in such a way under coordinate transformations is called a *scalar tensor density of weight* (-2). In general, a tensor density of weight w is an object that transforms as

$$T^{\mu_1'\dots\mu_p'}_{\nu_1'\dots\nu_q'} = \det\left(\frac{\partial y}{\partial x}\right)^w \frac{\partial y^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\mu_p'}}{\partial x^{\mu_p}}\frac{\partial x^{\nu_1}}{\partial y^{\nu_1'}}\cdots\frac{\partial x^{\nu_q}}{\partial y^{\nu_q'}}T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} \ . \tag{3.22}$$

In particular, this implies that $g^{w/2}T^{\dots}_{\dots}$ transforms as (and hence is) a tensor,

$$g'^{w/2}T^{\mu_1'\dots\mu_p'}_{\nu_1'\dots\nu_q'} = \frac{\partial y^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\mu_p'}}{\partial x^{\mu_p}}\frac{\partial x^{\nu_1}}{\partial y^{\nu_1'}}\cdots\frac{\partial x^{\nu_q}}{\partial y^{\nu_q'}}g^{w/2}T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} \ . \tag{3.23}$$

Conversely, therefore, any tensor density of weight w can be written as tensor times $g^{-w/2}$. The relevance of tensor densities arises from the fundamental theorem of integral calculus that says that the integral measure d^4x (more generally d^nx in dimension n) transforms as

$$d^4y = \det\left(\frac{\partial y}{\partial x}\right)d^4x \ , \tag{3.24}$$

i.e. as a scalar density of weight $(+1)$. Thus $g^{1/2}d^4x$ is a volume element which is invariant under coordinate transformations and can be used to define integrals of scalars (functions) in a general metric (curved) space in a coordinate-independent way as

$$\int f := \int \sqrt{g}d^4x f(x) \ . \tag{3.25}$$

This will of course be important in order to formulate action principles etc. in a metric space in a generally covariant way.

There is one more important tensor density which - like the Kronecker tensor - has the same components in all coordinate systems. This is the totally anti-symmetric Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ which, as you can check, is a tensor density of weight (-1) so that $g^{-1/2}\epsilon^{\mu\nu\rho\sigma}$ is a tensor (strictly speaking it is a pseudo-tensor because of its behaviour under reversal of orientation but this will not concern us here).

The algebraic rules for tensor densities are strictly analogous to those for tensors. Thus, for example, the sum of two (p,q) tensor densities of weight w (let us call this a $(p,q;w)$ tensor) is again a $(p,q;w)$ tensor, and the direct product of a $(p_1,q_1;w_1)$ and a $(p_2,q_2;w_2)$ tensor is a $(p_1+p_2,q_1+q_2;w_1+w_2)$ tensor. Contractions and the raising and lowering of indices of tensor densities can also be defined just as for ordinary tensors.

* A Coordinate-Independent Interpretation of Tensors

There is a more invariant and coordinate-independent way of looking at tensors than we have developed so far. The purpose of this section is to explain this point of view even though it is not indispensable for an understanding of the remainder of the course.

Consider first of all the derivative df of a function (scalar field) $f = f(x)$. This is clearly a coordinate-independent object, not only because we didn't have to specify a coordinate system to write df but also because

$$df = \frac{\partial f(x)}{\partial x^\mu} dx^\mu = \frac{\partial f(y(x))}{\partial y^{\mu'}} dy^{\mu'} \quad , \tag{3.26}$$

which follows from the fact that $\partial_\mu f$ (a covector) and dx^μ (the coordinate differentials) transform inversely to each other under coordinate transformations. This suggests that it is useful to regard the quantities $\partial_\mu f$ as the coefficients of the coordinate independent object df in a particular coordinate system, namely when df is expanded in the basis $\{dx^\mu\}$.

We can do the same thing for any covector U_μ. If U_μ is a covector (i.e. transforms like one under coordinate transformations), then $U := U_\mu(x)dx^\mu$ is coordinate-independent, and it is useful to think of the U_μ as the coefficients of the covector U when expanded in a coordinate basis, $U = U_\mu dx^\mu$.

We can even do the same thing for a general covariant tensor $T_{\mu\nu}....$ Namely, if $T_{\mu_1\cdots\mu_q}$ is a $(0,q)$-tensor, then

$$T := T_{\mu_1\cdots\mu_q} dx^{\mu_1} \ldots dx^{\mu_q} \tag{3.27}$$

is coordinate independent. In the particular case of the metric tensor we have already known and used this. In that case, T is what we called ds^2, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, which we know to be invariant under coordinate transformations.

Now, can we do something similar for vectors and other contravariant (or mixed) tensors? The answer is yes. Just as covectors transform inversely to coordinate differentials, vectors V^μ transform inversely to partial derivatives ∂_μ. Thus

$$V := V^\mu(x)\frac{\partial}{\partial x^\mu} \tag{3.28}$$

is coordinate dependent - a coordinate-independent linear first-order differential operator. One can thus always think of a vector field as a differential operator and this is a very fruitful point of view.

Acting on a function (scalar) f, V produces the derivative of f along V,

$$Vf = V^\mu \partial_\mu f \quad . \tag{3.29}$$

This is also a coordinate independent object, a scalar, arising from the contraction of a vector and a covector. And this is as it should be because, after all, both a function and a vector field can be specified on a space-time without having to introduce coordinates (e.g. by simply drawing the vector field and the profile of the function). Therefore also the change of the function along a vector field should be coordinate independent and, as we have seen, it is.

Also this can, in principle, be extended to higher rank tensors, but at this point it would be very useful to introduce the notion of tensor product, something I will not do. Fact of the matter is, however, that any (p,q)-tensor $T^{\mu_1...\mu_p}_{\ \ \nu_1...\nu_q}$ can be thought of as the collection of components of a coordinate independent object T when expanded in a particular coordinate basis in terms of the dx^μ and $(\partial/\partial x^\mu)$.

4 TENSOR ANALYSIS

TENSOR ANALYSIS: PRELIMINARY REMARKS

Tensors transform in a nice and simple way under general coordinate transformations. Thus these appear to be the right objects to construct equations from that satisfy the Principle of General Covariance.

However, the laws of physics are differential equations, so we need to know how to differentiate tensors. The problem is that the ordinary partial derivative does not map tensors to tensors, the partial derivative of a (p,q)-tensor is not a tensor unless $p = q = 0$.

This is easy to see: take for example a vector V^μ. Under a coordinate transformation, its partial derivative transforms as

$$
\begin{aligned}
\partial_{\nu'} V^{\mu'} &= \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial}{\partial x^\nu} \frac{\partial y^{\mu'}}{\partial x^\mu} V^\mu \\
&= \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial y^{\mu'}}{\partial x^\mu} \partial_\nu V^\mu + \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial^2 y^{\mu'}}{\partial x^\mu \partial x^\nu} V^\mu \ \ .
\end{aligned}
\tag{4.1}
$$

The appearance of the second term shows that the partial derivative of a vector is not a tensor.

As the second term is zero for linear transformations, you see that partial derivatives transform in a tensorial way e.g. under Lorentz transformations, so that partial derivatives are all one usually needs in special relativity.

THE COVARIANT DERIVATIVE FOR VECTOR FIELDS

We also see that the lack of covariance of the partial derivative is very similar to the lack of covariance of the equation $\ddot{x}^\mu = 0$, and this suggests that the problem can be cured in the same way - by introducing Christoffel symbols. This is indeed the case. To arrive at the correct definition, we proceed as follows.

Let $\{\xi^A\}$ be an inertial coordinate system. In an inertial coordinate system we can just use the ordinary partial derivative $\partial_B V^A$. We now define the new (improved, covariant)

derivative $\nabla_\nu V^\mu$ in any other coordinate system $\{x^\mu\}$ by demanding that it transforms as a (1,1)-tensor, i.e. we define

$$\nabla_\nu V^\mu := \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial \xi^B}{\partial x^\nu} \partial_B V^A \quad . \tag{4.2}$$

By a straightforward calculation one finds that

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda \quad , \tag{4.3}$$

where $\Gamma^\mu_{\nu\lambda}$ is our old friend

$$\Gamma^\mu_{\nu\lambda} = \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} \quad . \tag{4.4}$$

We thus adopt (4.3) as our definition of the *covariant derivative* in a general metric space or space-time (with the Christoffel symbols calculated from the metric in the usual way). Given the known behaviour of the Christoffel symbols under coordinate transformations, it is of course straightforward to check that (4.3) indeed defines a (1,1)-tensor, but this also follows from the way we arrived at the definition of the covariant derivative. Indeed, imagin transforming from inertial coordinates to another coordinate system $\{y^{\mu'}\}$. Then (4.2) is replaced by

$$\nabla_{\nu'} V^{\mu'} := \frac{\partial y^{\mu'}}{\partial \xi^A} \frac{\partial \xi^B}{\partial y^{\nu'}} \partial_B V^A \quad . \tag{4.5}$$

Comparing this with (4.2), we see that the two are related by

$$\nabla_{\nu'} V^{\mu'} := \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^{\nu'}} \nabla_\nu V^\mu \quad , \tag{4.6}$$

as required.

Frequently, the covariant derivative $\nabla_\nu V^\mu$ is also denoted by a semicolon, $\nabla_\nu V^\mu = V^\mu{}_{;\nu}$. Just as for functions, one can also define the covariant directional derivative of a vector field V along another vector field X^μ by

$$\nabla_X V^\mu \equiv X^\nu \nabla_\nu V^\mu \quad . \tag{4.7}$$

* INVARIANT INTERPRETATION OF THE COVARIANT DERIVATIVE

The appearance of the Christoffel-term in the definition of the covariant derivative may at first sight appear a bit unusual (even though it also appears when one just transforms Cartesian partial derivatives to polar coordinates etc.). There is a more invariant way of explaining the appearance of this term, related to the more coordinate-independent way of looking at tensors explained above. Namely, since the $V^\mu(x)$ are really just the coefficients of the vector field $V(x) = V^\mu(x)\partial_\mu$ when expanded in the basis ∂_μ, a meanigful definition of the derivative of a vector field must take into account not only

the change in the coefficients but also the fact that the basis changes from point to point - and this is precisely what the Christoffel symbols do. Writing

$$\begin{aligned} \nabla_\nu V &= \nabla_\nu(V^\mu \partial_\mu) \\ &= (\partial_\nu V^\mu)\partial_\mu + V^\lambda(\nabla_\nu \partial_\lambda) \ , \end{aligned} \tag{4.8}$$

we see that we reproduce the definition of the covariant derivative if we set

$$\nabla_\nu \partial_\lambda = \Gamma^\mu_{\ \nu\lambda}\partial_\mu \ . \tag{4.9}$$

Indeed we then have

$$\nabla_\nu V \equiv (\nabla_\nu V^\mu)\partial_\mu = (\partial_\nu V^\mu + \Gamma^\mu_{\ \nu\lambda}V^\lambda)\partial_\mu \ , \tag{4.10}$$

which agrees with the above definition.

It is instructive to check in some examples that the Christoffel symbols indeed describe the change of the tangent vectors ∂_μ. For instance on the plane, in polar coordinates, one has

$$\nabla_r \partial_r = \Gamma^\mu_{\ rr}\partial_\mu = 0 \ , \tag{4.11}$$

which is correct because ∂_r indeed does not change when one moves in the radial direction. ∂_r does change, however, when one moves in the angular direction given by ∂_ϕ. In fact, it changes its direction proportional to ∂_ϕ but this change is stronger for small values of r than for larger ones (draw a picture!). This is precisely captured by the non-zero Christoffel symbol $\Gamma^\phi_{\ r\phi}$,

$$\nabla_\phi \partial_r = \Gamma^\phi_{\ r\phi}\partial_\phi = \frac{1}{r}\partial_\phi \ . \tag{4.12}$$

EXTENSION OF THE COVARIANT DERIVATIVE TO OTHER TENSOR FIELDS

We will see that, demanding that the covariant derivative satisfies the Leibniz rule, there is a unique extension of the covariant derivative on vector fields to a differential operator on general tensor fields, mapping (p, q)- to $(p, q + 1)$-tensors.

We start with scalars ϕ. As $\partial_\mu \phi$ is already a covector, we set

$$\nabla_\mu \phi = \partial_\mu \phi \ . \tag{4.13}$$

To define the covariant derivative for covectors U_μ, we note that $U_\mu V^\mu$ is a scalar for any vector U^μ so that

$$\nabla_\mu(U_\nu V^\nu) = \partial_\mu(U_\nu V^\nu) = (\partial_\mu U_\nu)V^\nu + U_\nu(\partial_\mu V^\nu) \ , \tag{4.14}$$

and we demand

$$\nabla_\mu(U_\nu V^\nu) = (\nabla_\mu U_\nu)V^\nu + U_\nu \nabla_\mu V^\nu \ . \tag{4.15}$$

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As we know $\nabla_\mu V^\nu$, these two equations determine $\nabla_\mu U_\nu$ to be

$$\nabla_\mu U_\nu = \partial_\mu U_\nu - \Gamma^\lambda_{\mu\nu} U_\lambda \ . \tag{4.16}$$

That this is indeed a $(0,2)$-tensor can either be checked directly or, alternatively, is a consequence of the quotient theorem.

The extension to other (p,q)-tensors is now immediate. If the (p,q)-tensor is the direct product of p vectors and q covectors, then we already know its covariant derivative (using the Leibniz rule again). We simply adopt the same resulting formula for an arbitrary (p,q)-tensor. The result is that the covariant derivative of a general (p,q)-tensor is the sum of the partial derivative, a Christoffel symbol with a positive sign for each of the p upper indices, and a Christoffel with a negative sign for each of the q lower indices. In equations

$$\begin{aligned}
\nabla_\mu T^{\nu_1 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} &= \partial_\mu T^{\nu_1 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} \\
&+ \underbrace{\Gamma^{\nu_1}_{\mu\lambda} T^{\lambda \nu_2 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} + \ldots + \Gamma^{\nu_p}_{\mu\lambda} T^{\nu_1 \cdots \nu_{p-1} \lambda}{}_{\rho_1 \cdots \rho_q}}_{p \text{ terms}} \\
&- \underbrace{\Gamma^\lambda_{\mu\rho_1} T^{\nu_1 \cdots \nu_p}{}_{\lambda \rho_2 \cdots \rho_q} - \ldots - \Gamma^\lambda_{\mu\rho_q} T^{\nu_1 \cdots \nu_p}{}_{\rho_1 \cdots \rho_{q-1} \lambda}}_{q \text{ terms}}
\end{aligned} \tag{4.17}$$

Having defined the covariant derivative for arbitrary tensors, we are also ready to define it for tensor densities. For this we recall that if T is a $(p,q;w)$ tensor density, then $g^{w/2}T$ is a (p,q)-tensor. Thus $\nabla_\mu(g^{w/2}T)$ is a $(p,q+1)$-tensor. To map this back to a tensor density of weight w, we multiply this by $g^{-w/2}$, arriving at the definition

$$\nabla_\mu T := g^{-w/2} \nabla_\mu (g^{w/2} T) \ . \tag{4.18}$$

Working this out explictly, one finds

$$\nabla_\mu T = \frac{w}{2g}(\partial_\mu g)T + \nabla_\mu^{\text{tensor}} T \ , \tag{4.19}$$

where $\nabla_\mu^{\text{tensor}}$ just means the usual covariant derivative for (p,q)-tensors defined above. In particular for a scalar density ϕ one has

$$\nabla_\mu \phi = \partial_\mu \phi + \frac{w}{2g}(\partial_\mu g)\phi \ . \tag{4.20}$$

MAIN PROPERTIES OF THE COVARIANT DERIVATIVE

1. Linearity

   For a and b real numbers and A and B two (p,q)-tensors (I will sometimes drop the indices in the following) one has

   $$\nabla_\mu(aA + bB) = a\nabla_\mu A + b\nabla_\mu B \ . \tag{4.21}$$

2. Leibniz rule

   For A and B tensors (not necessarily of the same type), the covariant derivative of their direct product is

   $$\nabla_\mu(AB) = (\nabla_\mu A)B + A(\nabla_\mu B) \ . \tag{4.22}$$

3. Commutes with Contraction

   This means that if A is a (p, q)-tensor and B is the $(p-1, q-1)$-tensor obtained by contraction over two particular indices, then the covariant derivative of B is the same as the covariant derivative of A followed by contraction over these two indices. This comes about because of a cancellation between the corresopdning two Christoffel symbols with opposite signs. Consider e.g. a $(1,1)$-tensor $A^\nu{}_\rho$ and its contraction $A^\nu{}_\nu$. The latter is a scalar and hence its covariant derivative is just the partial derivative. This can also be obtained by taking first the covariant derivative of A,

   $$\nabla_\mu A^\nu{}_\rho = \partial_\mu A^\nu{}_\rho + \Gamma^\nu{}_{\mu\lambda} A^\lambda{}_\rho - \Gamma^\lambda{}_{\mu\rho} A^\nu{}_\lambda \ , \tag{4.23}$$

   and then contracting:

   $$\nabla_\mu A^\nu{}_\nu = \partial_\mu A^\nu{}_\nu + \Gamma^\nu{}_{\mu\lambda} A^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu} A^\nu{}_\lambda = \partial_\mu A^\nu{}_\nu \ . \tag{4.24}$$

4. The Metric is Covariantly Constant: $\nabla_\mu g_{\nu\lambda} = 0$

   This is one of the key properties of the covariant derivative ∇_μ we have defined. I will give two arguments to establish this:

   (a) Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, we can choose any coordinate system we like to establish if this tensor is zero or not at a given point x. Choose an inertial coordinate system at x. Then the partial derivatives of the metric and the Christoffel symbols are zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true in every coordinate system.

   (b) The other argument is by direct calculation. Recalling the identity

   $$\partial_\mu g_{\nu\lambda} = \Gamma_{\nu\lambda\mu} + \Gamma_{\lambda\nu\mu} \ , \tag{4.25}$$

   we calculate

   $$\begin{aligned} \nabla_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \Gamma^\rho{}_{\mu\nu} g_{\rho\lambda} - \Gamma^\rho{}_{\mu\lambda} g_{\nu\rho} \\ &= \Gamma_{\nu\lambda\mu} + \Gamma_{\lambda\nu\mu} - \Gamma_{\lambda\mu\nu} - \Gamma_{\nu\mu\lambda} \\ &= 0 \ . \end{aligned} \tag{4.26}$$

5. Commutes with Raising and Lowering of Indices

   This is really a direct consequence of the covariant constancy of the metric. For example, if V_μ is the covector obtained by lowering an index of the vector V^μ, $V_\mu = g_{\mu\nu}V^\nu$, then

   $$\nabla_\lambda V_\mu = \nabla_\lambda(g_{\mu\nu}V^\nu) = g_{\mu\nu}\nabla_\lambda V^\nu \quad . \tag{4.27}$$

6. Covariant Derivatives Commute on Scalars

   This is of course a familiar property of the ordinary partial derivative, but it is also true for the second covariant derivatives of a scalar and is a consequence of the symmetry of the Christoffel symbols in the second and third indices and is also knowns as the *no torsion* property of the covariant derivative. Namely, we have

   $$\begin{aligned}
   \nabla_\mu\nabla_\nu\phi - \nabla_\nu\nabla_\mu\phi &= \nabla_\mu\partial_\nu\phi - \nabla_\nu\partial_\mu\phi \\
   &= \partial_\mu\partial_\nu\phi - \Gamma^\lambda_{\mu\nu}\phi - \partial_\nu\partial_\mu\phi + \Gamma^\lambda_{\nu\mu}\phi = 0 \quad . \tag{4.28}
   \end{aligned}$$

   Note that the second covariant derivatives on higher rank tensors do *not* commute - we will come back to this in our discussion of the curvature tensor later on.

The Principle of Minimal Coupling

The fact that the covariant derivative ∇ maps tensors to tensors and reduces to the ordinary partial derivative in a locally inertial coordinate system suggests the following algorithm for assessing the effects of gravitation on physical systems and obtaining equations satisfying the Principle of General Covariance:

1. Write down the Lorentz invariant equations of Special Relativity (e.g. those of relativistic mechanics, Maxwell theory, relativistic hydrodynamics, . . .).

2. Wherever the Minkoski metric $\eta_{\mu\nu}$ appears, replace it by $g_{\mu\nu}$.

3. Wherever a partial derivative ∂_μ appears, replace it by the covariant derivative ∇_μ

By construction, these equations are tensorial (generally covariant) and true in the absence of gravity and hence satisfy the Principle of General Covariance. Hence they will be true in the presence of gravitational fields (at least on scales small compared to that of the gravitational fields - if one considers higher derivatives of the metric tensor then there are other equations that one can write down, involving e.g. the curvature tensor, that are tensorial but reduce to the same equations in the absence of gravity).

Tensor Analysis: Some Special Cases

In this section I will, without proof, give some useful special cases of covariant derivatives - covariant curl and divergence etc. - you should make sure that you can derive all of these yourself without any problems.

1. The Covariant Curl of a Covector

   One has

   $$\nabla_\mu U_\nu - \nabla_\nu U_\mu = \partial_\mu U_\nu - \partial_\nu U_\mu \;\;, \tag{4.29}$$

   because the symmetric Christoffel symbols drop out in this antisymmetric linear combination. Thus the Maxwell field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor under general coordinate transformations, no metric or covariant derivative is needed to make it a tensor in a general space time.

2. The Covariant Divergence of a Vector

   By the covariant divergence of a vector field one means the scalar

   $$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \;\;. \tag{4.30}$$

   Now a useful identity for the contracted Christoffel symbol is

   $$\Gamma^\mu_{\mu\lambda} = g^{-1/2} \partial_\lambda (g^{+1/2}) \;\;. \tag{4.31}$$

   Thus

   $$\nabla_\mu V^\mu = g^{-1/2} \partial_\mu (g^{+1/2} V^\mu) \;\;, \tag{4.32}$$

   and one only needs to calculate g and its derivative, not the Christoffel symbols themselves, to calculate the covariant divergence of a vector field.

3. The Covariant Laplacian of a Scalar

   . . . follows from the expression for the covariant divergence of a vector and is given by

   $$\begin{aligned} \Box \phi \;\; &:= \;\; g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \\ &= \;\; \nabla_\mu (g^{\mu\nu} \partial_\nu \phi) \\ &= \;\; g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu \phi) \;\;. \end{aligned} \tag{4.33}$$

4. The Covariant Form of Gauss' Theorem

   Let V^μ be a vector field, $\nabla_\mu V^\mu$ its divergence and recall that integrals in curved space are defined with respect to the integration measure $g^{1/2} d^4 x$. Thus one has

   $$\int g^{1/2} d^4 x \nabla_\mu V^\mu = \int d^4 x \partial_\mu (g^{1/2} V^\mu) \;\;. \tag{4.34}$$

   Now the second term is an ordinary total derivative and thus, if V^μ vanishes sufficiently rapidly at infinity, one has

   $$\int g^{1/2} d^4 x \nabla_\mu V^\mu = 0 \;\;. \tag{4.35}$$

5. The Covariant Divergence of an Antisymmetric Tensor

For a $(p, 0)$-tensor $T^{\mu\nu\cdots}$ one has

$$
\begin{aligned}
\nabla_\mu T^{\mu\nu\cdots} &= \partial_\mu T^{\mu\nu\cdots} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu\cdots} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda\cdots} + \cdots \\
&= g^{-1/2} \partial_\mu (g^{1/2} T^{\mu\nu\cdots}) + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda\cdots} + \cdots \quad .
\end{aligned}
\tag{4.36}
$$

In particular, if $T^{\mu\nu\cdots}$ is completely antisymmetric, the Christoffel terms disappear and one is left with

$$
\nabla_\mu T^{\mu\nu\cdots} = g^{-1/2} \partial_\mu (g^{1/2} T^{\mu\nu\cdots}) \quad .
\tag{4.37}
$$

6. The Covariant Curl of an Antisymmetric Tensor

Let $A_{\nu\lambda\cdots}$ be completely antisymmetric. Then, as for the curl of covectors, the metric and Christoffel symbols drop out of the expression for the curl, i.e. one has

$$
\nabla_{[\mu} A_{\nu\lambda\cdots]} = \partial_{[\mu} A_{\nu\lambda\cdots]} \quad .
\tag{4.38}
$$

Here the sqaure brackets denote complete antisymmetrization. In particular, the Bianchi identity for the Maxwell field strength tensor is independent of the metric also in a general metric space time.

You will have noticed that many equations simplify considerably for completely anti-symmetric tensors. In particular, their curl can be defined in a tensorial way without reference to any metric. This observation is at the heart of the coordinate independent calculus of *differential forms*. In this context, the curl is known as the *exterior derivative*.

COVARIANT DIFFERENTIATION ALONG A CURVE

So far, we have defined covariant differentiation for tensors defined everywhere in space time. Frequently, however, one encounters tensors that are only defined on curves - like the momentum of a particle which is only defined along its world line. In this section we will see how to define covariant differentiation along a curve. Thus consider a curve $x^\mu(\tau)$ (where τ could be, but need not be, proper time) and the tangent vector field $X^\mu(x(\tau)) = \dot{x}^\mu(\tau)$. Now define the covariant derivative $D/D\tau$ along the curve by

$$
\frac{D}{D\tau} = X^\mu \nabla_\mu = \dot{x}^\mu \nabla_\mu \quad .
\tag{4.39}
$$

For example, for a vector one has

$$
\begin{aligned}
\frac{DV^\mu}{D\tau} &= \dot{x}^\nu \partial_\nu V^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\lambda} V^\lambda \\
&= \frac{d}{d\tau} V^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu V^\lambda \quad .
\end{aligned}
\tag{4.40}
$$

For this to make sense, V^μ needs to be defined only along the curve and not necessarily everywhere in space time.

We now come to the important notion of *parallel transport* of a tensor along a curve. Note that in a general (curved) metric space time, it does not make sense to ask if two vectors defined at points x and y are parallel to each other or not. However, given a metric and a curve connecting these two points, one can compare the two by dragging one along the curve to the other using the covariant derivative.

We say that a tensor $T^{...}_{...}$ is *parallel transported along the curve* $x^\mu(\tau)$ if

$$\frac{DT^{...}_{...}}{D\tau} = 0 \ . \tag{4.41}$$

Here are some immediate consequences of this definition:

1. In a locally inertial coordinate system, this condition reduces to $dT/d\tau = 0$, i.e. to the statement that the tensor does not change along the curve. Thus the above is indeed an appropriate tensorial generalization of the intuitive notion of parallel transport to a general metric space time.

2. The parallel transport condition is a first order differential equation and thus defined $T^{...}_{...}(\tau)$ given an initial value $T^{...}_{...}(\tau_0)$.

3. Taking T to be the tangent vector $X^\mu = \dot{x}^\mu$ to the curve itself, the condition for parallel transport becomes

$$\frac{DX^\mu}{D\tau} = 0 \Leftrightarrow \ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \ , \tag{4.42}$$

   i.e. the geodesic equation. Thus geodesics are such curves for which their tangent vectors are parallel transported (do not change) along the curve. For this reason geodesics are also known as *autoparallels*.

4. Since the metric is covariantly constant, it is in particular parallel along any curve. Thus, in particular, if V^μ is parallel transported, also its length remains constant along the curve,

$$\frac{DV^\mu}{D\tau} = 0 \Rightarrow \frac{D}{D\tau}(g_{\mu\nu}V^\mu V^\nu) = 0 \ . \tag{4.43}$$

5. Now let $x^\mu(\tau)$ be a geodesic and V^μ parallel along this geodesic. Then, as one might intuitively expect, also the angle between V^μ and the tangent vector to the curve X^μ remains constant. This is a consequence of the fact that both the norm of V and the norm of X are constant along the curve and that

$$\begin{aligned} \frac{d}{d\tau}(g_{\mu\nu}X^\mu V^\nu) &= \frac{D}{D\tau}(g_{\mu\nu}X^\mu V^\nu) \\ &= g_{\mu\nu}\frac{D}{D\tau}(X^\mu)V^\nu + g_{\mu\nu}X^\mu\frac{D}{D\tau}V^\nu \\ &= 0 + 0 = 0 \end{aligned} \tag{4.44}$$

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6. The physical meaning of parallel transport of a vector along a curve is that it corresponds to a physically invariant direction as determined e.g. by a Foucault pendulum or a perfect gyroscope.

* Generalizations

Recall that the transformation behaviour of the Christoffel symbols, equation (1.17), was the key ingredient in the proof that the geodesic equation transforms like a vector under general coordinate transformations. Likewise, to show that the covariant derivative of a tensor is again a tensor all one needs to know is that the Christoffel symbols transform in this way. Thus any other object $\tilde{\Gamma}^{\mu}_{\nu\lambda}$ could also be used to define a covariant derivative (generalizing the partial derivative and mapping tensors to tensors) provided that it transforms in the same way as the Christoffel symbols, i.e. provided that one has

$$\tilde{\Gamma}^{\mu'}_{\nu'\lambda'} = \tilde{\Gamma}^{\mu}_{\nu\lambda} \frac{\partial y^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial y^{\nu'}} \frac{\partial x^{\lambda}}{\partial y^{\lambda'}} + \frac{\partial y^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial y^{\nu'} \partial y^{\lambda'}} \quad . \tag{4.45}$$

But this implies that the difference

$$\Delta^{\mu}_{\nu\lambda} = \tilde{\Gamma}^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\nu\lambda} \tag{4.46}$$

transforms as a tensor. Thus, any such $\tilde{\Gamma}$ is of the form

$$\tilde{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} + \Delta^{\mu}_{\nu\lambda} \tag{4.47}$$

where Δ is a (1,2)-tensor and the question arises if or why the Christoffel symbols we have been using are somehow singled out or preferred.

In some sense, the answer is an immediate yes because it is this particular connection that enters in determining the paths of freely falling particles (the geodesics which extremize proper time).

Moreover, the covariant derivative as we have defined it has two important properties, namely

1. that the metric is covariantly constant, $\nabla_{\mu} g_{\nu\lambda} = 0$, and

2. that the torsion is zero, i.e. that the second covariant derivatives of a scalar commute.

In fact, it turns out that these two conditions uniquely determine the $\tilde{\Gamma}$ to be the Christoffel symbols: the second condition implies that the $\tilde{\Gamma}^{\mu}_{\nu\lambda}$ are symmetric in the two lower indices, and then the first condition allows one to express them in terms of derivatives of the metric, leading to the familiar expression for the Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$. This unique metric-compatible and torsion-free connection is also known as the *Levi-Cività connection.*

It is of course possible to relax either of the conditions (1) or (2), or both of them. Relaxing (1), however, is probably physically not very meaningful (for more or less the same reasons for which Einstein rejected Weyl's original gauge theory).

It is possible, however, to relax (2), and such connections with torsion play a role in certain generalized theories of gravity. In general, for such a connection, the notions of geodesics and autoparallels no longer coincide. However, this difference disappears if Δ happens to be antisymmetric in its lower indices, as one then has

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda} = \ddot{x}^{\mu} + \tilde{\Gamma}^{\mu}_{\nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda} \ , \tag{4.48}$$

so that the presence of torsion may not readily be experimentally detectable.

5 PHYSICS IN A GRAVITATIONAL FIELD

PARTICLE MECHANICS IN A GRAVITATIONAL FIELD REVISITED

We can see the power of the formalism we have developed so far by rederiving the laws of particle mechanics in a general gravitational field. In Special Relativity, the motion of a particle with mass m moving under the influence of some external force is governed by the equation

$$\text{SR:} \quad \frac{dX^{\mu}}{d\tau} = \frac{f^{\mu}}{m} \ , \tag{5.1}$$

where f^{μ} is the force four-vector and $X^{\mu} = \dot{x}^{\mu}$. Thus, using the principle of minimal coupling, the equation in a general gravitational field is

$$\text{GR:} \quad \frac{DX^{\mu}}{D\tau} = \frac{f^{\mu}}{m} \ , \tag{5.2}$$

Of course, the left hand side is just the familiar geodesic equation, but we see that it follows much faster from demanding general covariance (the principle of minimal coupling) than from our previous considerations.

ELECTRODYNAMICS IN A GRAVITATIONAL FIELD

Here is where the formalism we have developed really pays off. We will see once again that, using the minimal coupling rule, we can immediately rewrite the Maxwell equations in a form in which they are valid in an arbitrary gravitational field.

Given the vector potential A_{μ}, the Maxwell field strength tensor in Special Relativity is

$$\text{SR:} \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \ . \tag{5.3}$$

Therefore in a general metric space time (gravitational field) we have

$$\text{GR:} \quad F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \ . \tag{5.4}$$

Actually, this is a bit misleading. The field strength tensor (two-form) in any, Abelian or non-Abelian, gauge theory is always given in terms of the gauge-covariant exterior derivative of the vector potential (connection), and as such has nothing whatsoever to do with the metric on space-time. So you should not really regard the first equality in the above equation as the definition of $F_{\mu\nu}$, but you should regard the second equality as a proof that $F_{\mu\nu}$, *always* defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is a tensor. The mistake of adopting $\nabla_\mu A_\nu - \nabla_\nu A_\mu$ as the definition of $F_{\mu\nu}$ in a curved space time has led some poor souls to believe, and even claim in published papers, that in a space time with torsion, for which the second equality does not hold, the Maxwell field strength tensor is modified by the torsion. This is nonsense.

In Special Relativity, the Maxwell equations read

$$\text{SR:} \qquad \partial_\mu F^{\mu\nu} = -J^\nu$$
$$\partial_{[\mu} F_{\nu\lambda]} = 0 \ . \qquad (5.5)$$

Thus in a general gravitational field (curved space time) these equations become

$$\text{GR:} \qquad \nabla_\mu F^{\mu\nu} = -J^\nu$$
$$\nabla_{[\mu} F_{\nu\lambda]} = 0 \ , \qquad (5.6)$$

where now of course all indices are raised and lowered with the metric $g_{\mu\nu}$ (and with the same *caveat* as above regarding the use of the covariant derivative in the second equation). Using the results derived above, we can rewrite these two equations as

$$\text{GR:} \qquad \partial_\mu(g^{1/2} F^{\mu\nu}) = -g^{1/2} J^\nu$$
$$\partial_{[\mu} F_{\nu\lambda]} = 0 \ . \qquad (5.7)$$

In Special Relativity, the equations of motion follow from the action

$$\text{SR:} \quad S = \int d^4x \ F_{\mu\nu} F^{\mu\nu} \ , \qquad (5.8)$$

so the action in a general gravitational field is

$$\text{GR:} \quad S = \int \sqrt{g} d^4x \ g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \ , \qquad (5.9)$$

The electromagnetic force acting on a particle of charge e is given in Special Relativity by

$$\text{SR:} \quad f^\mu = e F^\mu_{\ \nu} \dot{x}^\nu \ . \qquad (5.10)$$

Thus in General Relativity it becomes

$$\text{GR:} \quad f^\mu = e g^{\mu\lambda} F_{\lambda\nu} \dot{x}^\nu \ . \qquad (5.11)$$

The energy-momentum tensor of Maxwell theory is

$$\text{SR:} \quad T^{\mu\nu} = F^\mu_{\ \lambda} F^{\nu\lambda} - \tfrac{1}{4} \eta^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \ . \qquad (5.12)$$

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Therefore in General Relativity it reads

$$\text{SR:} \quad T^{\mu\nu} = F^\mu_{\ \lambda}F^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} \ , \tag{5.13}$$

where all indices are raised with the inverse metric $g^{\mu\nu}$. The conservation equation

$$\text{SR:} \quad \partial_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \ , \tag{5.14}$$

(deriving this requires using both sets of Maxwell equations) becomes the covariant conservation law

$$\text{GR:} \quad \nabla_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \ , \tag{5.15}$$

We will discuss below in which sense or under which conditions this equation leads to conserved quantities in the ordinary sense.

Conserved Quantities from Covariantly Conserved Currents

In Special Relativity a conserved current J^μ is characetrized by the vanishing of its divergence, i.e. by $\partial_\mu J^\mu = 0$. It leads to a conserved charge Q by integrating J^μ over a space-like hypersurface, say the one described bu $t = t_0$. This is usually written as something like

$$Q = \int_{t=t_0} J^\mu dS_\mu \ , \tag{5.16}$$

where dS_μ is the induced volume element on the hypersurface. That Q is conserved, i.e. independent of t_0, is a consequence of

$$Q(t_1) - Q(t_0) = \int_V d^4x \ \partial_\mu J^\mu = 0 \ , \tag{5.17}$$

where V is the four-volume $\mathbb{R}^3 \times [t_0, t_1]$. This holds provided that J vanishes at spatial infinity.

Now in General Relativity, the conservation law will be replaced by the covariant conservation law $\nabla_\mu J^\mu = 0$, and one may wonder if this also leads to some conserved charges in the ordinary sense. The answer is yes because, recalling the formula for the covariant divergence of a vector,

$$\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \ , \tag{5.18}$$

we see that

$$\nabla_\mu J^\mu = 0 \Leftrightarrow \partial_\mu(g^{1/2}J^\mu) = 0 \ , \tag{5.19}$$

so that $g^{1/2}J^\mu$ is a conserved current in the ordinary sense. We then obtain conserved quantities in the ordinary sense by integrating J^μ over a space-like hypersurface Σ. Using the generalized Gauss' theorem appropriate for metric space-times, one can see that Q is invariant under deformations of Σ.

In order to write down more precise equations for the charges in this case, we would have to understand how a metric on space-time induces a metric (and hence volume

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element) on a space-like hypersurface. This would require developing a certain amount of formalism, useful for certain purposes in Cosmology and for developing a canonical formalism for General Relativity. But as this lies somewhat outside of the things we will do in this course, I will skip this. Suffice it to say here that the first step would be the introduction of a normalized normal vector n^μ to the hypersurface Σ, $n^\mu n_\mu = -1$ and to consider the object $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. As $h_{\mu\nu} n^\nu = 0$ while $h_{\mu\nu} X^\nu = g_{\mu\nu} X^\nu$ for any vector X^μ normal to n^μ, $h_{\mu\nu}$ induces a metric and volume element on Σ.

The factor $g^{1/2}$ apearing in the current conservation law can be understood physically. To see what it means, split J^μ into its space-time direction u^μ, with $u^\mu u_\mu = -1$, and its magnitude ρ as

$$J^\mu = \rho u^\mu \ . \tag{5.20}$$

This defines the average four-velocity of the conserved quantity represented by J^μ and its density ρ measured by an observer moving at that average velocity (rest mass density, charge density, number density, ...). Since u^μ is a vector, in order for J^μ to be a vector, ρ has to be a scalar. Therefore this density is defined as per unit proper volume. The factor of $g^{1/2}$ transforms this into density per coordinate volume and this quantity is conserved (in a comoving coordinate system where $J^0 = \rho$, $J^i = 0$).

We will come back to this in the context of cosmology later on in this course (see section 16) but for now just think of the following picture (Figure 19): take a balloon, draw lots of dots on it at random, representing particles or galaxies. Next choose some coordinate system on the balloon and draw the coordinate grid on it. Now inflate or deflate the balloon. This represents a time dependent metric, roughly of the form $ds^2 = r^2(t)(d\theta^2 + \sin^2\theta d\phi^2)$. You see that the number of dots per coordinate volume element does not change, whereas the number of dots per unit proper volume will.

<center>CONSERVED QUANTITIES FROM COVARIANTLY CONSERVED TENSORS?</center>

In Special Relativity, if $T^{\mu\nu}$ is the energy-momentum tensor of a physical system, it satisfies an equation of the form

$$\partial_\mu T^{\mu\nu} = G^\nu \ , \tag{5.21}$$

where G^μ represents the density of the external forces acting on the system. In particular, if there are no external forces, the divergence of the energy-momentum tensor is zero. For example, in the case of Maxwell theory and a current corresponding to a charged particle we have

$$G^\nu = J_\mu F^{\mu\nu} = -F^\nu_{\ \mu} J^\mu = -e F^\nu_{\ \mu} \dot{x}^\mu \ , \tag{5.22}$$

which is indeed the relevant external (Lorentz) force density.

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Now, in General Relativity we will instead have

$$\nabla_\mu T^{\mu\nu} = G^\nu \Leftrightarrow g^{-1/2} \partial_\mu(g^{1/2}T^{\mu\nu}) = G^\nu - \Gamma^\nu_{\mu\lambda}T^{\mu\lambda} \ . \tag{5.23}$$

Thus the second term on the right hand side represents the gravitational force density. As expected, it depends on the system on which it acts via the energy momentum tensor. And, as expected, this contribution is not generally covariant.

Now, in analogy with Special Relativity, one might like to define quantities like energy and momentum, P^μ, and angular momentum $J^{\mu\nu}$, by integrals of $T^{\mu 0}$ or $x^\mu T^{\nu 0} - x^\nu T^{\mu 0}$ over space-like hypersurfaces. However, these quantities are rather obviously not covariant, and nor are they conserved. This should perhaps not be too surprising because in Minkowski space these quantities are preserved as a consequence of Poincaré invariance, i.e. because of the symmetries (isometries) of the Minkowski metric. As a generic metric will have no such isometries, we do not expect to find associated conserved quantities in general. However, if there are symmetries then one can indeed define conserved quantities (think of Noether's theorem), one for each symmetry generator. In order to implement this we need to understand how to define and detect isometries of the metric. For this we need the concepts of Lie derivatives and Killing vectors.

6 THE LIE DERIVATIVE, SYMMETRIES AND KILLING VECTORS

SYMMETRIES (ISOMETRIES) OF A METRIC: PRELIMINARY REMARKS

Before trying to figure out how to detect symmetries of a metric, or so-called *isometries*, let us decide what we mean by symmetries of a metric. For example, we would say that the Minkowski metric has the Poincaré group as a group of symmetries, because the corresponding coordinate transformations leave the metric invariant.

Likewise, we would say that the standard metrics on the two- or three-sphere have rotational symmetries because they are invariant under rotations of the sphere. We can look at this in one of two ways: either as an *active* transformation, in which we rotate the sphere and note that nothing changes, or as a *passive* transformation, in which we do not move the sphere, all the points remain fixed, and we just rotate the coordinate system. So this is tantamount to a relabelling of the points. From the latter (passive) point of view, the symmetry is again understood as an invariance of the metric under a particular family of coordinate transformations.

Thus consider a metric $g_{\mu\nu}(x)$ in a coordinate system $\{x^\mu\}$ and a change of coordinates $x^\mu \to y^\mu(x^\nu)$ (for the purposes of this and the following section it will be convenient not to label the two coordinate systems by different sets of indices). Of course, under such a coordinate transformation we get a new metric $g'_{\mu\nu}$, with

$$g'_{\mu\nu}(y(x)) = \frac{\partial x^\rho}{\partial y^\mu} \frac{\partial x^\lambda}{\partial y^\nu} g_{\rho\lambda}(x) \ . \tag{6.1}$$

From the above discussion we deduce that what we mean by a symmetry, i.e. invariance of the metric under a coordinate transformation, is the statement

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) \quad . \tag{6.2}$$

Indeed, from the passive point of view, in which a coordinate transformation represents a relabelling of the points of the space, this equation compares the old metric at a point P (with coordinates x^μ) with the new metric at the point P' which has the same values of the new coordinates as the point P had in the old coordinate system, $y^\mu(P') = x^\mu(P)$. The above equality states that the new metric at the point P' has the same functional dependence on the new coordinates as the old metric on the old coordinates at the point P. Thus a neighbourhood of P' in the new coordinates looks identical to a neighbourhood of P in the old coordinates, and they can be mapped into each other *isometrically*, i.e. such that all the metric properties, like distances, are preserved. Note that to detect a continuous symmetry in this way, we only need to consider infinitesimal coordinate transformations. In that case, the above amounts to the statement that metrically the space time looks the same when one moves infinitesimally in the direction given by the coordinate transformation.

<center>THE LIE DERIVATIVE FOR SCALARS</center>

We now want to translate the above discussion into a condition for an infinitesimal coordinate transformation

$$x^\mu \to y^\mu(x) = x^\mu + \epsilon V^\mu(x) \tag{6.3}$$

to generate a symmetry of the metric. Here you can and should think of V^μ as a vector field because, even though coordinates themselves of course do not transform like vectors, their infinitesimal variations δx^μ do,

$$z^{\mu'} = z^{\mu'}(x) \to \delta z^{\mu'} = \frac{\partial z^{\mu'}}{\partial x^\mu} \delta x^\mu \tag{6.4}$$

and we think of δx^μ as ϵV^μ.

In fact, we will do something slightly more general than just trying to detect symmetries of the metric. After all, we can also speak of functions or vector fields with symmetries, and this can be extended to arbitrary tensor fields (although that may be harder to visualize). So, for a general tensor field T we will want to compare $T'(y(x))$ with $T(y(x))$ - this is of course equivalent to, and only technically a bit easier than, comparing $T'(x)$ with $T(x)$.

As usual, we start the discussion with scalars. In that case, we want to compare $\phi(y(x))$ with $\phi'(y(x)) = \phi(x)$. We find

$$\phi(y(x)) - \phi'(y(x)) = \phi(x + \epsilon V) - \phi(x) = \epsilon V^\mu \partial_\mu \phi + \mathcal{O}(\epsilon^2) \quad . \tag{6.5}$$

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We now define the *Lie derivative* of ϕ along the vector field V^μ to be

$$L_V \phi := \lim_{\epsilon \to 0} \frac{\phi(y(x)) - \phi'(y(x))}{\epsilon} = V^\mu \partial_\mu \phi \ . \tag{6.6}$$

Thus for a scalar, the Lie derivative is just the ordinary directional derivative, and this is as it should be since saying that a function has a certain symmetry amounts to the assertion that its derivative in a particular direction vanishes.

THE LIE DERIVATIVE FOR VECTOR FIELDS

We now follow the same procedure for a vector field W^μ. We will need the matrix $(\partial y^\mu / \partial x^\nu)$ and its inverse for the above infinitesimal coordinate transformation. We have

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu + \epsilon \partial_\nu V^\mu \ , \tag{6.7}$$

and

$$\frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_\nu - \epsilon \partial_\nu V^\mu + \mathcal{O}(\epsilon^2) \ . \tag{6.8}$$

Thus we have

$$\begin{aligned}
W'^\mu(y(x)) &= \frac{\partial y^\mu}{\partial x^\nu} W^\nu(x) \\
&= W^\mu(x) + \epsilon W^\nu(x) \partial_\nu V^\mu(x) \ ,
\end{aligned} \tag{6.9}$$

and

$$W^\mu(y(x)) = W^\mu(x) + \epsilon V^\nu \partial_\nu W^\mu(x) + \mathcal{O}(\epsilon^2) \ . \tag{6.10}$$

Hence, defining the Lie derivative $L_V W$ of W by V by

$$L_V W^\mu := \lim_{\epsilon \to 0} \frac{W^\mu(y(x)) - W'^\mu(y(x))}{\epsilon} \ , \tag{6.11}$$

we find

$$L_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu \ . \tag{6.12}$$

There are several important things to note about this expression:

1. The result looks non-covariant, i.e. non-tensorial. But as a difference of two vectors at the same point (recall the limit $\epsilon \to 0$) the result should again be a vector. This is indeed the case. One way to make this manifest is to rewrite (6.12) in terms of covariant derivatives, as

$$\begin{aligned}
L_V W^\mu &= V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu \\
&= \nabla_V W^\mu - \nabla_W V^\mu \ .
\end{aligned} \tag{6.13}$$

This shows that $L_W W^\mu$ is again a vector field. Note, however, that the Lie derivative, in contrast to the covariant derivative, is defined without reference to any metric.

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2. Note that (6.12) is antisymmetric in V and W. Hence it defines a commutator $[V, W]$ on the space of vector fields,

$$[V, W]^\mu := L_V W^\mu = -L_W V^\mu \ . \tag{6.14}$$

This is actually a Lie bracket, i.e. it satisfies the Jacobi identity

$$[V, [W, X]]^\mu + [X, [V, W]]^\mu + [W, [X, V]]^\mu = 0 \ . \tag{6.15}$$

This can also be rephrased as the statement that the Lie derivative is also a derivation of the Lie bracket, i.e. that one has

$$L_V [W, X]^\mu = [L_V W, X]^\mu + [W, L_V X]^\mu \ . \tag{6.16}$$

3. I want to reiterate at this point that it is extremely useful to think of vector fields as first order linear differential operators, via $V^\mu \to V = V^\mu \partial_\mu$. In this case, the Lie bracket $[V, W]$ is simply the ordinary commutator of differential operators,

$$
\begin{aligned}
[V, W] &= [V^\mu \partial_\mu, W^\nu \partial_\nu] \\
&= V^\mu (\partial_\mu W^\nu) \partial_\nu + V^\mu W^\nu \partial_\mu \partial_\nu - W^\nu (\partial_\nu V^\mu) \partial_\mu - W^\nu V^\mu \partial_\nu \partial_\nu \\
&= (V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu) \partial_\mu \\
&= (L_V W)^\mu \partial_\mu = [V, W]^\mu \partial_\mu \ .
\end{aligned}
\tag{6.17}
$$

From this point of view, the Jacobi identity is obvious.

4. Having equipped the space of vector fields with a Lie algebra structure, in fact with the structure of an *infinite-dimensional Lie algebra*, it is fair to ask 'the Lie algebra of what group?'. Well, we have seen above that we can think of vector fields as infinitesimal generators of coordinate transformations. Hence, formally at least, the Lie algebra of vector fields is the Lie algebra of the group of coordinate transformations (passive point of view) or diffeomorphisms (active point of view).

THE LIE DERIVATIVE FOR OTHER TENSOR FIELDS

To extend the definition of the Lie derivative to other tensors, we can proceed in one of two ways. We can either extend the above procedure to other tensor fields by defining

$$L_V T^{...}_{...} := \lim_{\epsilon \to 0} \frac{T^{...}_{...}(y(x)) - T'^{...}_{...}(y(x))}{\epsilon} \ . \tag{6.18}$$

Or we can extend it to other tensors by proceeding as in the case of the covariant derivative, i.e. by demanding the Leibniz rule. In either case, the result can be rewritten in terms of covariant derivatives, The result is that the Lie derivative of a (p, q)-tensor T is, like the covariant derivative, the sum of three kinds of terms: the directional covariant derivative of T along V, p terms with a minus sign, involving the covariant

derivative of V contracted with each of the upper indices, and q terms with a plus sign, involving the convariant derivative of V contracted with each of the lower indices (note that the plus and minus signs are interchanged with respect to the covariant derivative). Thus, e.g., the Lie derivative of a (1,2)-tensor is

$$L_V T^\mu_{\nu\lambda} = V^\rho \nabla_\rho T^\mu_{\nu\lambda} - T^\rho_{\nu\lambda} \nabla_\rho V^\mu + T^\mu_{\rho\lambda} \nabla_\nu V^\rho + T^\mu_{\nu\rho} \nabla_\lambda V^\rho \quad . \tag{6.19}$$

The fact that the Lie derivative provides a representation of the Lie algebra of vector fields by first-order differential operators on the space of (p, q)-tensors is expressed by the identity

$$[L_V, L_W] = L_{[V,W]} \quad . \tag{6.20}$$

<center>THE LIE DERIVATIVE OF THE METRIC AND KILLING VECTORS</center>

The above general formula becomes particularly simple for the metric tensor $g_{\mu\nu}$. The first term is not there (because the metric is covariantly constant), so the Lie derivative is the sum of two terms (with plus signs) involving the covariant derivative of V,

$$L_V g_{\mu\nu} = g_{\lambda\nu} \nabla_\mu V^\lambda + g_{\mu\lambda} \nabla_\nu V^\lambda \quad . \tag{6.21}$$

Lowering the index of V with the metric, this can be written more succinctly as

$$L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \quad . \tag{6.22}$$

We are now ready to return to our discussion of isometries (symmetries of the metric). Evidently, an infinitesimal coordinate transformation is a symmetry of the metric if $L_V g_{\mu\nu} = 0$,

$$V \text{ generates an isometry} \Leftrightarrow \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \quad . \tag{6.23}$$

Vector fields V satisfying this equation are called *Killing vectors* - not because they kill the metric but after the 19th century mathematician W. Killing.

Since they are associated with symmetries of space time, and since symmetries are always of fundamental importance in physics, Killing vectors will play an important role in the following. Our most immediate concern will be with the conserved quantities associated with Killing vectors. We will return to a more detailed discussion of Killing vectors and symmetric space times in the context of Cosmology later on. For now, let us just note that by virtue of (6.20) Killing vectors form a Lie algebra, i.e. if V and W are Killing vectors, then also $[V, W]$ is a Killing vector,

$$L_V g_{\mu\nu} = L_W g_{\mu\nu} = 0 \Rightarrow L_{[V,W]} g_{\mu\nu} = 0 \quad . \tag{6.24}$$

Indeed one has

$$L_{[V,W]} g_{\mu\nu} = L_V L_W g_{\mu\nu} - L_W L_V g_{\mu\nu} = 0 \quad . \tag{6.25}$$

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This algebra is (a subalgebra of) the Lie algebra of the isometry group. For example, the collection of all Killing vectors of the Minkowski metric generate the Lie algebra of the Poincaré group.

Here is a simple example: as mentioned before, in some obvious sense the standard metric on the two-sphere is rotationally invariant. In particular, with our new terminology we would expect the vector field ∂_ϕ, i.e. the vector field with components $V^\phi = 1, V^\theta = 0$ to be Killing. Let us check this. With the metric $d\theta^2 + \sin^2\theta d\phi^2$, the corresponding covector V_μ, obtained by lowering the indices of the vector field V^μ, are

$$V_\theta = 0 \ , \qquad V_\phi = \sin^2\theta \ . \tag{6.26}$$

The Killing condition breaks up into three equations, and we verify

$$
\begin{aligned}
\nabla_\theta V_\theta &= \partial_\theta V_\theta - \Gamma^\mu_{\theta\theta} V_\mu \\
&= -\Gamma^\phi_{\theta\theta} \sin^2\theta = 0 \\
\nabla_\theta V_\phi + \nabla_\phi V_\theta &= \partial_\theta V_\phi - \Gamma^\mu_{\theta\phi} V_\mu + \partial_\phi V_\theta - \Gamma^\mu_{\theta\phi} V_\mu \\
&= 2\sin\theta\cos\theta - 2\cot\theta\sin^2\theta = 0 \\
\nabla_\phi V_\phi &= \partial_\phi V_\phi + \Gamma^\mu_{\phi\phi} V_\mu = 0 \ .
\end{aligned}
\tag{6.27}
$$

KILLING VECTORS AND CONSERVED QUANTITIES

We are used to the fact that symmetries lead to conserved quantities (Noether's theorem). For example, in classical mechanics, the angular momentum of a particle moving in a rotationally symmetric gravitational field is conserved. In the present context, the concept of 'symmetries of a gravitational field' is replaced by 'symmetries of the metric', and we therefore expect conserved charges associated with the presence of Killing vectors. Here are the two most important classes of examples of this phenomenon:

1. Killing Vectors, Geodesics and Conserved Quantities

   Let K^μ be a Killing vector field, and $x^\mu(\tau)$ be a geodesic. Then the quantity $K_\mu \dot{x}^\mu$ is constant along the geodesic. Indeed,

$$
\begin{aligned}
\frac{d}{d\tau}(K_\mu \dot{x}^\mu) &= (\frac{D}{D\tau}K_\mu)\dot{x}^\mu + K_\mu \frac{D}{D\tau}\dot{x}^\mu \\
&= \nabla_\nu K_\mu \dot{x}^\nu \dot{x}^\mu + 0 \\
&= \tfrac{1}{2}(\nabla_\nu K_\mu + \nabla_\mu K_\nu)\dot{x}^\mu \dot{x}^\nu = 0 \ .
\end{aligned}
\tag{6.28}
$$

2. Killing Vectors and Conserved Quantities from the Energy-Momentum Tensor

   Let K^μ be a Killing vector field, and $T^{\mu\nu}$ the covariantly conserved symmetric energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$. Then $J^\mu = T^{\mu\nu}K_\nu$ is a covariantly conserved current. Indeed,

$$
\begin{aligned}
\nabla_\mu J^\mu &= (\nabla_\mu T^{\mu\nu})K_\nu + T^{\mu\nu}\nabla_\mu K_\nu \\
&= 0 + \tfrac{1}{2}T^{\mu\nu}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0 \ .
\end{aligned}
\tag{6.29}
$$

Hence, as we now have a conserved current, we can associate with it a conserved charge in the way discussed above.

7   Curvature I: The Riemann Curvature Tensor

Curvature: Preliminary Remarks

We now come to one of the most important concepts of General Relativity and Riemannian Geometry, that of curvature and how to describe it in tensorial terms. Among other things, this will finally allow us to decide unambiguously if a given metric is just the (flat) Minkowski metric in disguise or the metric of a genuinely curved space. It will also lead us fairly directly to the Einstein equations, i.e. to the field equations for the gravitational field.

Recall that the equations that describe the behaviour of particles and fields in a gravitational field involve the metric and the Christoffel symbols determined by the metric. Thus the equations for the gravitational field should be generally covariant (tensorial) differential equations for the metric.

But at first, here we seem to face a dilemma. How can we write down covariant differential equations for the metric when the covariant derivative of the metric is identically zero? Having come to this point, Einstein himself expressed the opinion that therefore the field equations for gravity could not be generally covariant. He also gave some arguments in favour of this point of view which are obviously flawed and now only of historical interest. What he only realized later is that there are other tensors that can be constructed from (ordinary) derivatives of the metric which are not zero and which can be used to write down covariant differential equations for the metric.

The most important among these are the Riemann curvature tensor and its various contractions. In fact, it is known that these are the only tensors that can be constructed from the metric and its first and second derivatives, and they will therefore play a central role in all that follows.

The Riemann Curvature Tensor from the Commutator of Covariant Derivatives

Technically the most straightforward way of introducing the Riemann curvature tensor is via the commutator of covariant derivatives. As this is not geometrically the most intuitive way of introducing the concept of curvature, we will then, once we have defined it and studied its most important algebraic properties, study to which extent the curvature tensor reflects the geometric properties of space time.

As mentioned before, second covariant derivatives do not commute on (p,q)-tensors unless $p = q = 0$. However, the fact that they *do* commute on scalars has the pleasant consequence that e.g. the commutator of covariant derivatives acting on a vector field V^μ does not involve any derivatives of V^μ. In fact, I will first show, without actually calculating the commutator, that

$$[\nabla_\mu, \nabla_\nu]\phi V^\lambda = \phi[\nabla_\mu, \nabla_\nu]V^\lambda \tag{7.1}$$

for any scalar field ϕ. This implies that $[\nabla_\mu, \nabla_\nu]V^\lambda$ cannot depend on derivatives of V because if it did it would also have to depend on derivatives of ϕ. Hence, the commutator can be expressed purely algebraically in terms of V. As the dependence on V is clearly linear, there must therefore be an object $R^\lambda{}_{\sigma\mu\nu}$ such that

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda{}_{\sigma\mu\nu}V^\sigma \quad . \tag{7.2}$$

This can of course also be verified by a direct calculation, and we will come back to this below. For now let us just note that, since the left hand side of this equation is clearly a tensor for any V, the quotient theorem implies that $R^\lambda{}_{\sigma\mu\nu}$ has to be a tensor. It is the famous *Riemann-Christoffel Curvature Tensor*.

Let us first verify (7.1). We have

$$\nabla_\mu\nabla_\nu\phi V^\lambda = (\nabla_\mu\nabla_\nu\phi)V^\lambda + (\nabla_\nu\phi)(\nabla_\mu V^\lambda) + (\nabla_\mu\phi)(\nabla_\nu V^\lambda) + \phi\nabla_\mu\nabla_\nu V^\lambda \quad . \tag{7.3}$$

Thus, upon taking the commutator the second and third terms drop out and we are left with

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]\phi V^\lambda &= ([\nabla_\mu, \nabla_\nu]\phi)V^\lambda + \phi[\nabla_\mu, \nabla_\nu]V^\lambda \\
&= \phi[\nabla_\mu, \nabla_\nu]V^\lambda \quad ,
\end{aligned} \tag{7.4}$$

which is what we wanted to establish.

By explicitly calculating the commutator, one can confirm the structure displayed in (7.2). This explicit calculation shows that the Riemann tensor (for short) is given by

$$\boxed{R^\lambda{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\lambda{}_{\sigma\nu} - \partial_\nu\Gamma^\lambda{}_{\sigma\mu} + \Gamma^\lambda{}_{\mu\rho}\Gamma^\rho{}_{\nu\sigma} - \Gamma^\lambda{}_{\nu\rho}\Gamma^\rho{}_{\mu\sigma}} \quad . \tag{7.5}$$

Note how useful the quotient theorem is in this case. It would be quite unpleasant to have to verify the tensorial nature of this expression by explicitly checking its behaviour under coordinate transformations.

Note also that this tensor is clearly zero for the Minkowski metric written in Cartesian coordinates. Hence it is also zero for the Minkowski metric written in any other coordinate system. We will prove the converse, that vanishing of the Riemann curvature tensor implies that the metric is equivalent to the Minkowski metric, below.

It is straightforward to extend the above to an action of the commutator $[\nabla_\mu, \nabla_\nu]$ on arbitrary tensors. For covectors we have, since we can raise and lower the indices with the metric with impunity,

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V_\rho &= g_{\rho\lambda}[\nabla_\mu, \nabla_\nu]V^\lambda \\
&= g_{\rho\lambda}R^\lambda{}_{\sigma\mu\nu}V^\sigma \\
&= R_{\rho\sigma\mu\nu}V^\sigma \\
&= R_\rho{}^\sigma{}_{\mu\nu}V_\sigma \ .
\end{aligned}
\tag{7.6}
$$

We will see later that the Riemann tensor is antisymmetric in its first two indices. Hence we can also write

$$
[\nabla_\mu, \nabla_\nu]V_\rho = -R^\sigma{}_{\rho\mu\nu}V_\sigma \ .
\tag{7.7}
$$

The extension to arbitrary (p,q)-tensors now follows the usual pattern, with one Riemann curvature tensor, contracted as for vectors, appearing for each of the p upper indices, and one Riemann curvature tensor, contracted as for covectors, for each of the q lower indices. Thus, e.g. for a (1,1)-tensor $A^\lambda{}_\rho$ one would find

$$
[\nabla_\mu, \nabla_\nu]A^\lambda{}_\rho = R^\lambda{}_{\sigma\mu\nu}A^\sigma{}_\rho - R^\sigma{}_{\rho\mu\nu}A^\lambda{}_\sigma \ .
\tag{7.8}
$$

I will give two other versions of the fundamental formula (7.2) which are occasionally useful and used.

1. Instead of looking at the commutator $[\nabla_\mu, \nabla_\nu]$ of two derivatives in the coordinate directions x^μ and x^ν, we can look at the commutator $[\nabla_X, \nabla_Y]$ of two directional covariant derivatives. Evidently, in calculating this commutator one will pick up new terms involving $\nabla_X Y^\mu - \nabla_Y X^\mu$. Comparing with (6.13), we see that this is just $[X,Y]^\mu$. The correct formula for the curvature tensor in this case is

$$
([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})V^\lambda = R^\lambda{}_{\sigma\mu\nu}X^\mu Y^\nu V^\sigma \ .
\tag{7.9}
$$

   Note that, in this sense, the curvature measures the failure of the covariant derivative to provide a representation of the Lie algebra of vector fields.

2. Secondly, one can consider a net of curves $x^\mu(\sigma, \tau)$ parmetrizing, say, a two-dimensional surface, and look at the commutators of the covariant derivatives along the σ- and τ-curves. The formula one obtains in this case (it can be obtained from (7.9) by noting that X and Y commute in this case) is

$$
\left(\frac{D^2}{D\sigma D\tau} - \frac{D^2}{D\tau D\sigma} \right) V^\lambda = R^\lambda{}_{\sigma\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} V^\sigma \ .
\tag{7.10}
$$

SYMMETRIES AND ALGEBRAIC PROPERTIES OF THE RIEMANN TENSOR

A priori, the Riemann tensor has $256 = 4^4$ components in 4 dimensions. However, because of a large number of symmetries, the actual number of independent components is much smaller.

In general, to read off all the symmetries from the formula (7.5) is difficult. One way to simplify things is to look at the Riemann curvature tensor at the origin x_0 of a Riemann normal coordinate system (or some other inertial coordinate system). In that case, all the first derivatives of the metric disappear and only the first two terms of (7.5) contribute. One finds

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta}(x_0) &= g_{\alpha\lambda}(\partial_\gamma \Gamma^\lambda_{\beta\delta} - \partial_\delta \Gamma^\lambda_{\beta\gamma})(x_0) \\
&= (\partial_\gamma \Gamma_{\alpha\beta\delta} - \partial_\delta \Gamma_{\alpha\beta\gamma})(x_0) \\
&= \tfrac{1}{2}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma})(x_0) \ .
\end{aligned}
\tag{7.11}
$$

In principle, this expression is sufficiently simple to allow one to read off all the symmetries of the Riemann tensor. However, it is more insightful to derive these symmetries in a different way, one which will also make clear *why* the Riemann tensor has these symmetries.

1. $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$

   This is obviously true from the definition or by construction.

2. $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$

   This is a consequence of the fact that the metric is covariantly constant. In fact, we can calculate

   $$
   \begin{aligned}
   0 &= [\nabla_\gamma, \nabla_\delta] g_{\alpha\beta} \\
   &= -R^\lambda_{\alpha\gamma\delta} g_{\lambda\beta} - R^\lambda_{\beta\gamma\delta} g_{\alpha\lambda} \\
   &= -(R_{\beta\alpha\gamma\delta} + R_{\alpha\beta\gamma\delta}) \ .
   \end{aligned}
   \tag{7.12}
   $$

3. $R_{\alpha[\beta\gamma\delta]} = 0$

   This Bianchi identity is a consequence of the fact that there is no torsion. In fact, applying $[\nabla_\gamma, \nabla_\delta]$ to the covector $\nabla_\beta \phi$, ϕ a scalar, one has

   $$
   \nabla_{[\gamma} \nabla_\delta \nabla_{\beta]} \phi = 0 \Rightarrow R^\lambda_{[\beta\gamma\delta]} \nabla_\lambda \phi = 0 \ .
   \tag{7.13}
   $$

   As this has to be true for all scalars ϕ, this implies $R_{\alpha[\beta\gamma\delta]} = 0$ (to see this you could e.g. choose the (locally defined) coordinate functions $\phi(x) = x^\mu$ with $\nabla_\lambda \phi = \delta^\mu_\lambda$).

4. $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$

   This identity, stating that the Riemann tensor is symmetric in its two pairs of indices, is not an independent symmetry but can be deduced from the three other symmetries by some not particularly interesting algebraic manipulations.

We can now count how many independent components the Riemann tensor really has. (1) implies that the second pair of indices can only take $(4 \times 3)/2 = 6$ independent values. (2) implies the same for the first pair of indices. (4) thus says that the Riemann curvature tensor behaves like a symmetric (6×6) matrix and therefore has $(6 \times 7)/2 = 21$ components. (3) then provides one and only one more additional constraint so that the total number of independent components is 20.

Note that this agrees precisely with our previous counting of how many of the second derivatives of the metric cannot be set to zero by a coordinate transformation: the second derivative of the metric has 100 independent components, to be compared with the $4 \times (4 \times 5 \times 6)/(2 \times 3) = 80$ components of the matrix of third derivatives of the coordinates. This also leaves 20 components. We thus see very explicitly that the Riemann curvature tensor contains all the coordinate independent information about the geometry up to second derivatives of the metric. In fact, it can be shown that in a Riemann normal coordinate system one has

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 0 + \tfrac{1}{3} R_{\mu\lambda\sigma\nu}(x_0) x^\lambda x^\sigma + \mathcal{O}(x^3) \ . \tag{7.14}$$

Just for the record, I note here that in general dimension n the Riemann tensor has $n^2(n^2-1)/12$ independent components. This is easy to see for $n = 2$, where this formula predicts one independent component. Indeed, rather obviously the only independent non-vanishing component of the Riemann tensor in this case is R_{1212}.

Finally, a word of warning: there are a large number of sign conventions involved in the definition of the Riemann tensor (and its contractions we will discuss below), so whenever reading a book or article, in particular when you want to use results or equations presented there, make sure what conventions are being used and either adopt those or translate the results into some other convention. As a check: the conventions used here are such that $R_{\phi\theta\phi\theta}$ as well as the curvature scalar (to be introduced below) are *positive* for the standard metric on the two-sphere.

The Ricci Tensor and the Ricci Scalar

The Riemann tensor, as we have seen, is a four-index tensor. For many purposes this is not the most useful object. But we can create new tensors by contractions of the Riemann tensor. Due to the symmetries of the Riemann tensor, there is essentially only one possibility, namely the *Ricci tensor*

$$R_{\mu\nu} := R^\lambda_{\ \mu\lambda\nu} = g^{\lambda\sigma} R_{\sigma\mu\lambda\nu} \ . \tag{7.15}$$

It follows from the symmetries of the Riemann tensor that $R_{\mu\nu}$ is symmetric. Indeed

$$R_{\nu\mu} = g^{\lambda\sigma} R_{\sigma\nu\lambda\mu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} = R^\sigma_{\ \mu\sigma\nu} = R_{\mu\nu} \ . \tag{7.16}$$

Thus, for $n = 4$, the Ricci tensor has 10 independent components, for $n = 3$ it has 6, while for $n = 2$ there is only 1 because there is only one independent component of the Riemann curvature tensor to start off with.

There is one more contraction we can perform, namely on the Ricci tensor itself, to obtain what is called the *Ricci scalar* or *curvature scalar*

$$R := g^{\mu\nu} R_{\mu\nu} \ . \tag{7.17}$$

One might have thought that in four dimensions there is another way of constructing a scalar, by contracting the Riemann tensor with the Levi-Cività tensor, but

$$\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0 \tag{7.18}$$

because of the Bianchi identity.

Note that for $n = 2$ the Riemann curvature tensor has as many independent components as the Ricci scalar, namely one, and that in three dimensions the Ricci tensor has as many components as the Riemann tensor, whereas in four dimensions there are strictly less components of the Ricci tensor than of the Riemann tensor. This has profound implications for the dynamics of gravity in these dimensions. In fact, we will see that it is only in dimensions $n > 3$ that gravity becomes truly dynamical, where empty space can be curved, where gravitational waves can exist etc.

An Example: The Curvature Tensor of the Two-Sphere

To see how all of this can be done in practice, let us work out the example of the two-sphere of unit radius. We already know the Christoffel symbols,

$$\Gamma^\phi_{\phi\theta} = \cot\theta \ , \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ , \tag{7.19}$$

and we know that the Riemann curvature tensor has only one independent component. Let us therefore work out $R^\theta_{\phi\theta\phi}$. From the definition we find

$$R^\theta_{\phi\theta\phi} = \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\alpha} \Gamma^\alpha_{\phi\phi} - \Gamma^\theta_{\phi\alpha} \Gamma^\alpha_{\theta\phi} \ . \tag{7.20}$$

The second and third terms are manifestly zero, and we are left with

$$R^\theta_{\phi\theta\phi} = \partial_\theta(-\sin\theta\cos\theta) + \sin\theta\cos\theta\cot\theta = \sin^2\theta \ . \tag{7.21}$$

Thus we have

$$\begin{aligned}
R^\theta_{\phi\theta\phi} &= R_{\theta\phi\theta\phi} = \sin^2\theta \\
R^\phi_{\theta\phi\theta} &= 1 \ .
\end{aligned} \tag{7.22}$$

Therefore the Ricci tensor $R_{\mu\nu}$ has components

$$
\begin{aligned}
R_{\theta\theta} &= 1 \\
R_{\theta\phi} &= 0 \\
R_{\phi\phi} &= \sin^2\theta \ ,
\end{aligned}
\tag{7.23}
$$

and the Ricci scalar is

$$
\begin{aligned}
R &= g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\
&= 1 + \frac{1}{\sin^2\theta}\sin^2\theta \\
&= 2 \ .
\end{aligned}
\tag{7.24}
$$

In particular, we have here our first concrete example of a space with non-trivial, in fact positive, curvature.

Question: what is the curvature scalar of a sphere of radius a?

Rather than redoing the calculation in that case, let us observe first of all that the Christoffel symbols are invariant under constant rescalings of the metric because they are schematically of the form $g^{-1}\partial g$. Therefore the Riemann curvature tensor, which only involves derivatives and products of Christoffel symbols, is also invariant. Hence the Ricci tensor, which is just a contraction of the Riemann tensor, is also invariant. However, to construct the Ricci scalar, one needs the inverse metric. This introduces an explicit a-dependence and the result is that the curvature scalar of a sphere of radius a is $R = 2/a^2$. In particular, the curvature scalar of a large sphere is smaller than that of a small sphere.

This result could also have been obtained on purely dimensional grounds. The curvature scalar is constructed from second derivatives of the metric. Hence it has length-dimension (-2). Therefore for a sphere of radius a, R has to be proportional to $1/a^2$. Comparing with the known result for $a = 1$ determines $R = 2/a^2$, as before.

Bianchi Identities

So far, we have discussed algebraic properties of the Riemann tensor. But the Riemann tensor also satisfies some differential identities which, in particular in their contracted form, will be of fundamental importance in the following.

The first identity is easy to derive. As a (differential) operator the covariant derivative clearly satisfies the Jacobi identity

$$
[\nabla_{[\mu}, [\nabla_{\nu}, \nabla_{\lambda]}]] = 0
\tag{7.25}
$$

If you do not believe this, just write out the twelve relevant terms explicitly to see that this identity is true:

$$
\begin{aligned}
[\nabla_{[\mu}, [\nabla_\nu, \nabla_{\lambda]}]] \quad \sim \quad & \nabla_\mu \nabla_\nu \nabla_\lambda - \nabla_\mu \nabla_\lambda \nabla_\nu - \nabla_\nu \nabla_\lambda \nabla_\mu + \nabla_\lambda \nabla_\nu \nabla_\mu \\
+ \quad & \nabla_\lambda \nabla_\mu \nabla_\nu - \nabla_\lambda \nabla_\nu \nabla_\mu + \nabla_\nu \nabla_\mu \nabla_\lambda - \nabla_\mu \nabla_\nu \nabla_\lambda \\
+ \quad & \nabla_\nu \nabla_\lambda \nabla_\mu - \nabla_\nu \nabla_\mu \nabla_\lambda - \nabla_\lambda \nabla_\mu \nabla_\nu + \nabla_\mu \nabla_\lambda \nabla_\nu \\
= \quad & 0 \ .
\end{aligned}
\tag{7.26}
$$

Hence, recalling the definition of the curvature tensor in terms of commutators of covariant derivatives, we obtain

$$
\text{Jacobi Identity} \quad \Rightarrow \quad \text{Bianchi identity:} \quad R_{\alpha\beta[\mu\nu;\lambda]} = 0 \ .
\tag{7.27}
$$

Because of the antisymmetry of the Riemann tensor in the last two indices, this can also be written more explicitly as

$$
\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \ .
\tag{7.28}
$$

By contracting this with $g^{\alpha\mu}$ we obtain

$$
\nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} + \nabla_\mu R^\mu{}_{\beta\nu\lambda} = 0 \ .
\tag{7.29}
$$

To also turn the last term into a Ricci tensor we contract once more, with $g^{\beta\lambda}$ to obtain the contracted Bianchi identity

$$
\nabla_\lambda R^\lambda{}_\nu - \nabla_\nu R + \nabla_\mu R^\mu{}_\nu = 0 \ ,
\tag{7.30}
$$

or

$$
\boxed{\nabla^\mu (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) = 0}
\tag{7.31}
$$

The tensor appearing in this equation is the so-called *Einstein tensor* $G_{\mu\nu}$,

$$
G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \ .
\tag{7.32}
$$

It is the unique divergence-free tensor that can be built from the metric and its first and second derivatives (apart from $g_{\mu\nu}$ itself, of course), and this is why it will play *the* central role in the Einstein equations for the gravitational field.

ANOTHER LOOK AT THE PRINCIPLE OF GENERAL COVARIANCE

In the section on the principle of minimal coupling, I mentioned that this algorithm or the principle of general covariance do not necessarily fix the equations uniquely. In other words, there could be more than one generally covariant equation which reduces to a given equation in Minkowski space. Having the curvature tensor at our disposal now, we can construct an example of this kind. Consider a massive particle with spin,

characterized by a spin vector S^μ. We could imagine the possibility that in a gravitia-tional field there is a coupling between the spin and the curvature, so that the particle does not follow a geodesic, but rather obeys an equation of the type

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda + aR^\mu_{\nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda S^\rho \;\; . \tag{7.33}$$

This equation is clearly tensorial (generally covariant) and reduces to the equation for a straight line in Minkowski space, but differs from the geodeisc equation (which has the same properties) for $a \neq 0$. But, since the Riemann tensor is second order in derivatives, a has to be a dimensionful quantity (of length dimension 1) for this equation to make sense. Thus the rationale for usually not considering such additional terms is that they are irrelevant at scales large compared to some characteristic size of the particle, say its Compton wave length.

We will mostly be dealing with weak gravitational fields and other low-energy phenom-ena and under those circumstances the minimal coupling rule can be trusted. However, it is not ruled out that under extreme conditions (very strong or strongly fluctuating gravitational fields) such terms are actually present and relevant.

8 Curvature II: Geometry and Curvature

Intrinsic Geometry, Curvature and Parallel Transport

The Riemann curvature tensor and its relatives, introduced above, measure the intrinsic geometry and curvature of a space or space-time. This means that they can be calculated by making experiments and measurements on the space itself. Such experiments might involve things like checking if the interior angles of a triangle add up to π or not.

An even better method, the subject of this section, is to check the properties of parallel transport. The tell-tale sign (or smoking gun) of the presence of curvature is the fact that parallel transport is path dependent, i.e. that parallel transporting a vector V from a point A to a point B along two different paths will in general produce two different vectors at B. Another way of saying this is that parallel transporting a vector around a closed loop at A will in general produce a new vector at A which differs from the initial vector.

This is easy to see in the case of the two-sphere (see Figure 7). Since all the great circles on a two-sphere are geodesics, in particular the segments N-C, N-E, and E-C in the figure, we know that in order to parallel transport a vector along such a line we just need to make sure that its length and the angle between the vector and the geodesic line are constant. Thus imagine a vector 1 at the north pole N, pointing downwards along the line N-C-S. First parallel transport this along N-C to the point C. There we will obtain the vector 2, pointing downwards along C-S. Alternatively imagine parallel
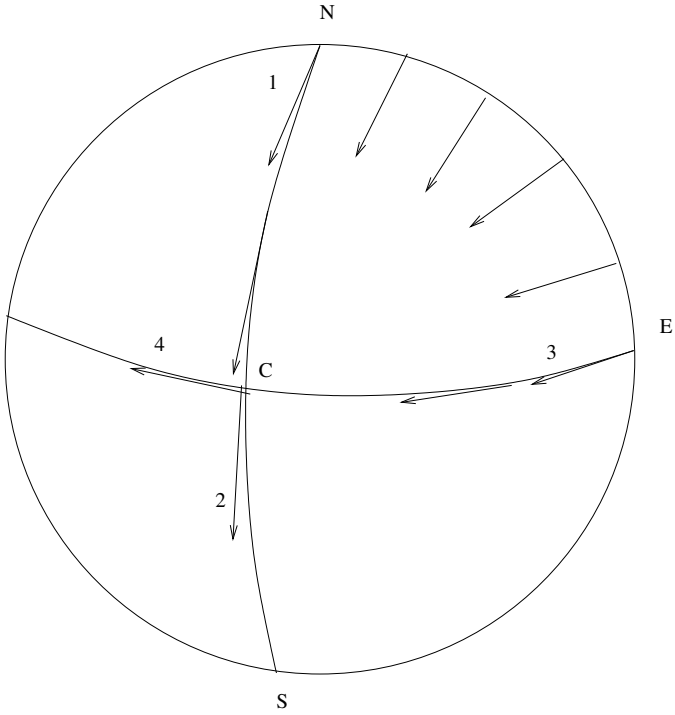
Figure 7: Figure illustrating the path dependence of parallel transport on a curved space: Vector 1 at N can be parallel transported along the geodesic N-S to C, giving rise to Vector 2. Alternatively, it can first be transported along the geodesic N-E (Vector 3) and then along E-C to give the Vector 4. Clearly these two are different. The angle between them reflects the curvature of the two-sphere.

transporting the vector 1 first to the point E. Since the vector has to remain at a constant (right) angle to the line N-E, at the point E parallel transport will produce the vector 3 pointing westwards along E-C. Now parallel transporting this vector along E-C to C will produce the vector 4 at C. This vector clearly differs from the vector 2 that was obtained by parallel transporting along N-C instead of N-E-C.

To illustrate the claim about closed loops above, imagine parallel transporting vector 1 along the closed loop N-E-C-N from N to N. In order to complete this loop, we still have to parallel transport vector 4 back up to N. Clearly this will give a vector, not indicated in the figure, different from (and pointing roughly at a right angle to) the vector 1 we started off with.

This intrinsic geometry and curvature described above should be contrasted with the extrinsic geometry which depends on how the space may be embedded in some larger space.

For example, a cylinder can be obtained by 'rolling up' $\mathbb{R}^2$. It clearly inherits the flat metric from $\mathbb{R}^2$ and if you calculate its curvature tensor you will find that it is zero. Thus, the intrinsic curvature of the cylinder is zero, and the fact that it looks curved to an outside observer is not something that can be detected by somebody living on the cylinder. For example, parallel transport is rather obviously path independent.

As we have no intention of embedding space-time into something higher dimensional, we will only be concerned with intrinsic geometry in the following. However, if you would for example be interested in the properties of space-like hypersurfaces in space-time, then aspects of both intrinsic and extrinsic geometry of that hypersurface would be relevant.

The precise statement regarding the relation between the path dependence of parallel transport and the presence of curvature is the following. If one parallel transports a vector V^μ along a closed infinitesimal loop $x^\mu(\tau)$ with, say, $x(\tau_0) = x(\tau_1) = x_0$, then one has

$$V^\mu(\tau_1) - V^\mu(\tau_0) \sim (\oint x^\alpha dx^\beta) R^\mu_{\lambda\alpha\beta}(x_0) V^\lambda(\tau_0) \ . \tag{8.1}$$

Thus an arbitrary vector V^μ will not change under parallel transport around an arbitrary small loop at x_0 only if the curvature tensor at x_0 is zero. This can of course be extended to finite loops, but the important point is that in order to detect curvature at a given point one only requires parallel transport along infinitesimal loops.

I will not prove the above equation here. I just want to note that intuitively it can be understood directly from the definition of the curvature tensor (7.2). Imagine that the infinitesimal loop is actually a tiny parallelogram made up of the coordinate lines x^1 and x^2. Parallel transport along x^1 is governed by the equation $\nabla_1 V^\mu = 0$, that along x^2 by $\nabla_2 V^\mu = 0$. The fact that parallel transporting first along x^1 and then along x^2 can be different from doing it the other way around is precisely the statement that ∇_1

and ∇_2 do not commute, i.e. that some of the components $R_{\mu\nu12}$ of the curvature tensor are non-zero.

<center>VANISHING RIEMANN TENSOR AND EXISTENCE OF FLAT COORDINATES</center>

We are now finally in a position to prove the converse to the statement that a flat space has vanishing Riemann tensor. Namely, we will see that when the Riemann tensor of a metric vanishes, there are coordinates in which the metric is the standard Minkowski metric.

So let us assume that we are given a metric with vanishing Riemann tensor. Then, by the above, parallel transport is path independent and we can, in particular, extend a vector $V^\mu(x_0)$ to a vector field everywhere in space-time: to define $V^\mu(x_1)$ we choose any path from x_0 to x_1 and use parallel transport along that path. In particular, the vector field V^μ, defined in this way, will be covariantly constant or *parallel*, $\nabla_\mu V^\nu = 0$. We can also do this for four linearly independent vectors V_a^μ at x_0 and obtain four covariantly constant (parallel) vector fields which are linearly independent at every point.

An alternative way of saying or seeing this is the following: The integrability condition for the equation $\nabla_\mu V^\lambda = 0$ is

$$\nabla_\mu V^\lambda = 0 \quad \Rightarrow \quad [\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \sigma\mu\nu}V^\sigma = 0 \ . \tag{8.2}$$

This means that the (4×4) matrices $M(\mu, \nu)$ with coefficients $M(\mu, \nu)^\lambda_{\ \sigma} = R^\lambda_{\ \sigma\mu\nu}$ have a zero eigenvalue. If this integrability condition is satisfied, a solution to $\nabla_\mu V^\lambda$ can be found. If one wants four linearly independent parallel vector fields, then the matrices $M(\mu, \nu)$ must have four zero eigenvalues, i.e. they are zero and therefore $R^\lambda_{\ \sigma\mu\nu} = 0$. If this condition is satisfied, all the integrability conditions are satisfied and there will be four linearly independent covariantly constant vector fields - the same conclusion as above.

We will now use this result in the proof, but for covectors instead of vectors. Clearly this makes no difference: if V^μ is a parallel vector field, then $g_{\mu\nu}V^\nu$ is a parallel covector field.

Fix some point x_0. At x_0, there will be an invertible matrix e_μ^a such that

$$g^{\mu\nu}(x_0)e_\mu^a e_\nu^b = \eta^{ab} \ . \tag{8.3}$$

Now we solve the equations

$$\nabla_\nu E_\mu^a = 0 \Leftrightarrow \partial_\nu E_\mu^a = \Gamma^\lambda_{\mu\nu}E_\lambda^a \tag{8.4}$$

with the initial condition $E_\mu^a(x_0) = e_\mu^a$. This gives rise to four linearly independent parallel covectors E_μ^a.

<center>66</center>

Now it follows from (8.4) that

$$\partial_\mu E_\nu^a = \partial_\nu E_\mu^a \ . \tag{8.5}$$

Therefore locally there are four scalars ξ^a such that

$$E_\mu^a = \frac{\partial \xi^a}{\partial x^\mu} \ . \tag{8.6}$$

These are already the flat coordinates we have been looking for. To see this, consider the expression $g^{\mu\nu} E_\mu^a E_\nu^b$. This is clearly constant because the metric and the E_μ^a are covariantly constant,

$$\partial_\lambda (g^{\mu\nu} E_\mu^a E_\nu^b) = \nabla_\lambda (g^{\mu\nu} E_\mu^a E_\nu^b) = 0 \ . \tag{8.7}$$

But at x_0, this is just the flat metric and thus

$$(g^{\mu\nu} E_\mu^a E_\nu^b)(x) = (g^{\mu\nu} E_\mu^a E_\nu^b)(x_0) = \eta^{ab} \ . \tag{8.8}$$

Summing this up, we have seen that, starting from the assumption that the Riemann curvature tensor of a metric $g_{\mu\nu}$ is zero, we have proven the existence of coordinates ξ^a in which the metric takes the Minkowski form,

$$g_{\mu\nu} = \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} \eta_{ab} \ . \tag{8.9}$$

THE GEODESIC DEVIATION EQUATION

In a certain sense the main effect of curvature (or gravity) is that initially parallel trajectories of freely falling non-interacting particles (dust, pebbles,...) do not remain parallel, i.e. that gravity has the tendency to focus (or defocus) matter. This statement find its mathematically precise formulation in the geodesic deviation equation.

Let us, as we will need this later anyway, recall first the situation in the Newtonian theory. One particle moving under the influence of a gravitational field is governed by the equation

$$\frac{d^2}{dt^2} x^i = -\partial^i \phi(x) \ , \tag{8.10}$$

where ϕ is the potential. Now consider a family of particles, or just two nearby particles, one at $x^i(t)$ and the other at $x^i(t) + \delta x^i(t)$. The other particle will of course obey the equation

$$\frac{d^2}{dt^2} (x^i + \delta x^i) = -\partial^i \phi(x + \delta x) \ . \tag{8.11}$$

From these two equations one can deduce an equation for δx itself, namely

$$\frac{d^2}{dt^2} \delta x^i = -\partial^i \partial_j \phi(x) \delta x^j \ . \tag{8.12}$$

It is the counterpart of this equation that we will be seeking in the context of General Relativity. The starting point is of course the geodesic equation for x^μ and for its nearby partner $x^\mu + \delta x^\mu$,

$$\frac{d^2}{d\tau^2} x^\mu + \Gamma^\mu_{\nu\lambda}(x) \frac{d}{d\tau} x^\nu \frac{d}{d\tau} x^\lambda = 0 \ , \tag{8.13}$$

and

$$\frac{d^2}{d\tau^2}(x^\mu + \delta x^\mu) + \Gamma^\mu{}_{\nu\lambda}(x + \delta x)\frac{d}{d\tau}(x^\nu + \delta x^\nu)\frac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0 \ . \tag{8.14}$$

As above, from these one can deduce an equation for δx, namely

$$\frac{d^2}{d\tau^2}\delta x^\mu + 2\Gamma^\mu{}_{\nu\lambda}(x)\frac{d}{d\tau}x^\nu\frac{d}{d\tau}\delta x^\lambda + \partial_\rho\Gamma^\mu{}_{\nu\lambda}(x)\delta x^\rho\frac{d}{d\tau}x^\nu\frac{d}{d\tau}x^\lambda = 0 \ . \tag{8.15}$$

Now this does not look particularly covariant. Thus instead of in terms of $d/d\tau$ we would like to rewrite this in terms of covariant operator $D/D\tau$, with

$$\frac{D}{D\tau}\delta x^\mu = \frac{d}{d\tau}\delta x^\mu + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\delta x^\lambda \ . \tag{8.16}$$

Calculating $(D/D\tau)^2\delta x^\mu$, replacing $\ddot{x}^\mu$ appearing in that expression by $-\Gamma^\mu{}_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda$ (because x^μ satisfies the geodesic equation) and using (8.15), one finds the nice covariant *geodesic deviation equation*

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu{}_{\nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda\delta x^\rho \ . \tag{8.17}$$

Note that for flat space(-time), this equation reduces to

$$\frac{d^2}{d\tau^2}\delta x^\mu = 0 \ , \tag{8.18}$$

which has the solution

$$\delta x^\mu = A^\mu\tau + B^\mu \ . \tag{8.19}$$

In particular, one recovers Euclid's parallel axiom that two straight lines intersect at most once and that when they are initially parallel they never intersect. This shows very clearly that intrinsic curvature leads to non-Euclidean geometry in which e.g. the parallel axiom is not necessarily satisifed.

9   Towards the Einstein Equations

Heuristics

We expect the gravitational field equations to be non-linear second order partial differential equations for the metric. If we knew more about the weak field equations of gravity (which should thus be valid near the origin of an inertial coordinate system) we could use the Einstein equivalence principle (or the principle of general covariance) to deduce the equations for strong fields.

However, we do not know a lot about gravity beyond the Newtonian limit of weak time-independent fields and low velocities, simply because gravity is so 'weak'. Hence, we cannot find the gravitational field equations in a completely systematic way and some guesswork will be required.

Nevertheless we will see that with some very few natural assumptions we will arrive at an essentially unique set of equations. Further theoretical (and aesthetical) confirmation for these equations will then come from the fact that they turn out to be the Euler-Lagrange equations of the absolutely simplest action principle for the metric imaginable.

To see at least roughly what we expect the gravitational field equations to look like, we begin with an analogy, a comparison of the geodesic deviation equations in Newton's theory and in General Relativity.

Recall that in Newton's theory we have

$$
\begin{aligned}
\frac{d^2}{dt^2}\delta x^i &= -K^i{}_j \delta x^j \\
K^i{}_j &= \partial^i \partial_j \phi \ ,
\end{aligned}
\tag{9.1}
$$

whereas in General Relativity we have

$$
\begin{aligned}
\frac{D^2}{Dt^2}\delta x^\mu &= -K^\mu{}_\nu \delta x^\nu \\
K^\mu{}_\nu &= R^\mu{}_{\lambda\nu\rho}\dot{x}^\lambda \dot{x}^\rho \ .
\end{aligned}
\tag{9.2}
$$

Now Newton's field equation is

$$
\text{Tr}\, K \equiv \Delta\phi = 4\pi G\rho \ ,
\tag{9.3}
$$

while in General Relativity we have

$$
\text{Tr}\, K = R_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \ .
\tag{9.4}
$$

This suggests that somehow in the gravitational field equations of General Relativity, $\Delta\phi$ should be replaced by the Ricci tensor $R_{\mu\nu}$.

Note that, at least roughly, the tensorial structure of this identification is compatible with the relation between ϕ and g_{00} in the Newtonian limit, the relation between ρ and the 0-0 component T_{00} of the energy momentum tensor, and the fact that for small velocities $R_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \sim R_{00}$.

Indeed, recall that the weak static field produced by a non-relativistic mass density ρ is

$$
g_{00} = -(1 + 2\phi) \ .
\tag{9.5}
$$

Moreover, for non-relativistic matter we have

$$
T_{00} = \rho \ ,
\tag{9.6}
$$

so that the Newtonian field equation can also be written as

$$
\Delta g_{00} = -8\pi G T_{00} \ .
\tag{9.7}
$$

This suggests that the weak-field equations for a general energy-momentum tensor take the form

$$E_{\mu\nu} = -8\pi G T_{\mu\nu} \ , \tag{9.8}$$

where $E_{\mu\nu}$ is constructed from the metric and its first and second derivatives. But by the Einstein equivalence principle, if this equation is valid for weak fields (i.e. near the origin of an inertial coordinate system) then also the equations which govern gravitational fields of arbitrary strength must be of this form, with $E_{\mu\nu}$ a tensor constructed from the metric and its first and second derivatives. We will now turn to a somewhat more precise argument along these lines which will enable us to determine $E_{\mu\nu}$.

A More Systematic Approach

Let us take stock of what we know about $E_{\mu\nu}$.

1. $E_{\mu\nu}$ is a tensor

2. $E_{\mu\nu}$ has the dimensions of a second derivative. If we assume that no new dimensionful constants enter in $E_{\mu\nu}$ then it has to be a linear combination of terms which are either second derivatives of the metric or quadratic in the first derivatives of the metric. (Later on, we will see that there is the possibility of a zero derivative term, but this requires a new dimensionful constant, the *cosmological constant* Λ. Higher derivative terms could in principle appear but would only be relevant at very high energies.)

3. $E_{\mu\nu}$ is symmetric since $T_{\mu\nu}$ is symmetric.

4. Since $T_{\mu\nu}$ is covariantly conserved, the same has to be true for $E_{\mu\nu}$,

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \nabla_\mu E^{\mu\nu} = 0 \ . \tag{9.9}$$

5. Finally, for a weak stationary gravitational field and non-relativistic matter we should find

$$E_{00} = \Delta g_{00} \ . \tag{9.10}$$

Now it turns out that these conditions (1)-(5) determine $E_{\mu\nu}$ *uniquely*! First of all, (1) and (2) tell us that $E_{\mu\nu}$ has to be a linear combination

$$E_{\mu\nu} = a R_{\mu\nu} + b g_{\mu\nu} R \ , \tag{9.11}$$

where $R_{\mu\nu}$ is the Ricci tensor and R the Ricci scalar. Then condition (3) is automatically satisfied.

To implement (4), we recall the contracted Bianchi identity (7.30,7.31),

$$2\nabla^\mu R_{\mu\nu} = \nabla_\nu R \ . \tag{9.12}$$

Hence

$$\nabla^\mu E_{\mu\nu} = (\frac{a}{2} + b)\nabla_\nu R \ . \tag{9.13}$$

We therefore have to require either $\nabla_\nu R = 0$ or $a = -2b$. That the first possibility is ruled out (inconsistent) can be seen by taking the trace of (9.8),

$$E^\mu_\mu = (a + 4b)R = -8\pi G T^\mu_\mu \ . \tag{9.14}$$

Thus, R is proportional to T^μ_μ and since this quantity need certainly not be constant for a general matter configuration, we are led to the conclusion that $a = -2b$. Thus we find

$$E_{\mu\nu} = a(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) = aG_{\mu\nu} \ , \tag{9.15}$$

where $G_{\mu\nu}$ is the Einstein tensor (7.32). We can now use the condition (5) to determine the constant a.

<center>THE WEAK-FIELD LIMIT</center>

By the above considerations we have determined the field equations to be of the form

$$aG_{\mu\nu} = -8\pi G T_{\mu\nu} \ , \tag{9.16}$$

with a some, as yet undetermined, constant. We will now consider the weak-field limit of this equation. We need to find that G_{00} is proportional to Δg_{00} and we can then use the condition (5) to fix the value of a. The following manipulations are somewhat analogous to those we performed when considering the Newtonian limit of the geodesic equation. The main difference is that now we are dealing with second derivatives of the metric rather than with just its first derivatives entering in the geodesic equation.

First of all, for a non-relativistic system we have $|T_{ij}| \ll T_{00}$ and hence $|G_{ij}| \ll |G_{00}|$. Therefore we conclude

$$|T_{ij}| \ll T_{00} \Rightarrow R_{ij} \sim \tfrac{1}{2}g_{ij}R \ . \tag{9.17}$$

Next, for a weak field we have $g_{\mu\nu} \sim \eta_{\mu\nu}$ and, in particular,

$$R \sim \eta^{\mu\nu}R_{\mu\nu} = R^k_k - R_{00} \ , \tag{9.18}$$

which, together with (9.17), translates into

$$R \sim \tfrac{3}{2}R - R_{00} \ . \tag{9.19}$$

or

$$R \sim 2R_{00} \ . \tag{9.20}$$

In the weak field limit, R_{00} in turn is given by

$$R_{00} = R^k_{0k0} = \eta^{ik}R_{i0k0} \ . \tag{9.21}$$

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Moreover, in this limit only the linear (second derivative) part of $R_{\mu\nu\lambda\sigma}$ will contribute, not the terms quadratic in first derivatives. Thus we can use the expression (7.11) for the curvature tensor. Additionally, in the static case we can ignore all time derivatives. Then only one term (the third) of (7.11) contributes and we find

$$R_{i0k0} = -\tfrac{1}{2}g_{00,ik} \quad , \tag{9.22}$$

and therefore

$$R_{00} = -\tfrac{1}{2}\Delta g_{00} \quad . \tag{9.23}$$

Thus, putting everything together, we get

$$
\begin{aligned}
E_{00} = aG_{00} &= a(R_{00} - \tfrac{1}{2}g_{00}R) \\
&= a(R_{00} - \tfrac{1}{2}\eta_{00}R) \\
&= a(R_{00} + \tfrac{1}{2}R) \\
&= a(R_{00} + R_{00}) \\
&= -a\Delta g_{00} \quad .
\end{aligned}
\tag{9.24}
$$

Thus we obtain the correct functional form of E_{00} and comparison with condition (5) determines $a = -1$ and therefore $E_{\mu\nu} = -G_{\mu\nu}$.

<center>THE EINSTEIN EQUATIONS</center>

We have finally arrived at the Einstein equations for the gravitational field (metric) of a matter-energy configuration described by the energy-momentum tensor $T_{\mu\nu}$. It is

$$\boxed{\boxed{R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}}} \tag{9.25}$$

In cgs units, the factor $8\pi G$ should be replaced by

$$\kappa = \frac{8\pi G}{c^2} = 1,865 \times 10^{-27} \text{ g}^{-1} \text{ cm} \quad . \tag{9.26}$$

Another common way of writing the Einstein equations is obtained by taking the trace of (9.25), which yields

$$R - 2R = 8\pi G T^{\mu}_{\mu} \quad , \tag{9.27}$$

and substituting this back into (9.25) to obtain

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}) \quad . \tag{9.28}$$

In particular, for the vacuum, $T_{\mu\nu} = 0$, the Einstein equations are simply

$$R_{\mu\nu} = 0 \quad . \tag{9.29}$$

A space-time metric satisfying this equation is, for obvious reasons, said to be Ricci-flat. And I should probably not have said 'simply' in the above because even the vacuum Einstein equations still constitute a complicated set of non-linear coupled partial differential equations whose general solution is not, and probably will never be, known. Usually, one makes some assumptions, in particular regarding the symmetries of the metric, which simplify the equations to the extent that they can be analyzed explicitly, either analytically, or at least qualitatively or numerically.

As we saw before, in two and three dimensions, vanishing of the Ricci tensor implies the vanishing of the Riemann tensor. Thus in these cases, the space-times are necessarily flat away from where there is matter, i.e. at points at which $T_{\mu\nu}(x) = 0$. Thus there are no gravitational fields and no gravitational waves.

In four dimensions, however, the situation is completely different. As we saw, the Ricci tensor has 10 independent components whereas the Riemann tensor has 20. Thus there are 10 components of the Riemann tensor which can curve the vacuum, as e.g. in the field around the sun, and a lot of interesting physics is already contained in the vacuum Einstein equations.

<center>SIGNIFICANCE OF THE BIANCHI IDENTITIES</center>

Because the Ricci tensor is symmetric, the Einstein equations consitute a set of ten algebraically independent second order differential equations for the metric $g_{\mu\nu}$. At first, this looks exactly right as a set of equations for the ten components of the metric.

But at second sight, this cannot be right. After all, the Einstein equations are generally covariant, so that they can at best determine the metric up to coordinate transformations. Therefore we should only expect six independent generally covariant equations for the metric. Here we should recall the contracted Bianchi identities. They tell us that

$$\nabla^\mu G_{\mu\nu} = 0 \ , \tag{9.30}$$

and hence, even though the ten Einstein equations are algebraically independent, there are four differential relations among them, so this is just right.

It is no coincidence, by the way, that the Bianchi identities come to the rescue of general covariance. We will see later that the Bianchi identities can in fact be understood as a consequence of the general covariance of the Einstein equations (and of the corresopnding action principle).

<center>* COMMENTS ON THE INITIAL VALUE PROBLEM AND THE CANONICAL FORMALISM</center>

The general covariance of the Einstein equations is reflected in the fact that only six of the ten equations are truly dynamical equations, namely (for the vacuum equations for

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simplicity)

$$G_{ij} = 0 \ , \tag{9.31}$$

where $i, j = 1, 2, 3$. The other four, namely

$$G_{\mu 0} = 0 \ , \tag{9.32}$$

are constraints that have to be satisfied by the initial data g_{ij} and, say, dg_{ij}/dt on some initial space-like hypersurface (Cauchy surface).

These constraints are analogues of the Gauss law constraint of Maxwell theory (which is a consequence of the $U(1)$ gauge invariance of the theory), but significantly more complicated. Over the years, a lot of effort has gone into developing a formalism and framework for the initial value and canonical (phase space) description of General Relativity. The most well known and useful of these is the so-called ADM (Arnowitt, Deser, Misner) formalism.

The canonical formalism has been developed in particular with an eye towards canonical quantization of gravity. Most of these approaches have not met with much success beyond certain toy-models (so-called mini-superspace models), partially because of technical problems with implementing the constraints as operator constraints in the quantum theory, but more fundamentally because of the non-renormalizability of perturbative quantum gravity. Recently, a new canonical formalism for gravity has been developed by Ashtekar and collaborators. This new formalism is much closer to that of non-Abelian gauge theories than the ADM formalism. In particular, the constraints in these new variables simplify quite drastically, and a lot of work has gone into developing a non-perturbative approach to quantum gravity on the basis of the Ashtekar variables. At present, this approach appears to be the only promising alternative to string theory as a way towards a quantum theory of gravity.

THE COSMOLOGICAL CONSTANT

As mentioned before, there is one more term that can be added to the Einstein equations provided that one relaxes conditions (2) that only terms quadratic in derivatives should appear. This term takes the form $\Lambda g_{\mu\nu}$ and is conserved provided that Λ is a constant, the *cosmological constant*. The Einstein equations with a cosmological constant read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{9.33}$$

To be compatible with condition (5) ((1), (3) and (4) are obviously satisfied), Λ has to be quite small (and observationally it is very small indeed).

Λ plays the role of a vacuum energy density, as can be seen by writing the vacuum Einstein equations as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} \ . \tag{9.34}$$

Comparing this with the energy-momentum tensor of, say, a perfect fluid (see the section on Cosmology),

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \quad , \tag{9.35}$$

we see that Λ corresponds to the energy-density and pressure values

$$\rho = -p = \frac{\Lambda}{8\pi G} \quad . \tag{9.36}$$

The cosmological constant was originally introduced by Einstein because he was unable to find static cosmological solutions without it. After Hubble's discovery of the expansion of the universe, a static universe fell out of fashion, the cosmological constant was no longer required and Einstein rejected it (supposedly calling the introduction of Λ in the first place his biggest blunder because he could have *predicted* the expansion of the universe if he had simply believed in his equations without the cosmological constant).

However, things are not as simple as that. In fact, one of the biggest puzzles in theoretical physics today is why the cosmological constant is so small. According to standard quantum field theory lore, the vacuum energy density should be many many orders of magnitude larger than astrophysical observations allow. Now usually in quantum field theory one does not worry too much about the vacuum energy as one can normal-order it away. However, as we know, gravity is unlike any other theory in that not only energy-differences but absolute energies matter (and cannot just be dropped). The question why the observed cosmological constant is so small (it may be exactly zero, but recent astrophysical observations appear to favour a tiny non-zero value) is known as the *Cosmological Constant Problem*. We will consider the possibility that $\Lambda \neq 0$ only in the section on Cosmology (in all other applications, Λ can indeed be neglected).

The Weyl Tensor

The Einstein equations

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{9.37}$$

can, taken at face value, be regarded as ten algebraic equations for certain traces of the Riemann tensor $R_{\mu\nu\rho\sigma}$. But $R_{\mu\nu\rho\sigma}$ has, as we know, twenty independent components, so how are the other ten determined? The obvious answer, already given above, is of course that we solve the Einstein equations for the metric $g_{\mu\nu}$ and then calculate the Riemann curvature tensor of that metric.

However, this answer leaves something to be desired because it does not really provide an explanation of how the information about these other components is encoded in the Einstein equations. It is interesting to understand this because it is precisely these components of the Riemann tensor wich represent the effects of gravity in vacuum, i.e. where $T_{\mu\nu} = 0$, like tidal forces and gravitational waves.

The more insightful answer is that the information is encoded in the Bianchi identities which serve as propagation equations for the trace-free parts of the Riemann tensor away from the regions where $T_{\mu\nu} \neq 0$.

Let us see how this works. First of all, we need to decompose the Riemann tensor into its trace parts $R_{\mu\nu}$ and R (determined directly by the Einstein equations) and its traceless part $C_{\mu\nu\rho\sigma}$, the *Weyl tensor*.

In any $n \geq 4$ dimensions, the Weyl tensor is defined by

$$
\begin{aligned}
C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} \\
&- \frac{1}{n-2}(g_{\mu\rho}R_{\nu\sigma} + R_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}R_{\mu\sigma} - R_{\nu\rho}g_{\mu\sigma}) \\
&+ \frac{1}{(n-1)(n-2)}R(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \ .
\end{aligned}
\tag{9.38}
$$

This definition is such that $C_{\mu\nu\rho\sigma}$ has all the symmetries of the Riemann tensor (this is manifest) and that all of its traces are zero, i.e.

$$
C^{\mu}_{\ \nu\mu\sigma} = 0 \ .
\tag{9.39}
$$

In the vacuum, $R_{\mu\nu} = 0$, and therefore

$$
T_{\mu\nu}(x) = 0 \ \Rightarrow \ R_{\mu\nu\rho\sigma}(x) = C_{\mu\nu\rho\sigma}(x) \ ,
\tag{9.40}
$$

and, as anticipated, the Weyl tensor encodes the information about the gravitational field in vacuum. The question thus is how $C_{\mu\nu\rho\sigma}$ is determined everywhere in space-time by an energy-momentum tensor which may be localized in some finite region of space-time.

Contracting the Bianchi identity, which we write as

$$
\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0 \ ,
\tag{9.41}
$$

over λ and ρ and making use of the symmetries of the Riemann tensor, one obtains

$$
\nabla^{\lambda}R_{\lambda\sigma\mu\nu} = \nabla_{\mu}R_{\nu\sigma} - \nabla_{\nu}R_{\mu\sigma} \ .
\tag{9.42}
$$

Expressing the Riemann tensor in terms of its contractions and the Weyl tensor, and using the Einstein equations to replace the Ricci tensor and Ricci scalar by the energy-momentum tensor, one now obtains a propagation equation for the Weyl tensor of the form

$$
\nabla^{\mu}C_{\mu\nu\rho\sigma} = J_{\nu\rho\sigma} \ ,
\tag{9.43}
$$

where $J_{\nu\rho\sigma}$ depends only on the energy-momentum tensor and its derivatives. Determining $J_{\nu\rho\sigma}$ in this way is straightforward and one finds

$$
J_{\nu\rho\sigma} = \kappa \frac{n-3}{n-2} \left[\nabla_{\rho}T_{\nu\sigma} - \nabla_{\sigma}T_{\nu\rho} - \frac{1}{n-1} \left[\nabla_{\rho}T^{\lambda}_{\lambda}g_{\nu\sigma} \nabla_{\sigma}T^{\lambda}_{\lambda}g_{\nu\rho} \right] \right] \ .
\tag{9.44}
$$

The equation (9.43) is reminscent of the Maxwell equation

$$\nabla^\mu F_{\mu\nu} = J_\nu \;\;,\tag{9.45}$$

and this is the starting point for a very fruitful analogy between the two subjects. Indeed it turns out that in many other respects as well $C_{\mu\nu\rho\sigma}$ behaves very much like an electro-magnetic field: one can define electric and magnetic components E and B, these satisfy $|E| = |B|$ for a gravitational wave, etc.

Finally, the Weyl tensor is also useful in other contexts as it is *conformally invariant*, i.e. $C^\mu_{\;\nu\rho\sigma}$ is invariant under conformal rescalings of the metric

$$g_{\mu\nu}(x) \to \mathrm{e}^{f(x)} g_{\mu\nu}(x) \;\;.\tag{9.46}$$

In particular, the Weyl tensor is zero if the metric is *conformally flat*, i.e. related by a conformal transformation to the flat metric, and conversely vanishing of the Weyl tensor is also a sufficient condition for a metric to be conformal to the flat metric.

10 THE EINSTEIN EQUATIONS FROM A VARIATIONAL PRINCIPLE

THE EINSTEIN-HILBERT ACTION

To increase our confidence that the Einstein equations we have derived above are in fact reasonable and almost certainly correct, we can adopt a more modern point of view. We can ask if the Einstein equations follow from an action principle or, alternatively, what would be a natural action principle for the metric.

After all, for example in the construction of the Standard Model, one also does not start with the equations of motion but one writes down the simplest possible Lagrangian with the desired field content and symmetries.

We will start with the gravitational part, i.e. the Einstein tensor $G_{\mu\nu}$ of the Einstein equations, and deal with the matter part, the energy-momentum tensor $T_{\mu\nu}$, later.

By general covariance, an action for the metric $g_{\mu\nu}$ will have to take the form

$$S = \int \sqrt{g}\,d^4x \; \Phi(g_{\mu\nu}) \;\;,\tag{10.1}$$

where Φ is a scalar constructed from the metric. So what is Φ going to be? Clearly, the simplest choice is the Ricci scalar R, and this is also the unique choice if one is looking for a scalar constructed from not higher than second derivatives of the metric. Therefore we postulate the beautifully simple and elegant Lagrangian

$$\boxed{S_{EH} = \int \sqrt{g}\,d^4x \; R}\tag{10.2}$$

known as the Einstein-Hilbert Lagrangian. It was presented by Hilbert practically on the same day that Einstein presented his final form (9.25) of the gravitational field equations. Discussions regarding who did what first and who deserves credit for what have been a favourite occupation of historians of science ever since. But Hilbert's work would certainly not have been possible without Einstein's realization that gravity should be regarded not as a force but as a property of space-time and that Riemannian geometry provides the correct framework for embodying the equivalence principle.

We will now prove that the Euler-Lagrange equations following from the Einstein-Hilbert Lagrangian indeed give rise to the Einstein tensor and the vacuum Einstein equations. It is truly remarkable, that such a simple Lagrangian is capable of explaining practically all known gravitational, astrophysical and cosmological phenomena (contrast this with the complexity of the Lagrangian of the Standard Model or any of its generalizations).

Since the Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$, it is simpler to consider variations $\delta g^{\mu\nu}$ of the inverse metric instead of $\delta g_{\mu\nu}$. Thus, as a first step we write

$$
\begin{aligned}
\delta S_{EH} &= \delta \int \sqrt{g} d^4 x \ g^{\mu\nu} R_{\mu\nu} \\
&= \int d^4 x \ (\delta \sqrt{g} g^{\mu\nu} R_{\mu\nu} + \sqrt{g} \delta g^{\mu\nu} R_{\mu\nu} + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu}) \ .
\end{aligned}
\tag{10.3}
$$

Now we recall (from the assignments) that

$$
\delta g^{1/2} = \tfrac{1}{2} g^{1/2} g^{\lambda\rho} \delta g_{\lambda\rho} = -\tfrac{1}{2} g^{1/2} g_{\lambda\rho} \delta g^{\lambda\rho} \ .
\tag{10.4}
$$

Hence,

$$
\begin{aligned}
\delta S_{EH} &= \int \sqrt{g} d^4 x \ [(-\tfrac{1}{2} g_{\mu\nu} R + R_{\mu\nu}) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}] \\
&= \int \sqrt{g} d^4 x \ (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} + \int \sqrt{g} d^4 x \ g^{\mu\nu} \delta R_{\mu\nu} \ .
\end{aligned}
\tag{10.5}
$$

The first term all by itself would already give the Einstein tensor. Thus we need to show that the second term is identically zero. I do not know of any particularly elegant argument to establish this, so this will require a little bit of work, but it is not difficult.

First of all, we need the explicit expression for the Ricci tensor in terms of the Christoffel symbols, whch can be obtained by contraction of (7.5),

$$
R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\nu\mu} - \Gamma^\lambda_{\nu\rho} \Gamma^\rho_{\lambda\mu} \ .
\tag{10.6}
$$

Now we need to calculate the variation of $R_{\mu\nu}$. We will not require the explicit expression in terms of the variations of the metric, but only in terms of the variations $\delta\Gamma^\mu_{\nu\lambda}$ induced by the variations of the metric. This simplifies things considerably. Obviously, $\delta R_{\mu\nu}$ will then be a sum of six terms,

$$
\delta R_{\mu\nu} = \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\nu\mu} + \Gamma^\lambda_{\lambda\rho} \delta\Gamma^\rho_{\nu\mu} - \delta\Gamma^\lambda_{\nu\rho} \Gamma^\rho_{\lambda\mu} - \Gamma^\lambda_{\nu\rho} \delta\Gamma^\rho_{\lambda\mu} \ .
\tag{10.7}
$$

Now the crucial observation is that $\delta\Gamma^{\mu}_{\nu\lambda}$ is a tensor. This follows from the arguments given at the end of section 4, under the heading 'Generalizations', but I will repeat it here in the present context. Of course, we know that the Christoffel symbols themselves are not tensors, because of the inhomogeneous (second derivative) term appearing in the transformation rule under coordinate transformations. But this term is independent of the metric. Thus the metric variation of the Christoffel symbols indeed transforms as a tensor, and it turns out that $\delta R_{\mu\nu}$ can be written rather compactly in terms of covariant derivatives of $\delta\Gamma^{\mu}_{\nu\lambda}$, namely as

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\lambda\mu} \ . \tag{10.8}$$

As a first check on this, note that the first term on the right hand side is manifestly symmetric and that the second term is also symmetric because of (4.31) and (5.18). To establish (10.8), one simply has to use the definition of the covariant derivative. The first term is

$$\nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} = \partial_\lambda \delta\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\lambda\rho} \delta\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda} \delta\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\rho}_{\nu\lambda} \delta\Gamma^{\lambda}_{\rho\mu} \ , \tag{10.9}$$

which takes care of the first, fourth, fifth and sixth terms of (10.7). The remaining terms are

$$-\partial_\nu \delta\Gamma^{\lambda}_{\mu\lambda} + \delta\Gamma^{\lambda}_{\lambda\rho} \Gamma^{\rho}_{\nu\mu} = -\nabla_\nu \Gamma^{\lambda}_{\mu\lambda} \ , \tag{10.10}$$

which establishes (10.8). Now what we really need is $g^{\mu\nu} \delta R_{\mu\nu}$,

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\lambda (g^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_\nu (g^{\mu\nu} \delta\Gamma^{\lambda}_{\lambda\mu}) \ . \tag{10.11}$$

Since both of these terms are covariant divergences of vector fields,

$$
\begin{aligned}
g^{\mu\nu} \delta R_{\mu\nu} &= \nabla_\lambda J^\lambda \\
J^\lambda &= g^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda} \delta\Gamma^{\nu}_{\nu\mu} \ ,
\end{aligned}
\tag{10.12}
$$

we can use Gauss' theorem to conclude that

$$\int \sqrt{g} d^4x \ g^{\mu\nu} \delta R_{\mu\nu} = \int \sqrt{g} d^4x \ \nabla_\lambda J^\lambda = 0 \ , \tag{10.13}$$

(since J^μ is constructed from the variations of the metric which, by assumption, vanish at infinity) as we wanted to show.

To sum this up, we have established that the variation of the Einstein-Hilbert action gives the gravitational part (left hand side) of the Einstein equations,

$$\delta \int \sqrt{g} d^4x \ R = \int \sqrt{g} d^4x \ (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \ . \tag{10.14}$$

We can also write this as

$$\frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S_{EH} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \ . \tag{10.15}$$

If one wants to include the cosmological constant Λ, then the action gets modified to

$$S_{EH,\Lambda} = \int \sqrt{g}d^4x \, (R - 2\Lambda) \ . \tag{10.16}$$

Of course, once one is working at the level of the action, it is easy to come up with covariant generalizations of the Einstein-Hilbert action, such as

$$S = \int \sqrt{g}d^4x \, (R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R\Box R + \ldots) \ , \tag{10.17}$$

but these invariably involve higher-derivative terms and are therefore irrelevant for low-energy physics and thus the word we live in. Such terms could be relevant for the early universe, however, and are also typically predicted by quantum theories of gravity like string theory.

<center>THE MATTER LAGRANGIAN</center>

In order to obtain the non-vacuum Einstein equations, we need to decide what the matter Lagrangian should be. Now there is an obvious choice for this. If we have matter, then in addition to the Einstein equations we also want the equations of motion for the matter fields. Thus we should add to the Einstein-Hilbert Lagrangian the standard matter Lagrangian L_M, of course suitably covariantized via the principle of minimal coupling. Thus the matter action for Maxwell theory would be (5.9), and for a scalar field ϕ we would choose

$$\begin{aligned} S_M &= \int \sqrt{g}d^4x \, (g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \ldots) \\ &= \int \sqrt{g}d^4x \, (-g^{\mu\nu}\phi\nabla_\mu\nabla_\nu\phi + \ldots) \ . \end{aligned} \tag{10.18}$$

Of course, the variation of the matter action with respect to the matter fields will give rise to the covariant equations of motion of the matter fields. But if we want to add the matter action to the Einstein-Hilbert action and treat the metric as an additional dynamical variable, then we have to ask what the variation of the matter action with respect to the metric is. The short answer is: the energy-momentum tensor. Indeed, even though there are other definitions of the energy-momentum tensor you may know (defined via Noether's theorem applied to translations in flat space, for example), this is the modern, and by far the most useful, definition of the energy-momentum tensor, namely as the response of the matter action to a variation of the metric,

$$\delta_{\text{metric}} S_M = -\int \sqrt{g}d^4x \, T_{\mu\nu}\delta g^{\mu\nu} \ , \tag{10.19}$$

or

$$T_{\mu\nu} := -\frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S_M \ . \tag{10.20}$$

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One of the many advantages of this definition is that it automatically gives a symmetric tensor (no improvement terms required) which is also automatically covariantly conserved. We will establish this fact below - it is simply a consequence of the general covariance of S_M.

Therefore, the complete gravity-matter action for General Relativity is

$$S = \frac{1}{8\pi G} S_{EH} + S_M \ .$$ (10.21)

If one were to try to deduce the gravitational field equations by starting from a variational principle, i.e. by constructing the simplest generally covariant action for the metric and the matter fields, then one would also invariably be led to the above action. The relative numerical factor $8\pi G$ between the two terms would of course not be fixed a priori (but could once again be determined by looking at the Newtonian limit of the resulting equations of motion).

Typically, the above action principle will lead to a very complicated coupled system of equations for the metric and the matter fields beacuse the metric also appears in the energy-momentum tensor and in the equations of motion for the matter fields.

Consequences of the Variational Principle

I mentioned before that it is no accident that the Bianchi identities come to the rescue of the general covariance of the Einstein equations in the sense that they reduce the number of independent equations from six to ten. We will now see that indeed the Bianchi identities are a consequence of the general covariance of the Einstein-Hilbert action. Virtually the same calculation will show that the energy-momentum tensor, as defined above, is automatically conserved (on shell) by virtue of the general covariance of the matter action.

Let us start with the Einstein-Hilbert action. We already know that

$$\delta S_{EH} = \int \sqrt{g} d^4 x \ G_{\mu\nu} \delta g^{\mu\nu}$$ (10.22)

for any metric variation. We also know that the Einstein-Hilbert action is invariant under coordinate transformations. In particular, therefore, the above variation should be identically zero for variations of the metric induced by an infinitesimal coordinate transformation. But we know from the discussion of the Lie derivative that such a variation is of the form

$$\delta_V g_{\mu\nu} = L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \ ,$$ (10.23)

or

$$\delta_V g^{\mu\nu} = L_V g^{\mu\nu} = -(\nabla^\mu V^\nu + \nabla^\nu V^\mu) \ ,$$ (10.24)

where the vector field V is the infinitesimal generator of the coordinate transformation. Thus, $\delta_V S_{EH}$ should be identically zero. Calculating this we find

$$
\begin{aligned}
0 &= \delta_V S_{EH} \\
&= -\int \sqrt{g}\, d^4x\; G_{\mu\nu}(\nabla^\mu V^\nu + \nabla^\nu V^\mu) \\
&= -2 \int \sqrt{g}\, d^4x\; G_{\mu\nu}\nabla^\mu V^\nu \\
&= 2 \int \nabla^\mu G_{\mu\nu} V^\nu \ .
\end{aligned}
\tag{10.25}
$$

Since this has to hold for all V we deduce

$$
\delta_V S_{EH} = 0 \quad \forall\, V \quad \Rightarrow \quad \nabla^\mu G_{\mu\nu} = 0 \ ,
\tag{10.26}
$$

and, as promised, the Bianchi identities are a consequence of the general covariance of the Einstein-Hilbert action.

Now let us play the same game with the matter action S_M. Let us denote the matter fields generically by Φ so that $L_M = L_M(\Phi, g_{\mu\nu})$. Once again, the variation $\delta_V S_M$, expressed in terms of the Lie derivatives $L_V g_{\mu\nu}$ and $\delta_V \Phi = L_V \Phi$ of the matter fields should be identically zero, by general covariance of the matter action. Proceeding as before, we find

$$
\begin{aligned}
0 &= \delta_V S_M \\
&= \int \sqrt{g}\, d^4x \left(-T_{\mu\nu}\delta_V g^{\mu\nu} + \frac{\delta L_M}{\delta \Phi}\delta_V \Phi\right) \\
&= -2 \int \sqrt{g}\, d^4x\; (\nabla^\mu T_{\mu\nu})V^\nu + \int \sqrt{g}\, d^4x\; \frac{\delta L_M}{\delta \Phi}\delta_V \Phi \ .
\end{aligned}
\tag{10.27}
$$

Now once again this has to hold for all V, and as the second term is identically zero 'on-shell', i.e. for Φ satisfying the matter equations of motion, we deduce that

$$
\delta_V S_M = 0 \quad \forall\, V \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0 \quad \text{on-shell} \ .
\tag{10.28}
$$

This should be contrasted with the Bianchi identities which are valid 'off-shell'.

Part II: Selected Applications of General Relativity

Until now, our treatment of the basic structures and properties of Riemannian geometry and General Relativity has been rather systematic. In the second half of the course, we will instead discuss some selected applications of General Relativity. These will include, of course, a discussion of the classical predictions and tests of General Relativity (the deflection of light by the sun and the perihelion shift of Mercury). Then we will go on to discuss various other things, like the causal structure of the Schwarzschild metric (and the relation to black holes), the so-called standard (Friedman-Robertson-Walker) model of Cosmology, issues related to the linearized theory of gravity and gravitational waves, as well as, time permitting, a brief outline of Kaluza-Klein theory (about which I will not reveal anything at present).

11 The Schwarzschild Metric

Introduction

Einstein himself suggested three tests of General Relativity, namely

1. the gravitational redshift

2. the deflection of light by the sun

3. the anomalous precession of the perihelion of the orbits of Mercury and Venus,

and calculated the theoretical predictions for these effects. In the meantime, other tests have also been suggested and performed, for example the time delay of radar echos passing the sun (the Shapiro effect).

All these tests have in common that they are carried out in empty space, with gravitational fields that are to an excellent aproximation static (time independent) and isotropic (spherically symmetric). Thus our first aim will have to be to solve the vacuum Einstein equations under the simplifying assumptions of isotropy and time-independence. This, as we will see, is indeed not too difficult.

Static Isotropic Metrics

Even though we have decided that we are interested in static isotropic metrics, we still have to determine what we actually mean by this statement. After all, a metric which looks time-independent in one coordinate system may not do so in another coordinate system. There are two ways of approaching this issue:

1. One can try to look for a covariant characterization of such metrics, in terms of Killing vectors etc. In the present context, this would amount to considering metrics which admit four Killing vectors, one of which is time-like, with the remaining three representing the Lie algebra of the rotation group $SO(3)$.

2. Or one works with 'preferred' coordinates from the outset, in which these symmetries are manifest.

While the former approach may be conceptually more satisfactory, the latter is much easier to work with and is hence the one we will adopt. We will implement the condition of time-independence by choosing all the components of the metric to be time-independent, and we will express the condition of isotropy by the requirement that, in terms of spatial polar coordinates (r, θ, ϕ) the metric can be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + 2rC(r)dr\ dt + D(r)r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{11.1}$$

This ansatz, depending on the four functions $A(r), B(r), C(r), D(r)$, can still be simplified a lot by choosing appropriate new time and radial coordinates.

First of all, let us introduce a new time coordinate t' by

$$t' = t + \psi(r)\ . \tag{11.2}$$

Then

$$dt'^2 = dt^2 + \psi'^2 dr^2 + 2\psi' dr\ dt\ . \tag{11.3}$$

Thus we can eliminate the off-diagonal term in the metric by choosing ψ to satisfy the differential equation

$$\frac{d\psi(r)}{dr} = -r\frac{C(r)}{A(r)}\ . \tag{11.4}$$

We can also eliminate $D(r)$ by introducing a new radial coordinate $r'^2 = D(r)r^2$. Thus we can assume that the line element of a static isotropic metric is of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{11.5}$$

This is known as the *standard form* of a static isotropic metric. Another useful presentation, related to the above by a coordinate transformation, is

$$ds^2 = -E(r)dt^2 + F(r)(dr^2 + r^2 d\Omega^2)\ . \tag{11.6}$$

This is the static isotropic metric in *isotropic form*. We will mostly be using the metric in the standard form (11.5). Let us note some immediate properties of this metric:

1. By our ansatz, the components of the metric are time-independent. Because we have been able to eliminate the $dtdr$-term, the metric is also invariant under time-reversal $t \to -t$.

2. The surfaces of constant t and r have the metric

$$ds^2|_{r=const.,t=const.} = r^2 d\Omega^2 \quad , \tag{11.7}$$

and hence have the geometry of two-spheres.

3. Because $B(r) \neq 1$, we cannot identify r with the proper radial distance. However, r has the clear geometrical meaning that the two-sphere of constant r has the area $A(S_r^2) = 4\pi r^2$.

4. Also, even though the coordinate time t is not directly measurable, it can be invariantly characterized by the fact that $\partial/\partial t$ is a time-like Killing vector.

5. The functions A and B are now to be found by solving the Einstein field equations.

6. If we want the solution to be asymptotically flat (i.e. that it approaches Minkowski space for $r \rightarrow \infty$), we need to impose the boundary conditions

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \quad . \tag{11.8}$$

We will come back to other aspects of measurements of space and time in such a geometry after we have solved the Einstein equations.

We have assumed from the outset that the metric is static. However, it can be shown with little effort (even though I will not do this here) that the vacuum Einstein equations imply that a spherically symmetric metric is static (for those who want to check this: this follows primarily from the rt-component $R_{rt} = 0$ of the Einstein equations).

This result is known as *Birkhoff's theorem*. It is the General Relativity analogue of the Newtonian result that a spherically symmetric body behaves as if all the mass were concentrated in its center. In the present context it means that the gravitational field not only of a static spherically symmetric body is static and spherically symmetric (as we have assumed), but that the same is true for a radially oscillating/pulsating object. This is a bit surprising because one would expect such a body to emit gravitational radiation. What Birkhoff's theorem shows is that this radiation cannot escape into empty space (because otherwise it would destroy the time-independence of the metric).

SOLVING THE EINSTEIN EQUATIONS FOR A STATIC ISOTROPIC METRIC

We will now solve the vacuum Einstein equations for the static isotropic metric in standard form, i.e. we look for solutions of $R_{\mu\nu} = 0$. You have already calculated all the Christoffel symbols of this metric, using the Euler-Lagrange equations for the geodesic equation, and should have found that they are given by (a prime denotes an r-derivative)

$$\Gamma^r_{rr} = \frac{B'}{2B} \qquad \Gamma^r_{tt} = \frac{A'}{2B}$$

$$\Gamma^r{}_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^r{}_{\phi\phi} = -\frac{r\sin^2\theta}{B}$$

$$\Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{\phi r} = \frac{1}{r} \qquad \Gamma^t{}_{tr} = \frac{A'}{2A}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi{}_{\phi\theta} = \cot\theta \tag{11.9}$$

Now we need to calculate the Ricci tensor of this metric. A silly way of doing this would be to blindly calculate all the components of the Riemann tensor and to then perform all the relevant contractions to obtain the Ricci tensor. A more intelligent and less time-consuming strategy is the following:

1. Instead of using the explicit formula for the Riemann tensor in terms of Christoffel symbols, one should use directly its contracted version

$$
\begin{aligned}
R_{\mu\nu} &= R^\lambda{}_{\mu\lambda\nu} \\
&= \partial_\lambda\Gamma^\lambda{}_{\mu\nu} - \partial_\nu\Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\lambda\rho}\Gamma^\rho{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\rho}\Gamma^\rho{}_{\mu\lambda}
\end{aligned}
\tag{11.10}
$$

and use the formula for $\Gamma^\lambda{}_{\mu\lambda}$ derived previously.

2. The high degree of symmetry of the Schwarzschild metric implies that many components of the Ricci tensor are automatically zero. For example, invariance of the Schwarzschild metric under $t \to -t$ implies that $R_{rt} = 0$. The argument for this is simple. Since the metric is invariant under $t \to -t$, the Ricci tensor should also be invariant. But under the coordinate transformation $t \to -t$, R_{rt} transforms as $R_{rt} \to -R_{rt}$. Hence, invariance requires $R_{rt} = 0$, and no further calculations for this component of the Ricci tensor are required.

3. Analogous arguments, now involving θ or ϕ instead of t, imply that

$$R_{r\theta} = R_{r\phi} = R_{t\theta} = R_{t\phi} = R_{\theta\phi} = 0 \ . \tag{11.11}$$

4. Since the Schwarzschild metric is spherically symmetric, its Ricci tensor is also spherically symmetric. It is easy to prove, by considering the effect of a coordinate transformation that is a rotation of the two-sphere defined by θ and ϕ (leaving the metric invariant), that this implies that

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \ . \tag{11.12}$$

One possible proof (there may be a shorter argument): Consider a coordinate transformation $(\theta, \phi) \to (\theta', \phi')$. Then

$$d\theta^2 + \sin^2\theta d\phi^2 = \left[(\frac{\partial\theta}{\partial\theta'})^2 + \sin^2\theta(\frac{\partial\phi}{\partial\theta'})^2 \right] d\theta'^2 + \dots \tag{11.13}$$

Thus, a necessary condition for the metric to be invariant is

$$(\frac{\partial\theta}{\partial\theta'})^2 + \sin^2\theta(\frac{\partial\phi}{\partial\theta'})^2 = 1 \ . \tag{11.14}$$

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Now consider the tranformation behaviour of $R_{\theta\theta}$ under such a transformation. Using $R_{\theta\phi} = 0$, one has

$$R_{\theta'\theta'} = (\frac{\partial\theta}{\partial\theta'})^2 R_{\theta\theta} + (\frac{\partial\phi}{\partial\theta'})^2 R_{\phi\phi} \ . \tag{11.15}$$

Demanding that this be equal to $R_{\theta\theta}$ (because we are considering a coordinate transformation which does not change the metric) and using the condition derived above, one obtains

$$R_{\theta\theta} = R_{\theta\theta}(1 - \sin^2\theta(\frac{\partial\phi}{\partial\theta'})^2) + (\frac{\partial\phi}{\partial\theta'})^2 R_{\phi\phi} \ , \tag{11.16}$$

which implies (11.12).

5. Thus the only components of the Ricci tensor that we need to compute are R_{rr}, R_{tt} and $R_{\theta\theta}$.

Now some unenlightning calculations lead to the result that these components of the Ricci tensor are given by

$$\begin{aligned}
R_{tt} &= \frac{A''}{2B} - \frac{A'}{4B}(\frac{A'}{A} + \frac{B'}{B}) + \frac{A'}{rB} \\
R_{rr} &= -\frac{A''}{2A} + \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) + \frac{B'}{rB} \\
R_{\theta\theta} &= 1 - \frac{1}{B} - \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) \ .
\end{aligned} \tag{11.17}$$

Inspection of these formulae reveals that there is a linear combination which is particularly simple, namely $BR_{tt} + AR_{rr}$, which can be written as

$$BR_{tt} + AR_{rr} = \tfrac{1}{rB}(A'B + B'A) \ . \tag{11.18}$$

Demanding that this be equal to zero, one obtains

$$A'B + B'A = 0 \quad \Rightarrow \quad A(r)B(r) = const. \tag{11.19}$$

Asymptotic flatness fixes this constant to be $= 1$, so that

$$B(r) = \frac{1}{A(r)} \ . \tag{11.20}$$

Plugging this result into the expression for $R_{\theta\theta}$, one obtains

$$R_{\theta\theta} = 0 \quad \Rightarrow \quad A - 1 + rA' = 0 \quad \Leftrightarrow \quad (Ar)' = 1 \ , \tag{11.21}$$

which has the solution $Ar = r + C$ or

$$A(r) = 1 + \frac{C}{r} \ . \tag{11.22}$$

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To fix C, we compare with the Newtonian limit which tells us that asymptotically $A(r) = -g_{00}$ should approach $(1 + 2\Phi)$, where $\Phi = -GM/r$ is the Newtonian potential for a static spherically symmetric star of mass M. Thus $C = -2MG$, and the final form of the metric is

$$\boxed{ds^2 = -(1 - \tfrac{2MG}{r})dt^2 + (1 - \tfrac{2MG}{r})^{-1}dr^2 + r^2 d\Omega^2} \; . \tag{11.23}$$

This is the famous *Schwarzschild metric*, obtained by the astronomer Schwarzschild in 1916, the very same year that Einstein published his field equations, while he was serving as a soldier in World War I.

We have seen that, by imposing appropriate symmetry conditions on the metric, and making judicious use of them in the course of the calculation, the complicated Einstein equations become rather simple and manageable.

Before discussing some of the remarkable properties of the solution we have just found, I want to mention that the coordinate transformation

$$r = \rho(1 + \frac{MG}{2\rho})^2 \tag{11.24}$$

puts the Schwarzschild metric into isotropic form,

$$ds^2 = -\frac{(1 - \frac{MG}{2\rho})^2}{(1 + \frac{MG}{2\rho})^2} + (1 + \frac{MG}{2\rho})^4(d\rho^2 + \rho^2 d\Omega^2) \; . \tag{11.25}$$

The advantage of this istropic form of the metric is that one can replace $d\rho^2 + \rho^2 d\Omega^2$ by e.g. the standard metric on $\mathbb{R}^3$ in Cartesian coordinates, or any other metric on $\mathbb{R}^3$. This is useful when one likes to think of the solar system as being essentially described by flat space, with some choice of coordinates.

BASIC PROPERTIES OF THE SCHWARZSCHILD METRIC - THE SCHWARZSCHILD RADIUS

The metric we have obtained is quite remarkable in several respects. As mentioned before, the vacuum Einstein equations imply that an isotropic metric is static. Furthermore, the metric contains only a single constant of integration, the mass M. This implies that the metric in the exterior of a spherical body is completely independent of the composition of that body. Whatever the energy-momentum tensor for a star may be, the field in the exterior of the star has always got the form (11.23). This considerably simplifies the physical interpretation of General Relativity. In particular, in the subsequent discussion of tests of General Relativity, which only involve the exterior of stars like the sun, we do not have to worry about solutions for the interior of the star and how those could be patched to the exterior solutions.

Let us take a look at the range of coordinates in the Schwarzschild metric. Clearly, t is unrestricted, $-\infty < t < \infty$, and the polar coordinates θ and ϕ have their standard range. However, the issue regarding r is more interesting. First of all, the metric is a vacuum metric. Thus, if the star has radius r_0, then the solution is only valid for $r > r_0$. However, (11.23) also shows that the metric has a singularity at the *Schwarzschild radius* r_S, given by (reintroducing c)

$$r_S = \frac{2GM}{c^2} \quad . \tag{11.26}$$

Thus, we also have to require $r > r_S$. Since one frequently works in units in which $G = c = 1$, the Schwarzschild radius is often just written as $r_S = 2m$.

Now, in practice the radius of a physical object is almost always much larger than its Schwarzschild radius. For example, for a proton, for the earth and for the sun one has approximately

$$
\begin{aligned}
M_{\text{proton}} &\sim 10^{-24} \text{ g} \quad \Rightarrow r_S \sim 2,5 \times 10^{-52} \text{ cm} \ll r_0 \sim 10^{-13}\text{cm} \\
M_{\text{earth}} &\sim 6 \times 10^{27} \text{ g} \quad \Rightarrow r_S \sim 1 \text{ cm} \ll r_0 \sim 6000\text{km} \\
M_{\text{sun}} &\sim 2 \times 10^{33} \text{ g} \quad \Rightarrow r_S \sim 3 \text{ km} \ll r_0 \sim 7 \times 10^5\text{km} \quad .
\end{aligned} \tag{11.27}
$$

However, for more compact objects, their radius can approach that of their Schwarzschild radius. For example, for neutron stars one can have $r_S \sim 0.1 r_0$, and it is an interesting question (we will take up again later on) what happens to an object whose size is equal to or smaller than its Schwarzschild radius.

One thing that does *not* occur at r_S, however, in spite of what (11.23) may suggest, is a singularity. The singularity in (11.23) is a pure coordinate singularity, an artefact of having chosen a poor coordinate system. One can already see from the metric in isotropic form that in these new coordinates there is no singularity at the Schwarzschild radius, given by $\rho = MG/2$ in the new coordinates. It is true that g_{00} vanishes at that point, but we will later on construct coordinates in which the metric is completely regular at r_S. The only true singularity of the Schwarzschild metric is at $r = 0$, but there the solution was not meant to be valid anyway, so this is not a problem. Nevertheless, as we will see later, something interesting does happen at $r = r_S$, even though there is no singularity and e.g. geodesics are perfectly well behaved there: r_S is an event-horizon, in a sense a point of no return. Once one has passed the Schwarzschild radius of an object with $r_0 < r_S$, there is no turning back, not on geodesics, but also not with any amount of acceleration.

MEASURING LENGTH AND TIME IN THE SCHWARZSCHILD METRIC

In order to learn how to visualize the Schwarzschild metric (for $r > r_0 > r_S$), we will discuss some basic properties of length and time in the Schwarzschild geometry.

Let us first consider proper time for a sationary observer, i.e. an observer at rest at fixed values of (r, θ, ϕ). Proper time is related to coordinate time by

$$d\tau = (1 - 2m/r)^{1/2} dt < dt \ . \tag{11.28}$$

Thus clocks go slower in a gravitational field - something we already saw in the discussion of the gravitational redshift, and also in the discussion of the so-called 'twin-paradox': it is this fact that makes the accelerating twin younger than his unaccelerating brother whose proper time would be dt. This formula again suggests that something interesting is happening at the Schwarzschild radius $r = 2m$ - we will come back to this below.

As regards spatial length measurements, thus $dt = 0$, we have already seen above that the slices $r = const.$ have the standard two-sphere geometry. However, as r varies, these two-spheres vary in a way different to the way concentric two-spheres vary in $\mathbb{R}^3$. To see this, note that the proper radius R, obtained from the spatial line element by setting $\theta = const., \phi = const$, is

$$dR = (1 - 2m/r)^{-1/2} dr > dr \ . \tag{11.29}$$

In other words, the (proper) distance between spheres of radius r and radius $r + dr$ is $dR > dr$ and hence larger than in flat space. Note that $dR \to dr$ for $r \to \infty$ so that, as expected, far away from the origin the space approximately looks like $\mathbb{R}^3$. One way to visualize this geometry is as a sort of throat or sink, as in Figure 8.

To get some more quantitative feeling for the distortion of the geometry produced by the gravitational field of a star, consider a long stick lying radially in this gravitational field, with its endpoints at the coordinate values $r_1 > r_2$. To compute its length L, we have to evaluate

$$L = \int_{r_2}^{r_1} dr (1 - 2m/r)^{-1/2} \ . \tag{11.30}$$

It is possible to evaluate this integral in closed form (by changing variables from r to $u = 1/r$), but for the present purposes it will be enough to treat $2m/r$ as a small perturbation and to only retain the term linear in m in the Taylor expansion. Then we find

$$L \approx \int_{r_2}^{r_1} dr (1 + m/r) = (r_1 - r_2) + m \log \frac{r_1}{r_2} > (r_1 - r_2) \ . \tag{11.31}$$

We see that the corrections to the Euclidean result are suppressed by powers of the Schwarzschild radius $r_S = 2m$ so that for most astronomical purposes one can simply work with coordinate distances!

12 PARTICLE AND PHOTON ORBITS IN THE SCHWARZSCHILD GEOMETRY

We now come to the heart of the matter, the study of planetary orbits and light rays in the gravitational field of the sun, i.e. the properties of time-like and null geodesics of

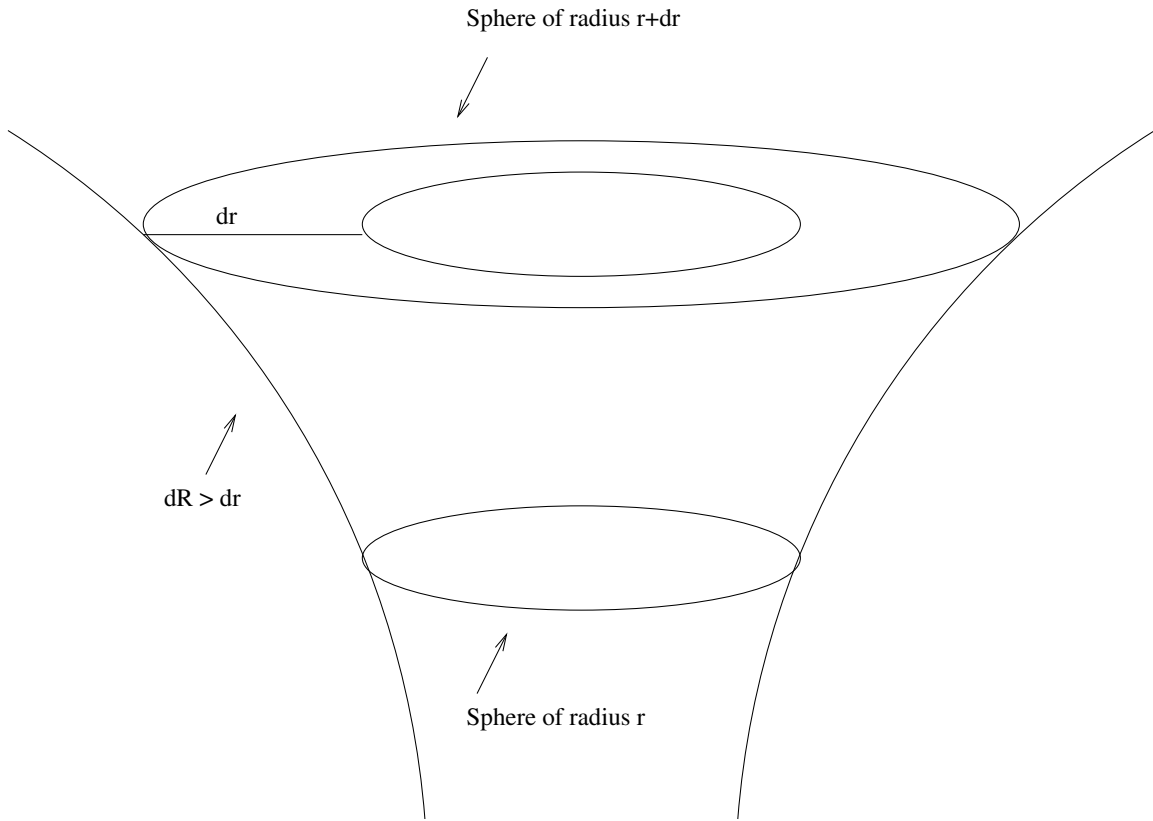Figure 8: Figure illustrating the geometry of the Schwarzschild metric. In $\mathbb{R}^3$, concentric spheres of radii r and $r + dr$ are a distance dr apart. In the Schwarzschild geometry, such spheres are a distance $dR > dr$ apart. This departure from Euclidean geometry becomes more and more pronounced for smaller values of r, i.e. as one travels down the throat towards the Schwarzschild radius $r = 2m$.

the Schwarzschild geometry. We shall see that, by once again making good use of the symmetries of the problem, we can reduce the geodesic equations to a single first order differential equation in one variable, analogous to that for a one-dimensional particle moving in a particular potential. Solutions to this equation can then readily be discussed qualitatively and also quantitatively (analytically).

FROM CONSERVED QUANTITIES TO THE EFFECTIVE POTENTIAL

A convenient starting point in general for discussing geodesics is, as I stressed before, the Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. For the Schwarzschild metric this is

$$\mathcal{L} = -(1 - 2m/r)\dot{t}^2 + (1 - 2m/r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \ , \tag{12.1}$$

where $2m = 2MG/c^2$. Rather than writing down and solving the (second order) geodesic equations, we will make use of the conserved quantities $K_\mu\dot{x}^\mu$ associated with Killing vectors. After all, conserved quantities correspond to first integrals of the equations of motion and if there are a sufficient number of them (there are) we can directly reduce the second order differential equations to first order equations.

So, how many Killing vectors does the Schwarzschild metric have? Well, since the metric is static, there is one time-like Killing vector, namely $\partial/\partial t$, and since the metric is spherically symmetric, there are spatial Killing vectors generating the Lie algebra of $SO(3)$, hence there are three of those, and therefore all in all four Killing vectors.

Now, since the gravitational field is isotropic (and hence there is conservation of angular momentum), the orbits of the particles or planets are planar. Without loss of generality, we can choose our coordinates in such a way that this plane is the equatorial plane $\theta = \pi/2$, so in particular $\dot{\theta} = 0$, and the residual Lagrangian to deal with is

$$\mathcal{L}' = -(1 - 2m/r)\dot{t}^2 + (1 - 2m/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \ . \tag{12.2}$$

This choice fixes the direction of the angular momentum (to be orthogonal to the plane) and leaves two conserved quantities, the energy (per unit rest mass) E and the magnitude L (per unit rest mass) of the angular momentum, corresponding to the cyclic variables t and ϕ, (or: corresponding to the Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$),

$$\frac{\partial\mathcal{L}}{\partial t} = 0 \quad \Rightarrow \quad \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{t}} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \Rightarrow \quad \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0 \ , \tag{12.3}$$

namely

$$E = (1 - 2m/r)\dot{t} \tag{12.4}$$
$$L = r^2\sin^2\theta\dot{\phi} = r^2\dot{\phi} \ . \tag{12.5}$$

There is also one more integral of the geodesic equation (corresponding to parametrization invariance of the Lagrangian), namely $\mathcal{L}$ itself,

$$\frac{d}{d\tau}\mathcal{L} = 2g_{\mu\nu}\dot{x}^{\mu}\frac{D}{D\tau}\dot{x}^{\nu} = 0 \quad . \tag{12.6}$$

Thus we set

$$\mathcal{L} = \epsilon \quad , \tag{12.7}$$

where $\epsilon = -1$ for time-like geodesics and $\epsilon = 0$ for null geodesics. Thus we have

$$-(1 - 2m/r)\dot{t}^2 + (1 - 2m/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \epsilon \quad , \tag{12.8}$$

and we can now express $\dot{t}$ and $\dot{\phi}$ in terms of the conserved quantities E and L to obtain a first order differential equation for r alone, namely

$$-(1 - 2m/r)^{-1}E^2 + (1 - 2m/r)^{-1}\dot{r}^2 + \frac{L^2}{r^2} = \epsilon \quad . \tag{12.9}$$

Multiplying by $(1 - 2m/r)/2$ and rearranging the terms, one obtains

$$\frac{E^2 + \epsilon}{2} = \frac{\dot{r}^2}{2} + \epsilon\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \quad . \tag{12.10}$$

Now this equation is of the familiar Newtonian form

$$\mathcal{E} = \frac{\dot{r}^2}{2} + V_{eff}(r) \quad , \tag{12.11}$$

with

$$\begin{aligned}
\mathcal{E} &= \frac{E^2 + \epsilon}{2}\\
V_{eff}(r) &= \epsilon\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \quad ,
\end{aligned} \tag{12.12}$$

describing the energy conservation in an effective potential. Except for $t \to \tau$, this is exactly the same as the Newtonian equation of motion in a potential

$$V(r) = \epsilon\frac{m}{r} - \frac{mL^2}{r^3} \quad , \tag{12.13}$$

the effective angular momentum term $L^2/r^2 = r^2\dot{\phi}^2$ arising, as usual, from the change to polar coordinates.

In particular, for $\epsilon = -1$, the general relativistic motion (as a function of τ) is exactly the same as the Newtonian motion (as a function of t) in the potential

$$\epsilon = -1 \quad \Rightarrow \quad V(r) = -\frac{m}{r} - \frac{mL^2}{r^3} \quad . \tag{12.14}$$

The first term is just the ordinary Newtonian potential, so the second term is apparently a general relativistic correction. We will later on treat this as a perturbation but note

that the above is an exact result, not an approximation (so, for example, there are no higher order corrections proportional to higher inverse powers of r). We expect observable consequences of this general relativistic correction because many properties of the Newtonian orbits (Kepler's laws) depend sensitively on the fact that the Newtonian potential is precisely $\sim 1/r$.

For null geodesics, on the other hand, the Newtonian part of the potential is zero, as one might expect for massless particles, but in General Relativity a photon feels a non-trivial potential

$$\epsilon = 0 \quad \Rightarrow \quad V(r) = -\frac{mL^2}{r^3} \quad . \tag{12.15}$$

In particular, in either case, if one is primarily interested in the shape of the trajectories (and this is what is astronomically observable), this means that one wants to know r as a function of ϕ. In that case, the difference between t and τ is irrelevant: In the Newtonian theory one uses $L = r^2 d\phi/dt$ to express t as a function of ϕ, $t = t(\phi)$ to obtain $r(\phi)$ from $r(t)$. In General Relativity, one uses the analogous equation $L = r^2 d\phi/d\tau$ to express τ as a function of ϕ, $\tau = \tau(\phi)$. But then obviously $t(\phi)$ and $\tau(\phi)$ are the same functions of ϕ. Hence the shapes of the General Relativity orbits are precisely the shapes of the Newtonian orbits in the potential (12.13). Thus we can use the standard methods of Classical Mechanics to discuss these general relativistic orbits and of course this simplifies matters considerably.

<center>TIMELIKE GEODESICS</center>

In that case, we need to understand the orbits in the effective potential

$$V_{eff}(r) = -\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \quad , \tag{12.16}$$

and the standard way to do this is to plot this potential as a function of r for various values of the parameters L and m. The basic properties of $V_{eff}(r)$ are the following:

1. Asymptotically, i.e. for $r \to \infty$, the potential tends to the Newtonian potential,

$$V_{eff}(r) \stackrel{r \to \infty}{\Longrightarrow} -\frac{m}{r} \quad . \tag{12.17}$$

2. At the Schwarzschild radius $r_S = 2m$, nothing special happens and the potential is completely regular there,

$$V_{eff}(r = 2m) = -\frac{1}{2} \quad . \tag{12.18}$$

Of course, for the discussion of planetary orbits in the solar system we can safely assume that the radius of the sun is much larger than its Schwarzschild radius, $r_0 \gg r_S$, but the above shows that even for these highly compact objects with

<center>94</center>
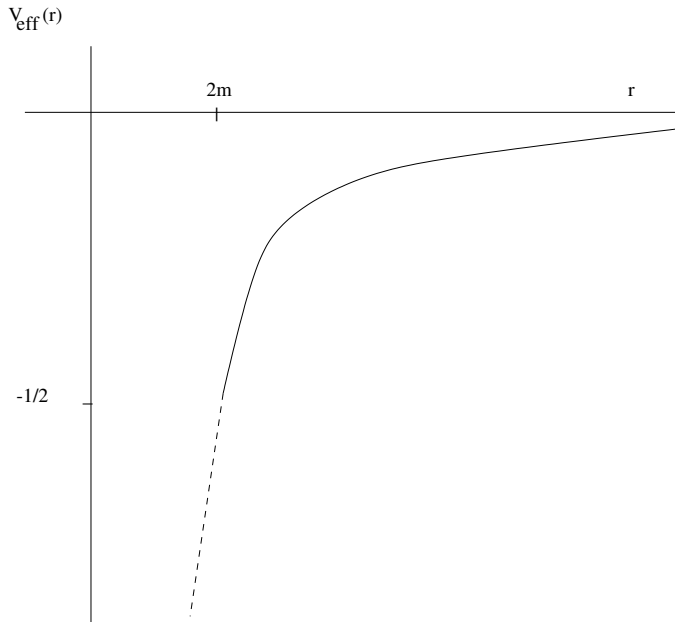
Figure 9: Effective potential for a massive particle with $L/m < \sqrt{12}$. The extrapolation to values of $r < 2m$ has been indicated by a dashed line.

> $r_0 < r_S$ geodesics are perfectly regular as one approaches r_S. Of course the particular numerical value of $V_{eff}(r = 2m)$ has no special significance because $V(r)$ can always be shifted by a constant.

3. The extrema of the potential, i.e. the points at which $dV_{eff}/dr = 0$, are at

$$r_\pm = (L^2/2m)[1 \pm \sqrt{1 - 12(m/L)^2}] \ , \tag{12.19}$$

and the potential has a maximum at r_- and a local minimum at r_+.

Thus there are qualitative differences in the shapes of the orbits between $L/m < \sqrt{12}$ and $L/m > \sqrt{12}$. Let us discuss these two cases in turn.

When $L/m < \sqrt{12}$, then there are no real turning points and the potential looks approximately like that in Figure 9. Note that we should be careful with extrapolating to values of r with $r < 2m$ because we know that the Schwarzschild metric has a coordinate singularity there. However, qualitatively the picture is also correct for $r < 2m$.

From this picture we can read off that there are no orbits for these values of the parameters. Any inward bound particle with $L < \sqrt{12}m$ will continue to fall inwards (provided that it moves on a geodesic). This should be contrasted with the Newtonian situation in which for any $L \neq 0$ there is always the centrifugal barrier reflecting incoming particles since the repulsive term $L^2/2r^2$ will dominate over the attractive $-m/r$ for small values of r. In General Relativity, on the other hand, it is the attractive term $-mL^2/r^3$ that dominates for small r.
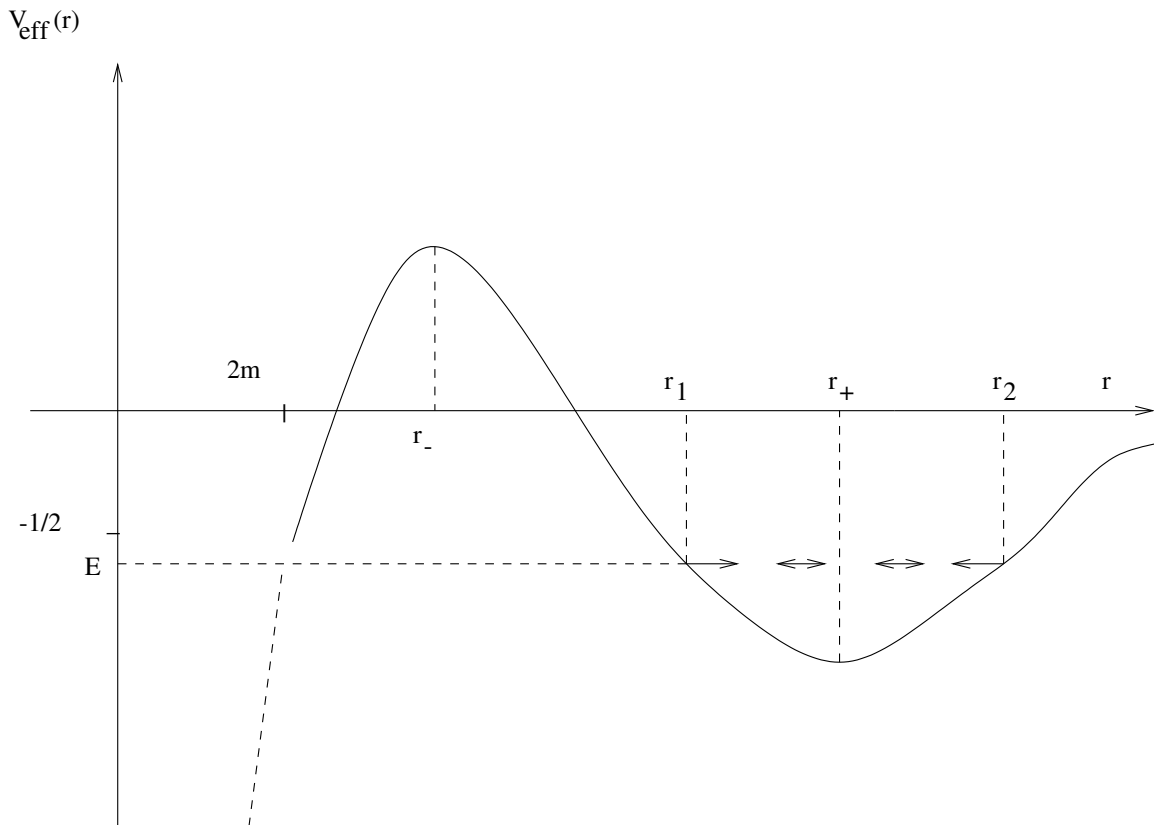
Figure 10: Effective potential for a massive particle with $L/m > \sqrt{12}$. Shown are the maximum of the potential at r_- (an unstable circular orbit), the minimum at r_+ (a stable circular orbit), and the orbit of a particle with $\mathcal{E} < 0$ with turning points r_1 and r_2.

Fortunately for the stability of the solar system, the situation is qualitatively quite different for sufficiently large values of the angular momentum, namely $L > \sqrt{12}m$ (see Figure 10).

In that case, there is a minimum and a maximum of the potential. The critical radii correspond to exactly circular orbits, unstable at r_- (on top of the potential) and stable at r_+ (the minimum of the potential). For $L \to \sqrt{12m}$ these two orbits approach each other, the critical radius tending to $r_\pm \to 6m$. Thus there are no stable circular orbits for $r < 6m$. On the other hand, for very large values of L the critical radii are (expand the square root to first order) to be found at

$$(r_+, r_-) \overset{L\to\infty}{\longrightarrow} (L^2/m, 3m) \ . \tag{12.20}$$

For given L, for sufficiently large values of $\mathcal{E}$ a particle will fall all the way down the potential. For $\mathcal{E} < 0$, there are bound orbits which are not circular and which range between the radii r_1 and r_2, the turning points at which $\dot{r} = 0$ and therefore $\mathcal{E} = V_{eff}(r)$. These orbits will not be closed (elliptical) because of the non-Newtonian term $\sim 1/r^3$.

In particular, therefore, the perihelion, the point of closest approach of the planet to the sun where the planet has distance r_1, will not remain constant. However, because r_1 is constant, and the planetary orbit is planar, this point will move on a circle of radius r_1 around the sun. In order to calculate this perihelion shift, one needs to express ϕ as a function of r using (12.5) and (12.12),

$$
\begin{aligned}
\frac{d\phi}{dr} &= \frac{d\phi}{dt}\left(\frac{dr}{dt}\right)^{-1} \\
&= \frac{L}{r^2}(2\mathcal{E} - 2V_{eff}(r))^{-1/2}
\end{aligned}
\tag{12.21}
$$

One can then calculate the total angle $\Delta\phi$ swept out by the planet during one revolution by integrating this from r_1 to r_2 and back again to r_1, or

$$
\Delta\phi = 2\int_{r_1}^{r_2} \frac{d\phi}{dr}dr \quad .
\tag{12.22}
$$

In the Newtonian theory, one would have $\Delta\phi = 2\pi$. The anomalous perihelion shift due to the effects of General Relativity is thus

$$
\delta\phi = \Delta\phi - 2\pi \quad .
\tag{12.23}
$$

Unfortunately, the above differential equation cannot be integrated in closed form (it leads to elliptic integrals) but can be calculated to first order in the perturbation, and the result is that

$$
\delta\phi = 6\pi\left(\frac{GM}{cL}\right)^2 \quad .
\tag{12.24}
$$

In terms of the eccentricity e and the semi-major axis a of an elliptical orbit, this can be written as

$$
\delta\phi = \frac{6\pi G}{c^2}\frac{M}{a(1 - e^2)} \quad .
\tag{12.25}
$$

As these paremters are known for the planetary orbits, $\delta\phi$ can be evaluated. For example, for Mercury, where this effect is largest (because it has the largest eccentricity) one finds $\delta\phi = 0, 1''$ per revolution. This is of course a tiny effect (1 second, $1''$, is one degree divided by 3600) and not *per se* detectable. However,

1. this effect is cumulative, i.e. after N revolutions one has an anomalous perihelion shift $N\delta\phi$;

2. Mercury has a very short solar year, with about 415 revolutions per century;

3. and accurate observations of the orbit of Mercury go back over 200 years.

Thus the cumulative effect is approximately $850\delta\phi$ and this *is* sufficiently large to be observable. The prediction of General Relativity for this effect is

$$
\delta\phi_{GR} = 43, 03''/\text{century} \quad .
\tag{12.26}
$$

And indeed such an effect is observed (and had for a long time presented a puzzle, an anomaly, for astronomers). In actual fact, the perihelion of Mercury's orbit shows a precession rate of $5601''$ per century, so this does not yet look like a brilliant confirmation of General Relativity. However, of this effect about $5025''$ are due to fact that one is using a non-inertial geocentric coordinate system (precession of the equinoxes). $532''$ are due to perturbations of Mercury's orbit caused by the (Newtonian) gravitational attraction of the other planets of the solar system (chiefly Venus, earth and Jupiter). This much was known prior to General Relativity and left an unexplained anomalous perihelion shift of

$$\delta\phi_{anomalous} = 43,11'' \pm 0,45''/\text{century} \ . \tag{12.27}$$

Now the agreement with the result of General Relativity is truly impressive and this is one of the most important experimental verifications of General Relativity. Other observations, involving e.g. the mini-planet Icarus, discovered in 1949, with a huge eccentricity $e \sim 0,827$, or binary pulsar systems, have provided further confirmation of the agreement between General Relativity and experiment.

<center>NULL GEODESICS</center>

To study the behaviour of massless particles (photons) in the Schwarzschild geometry, we need to study the effective potential

$$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{mL^2}{r^3} = \frac{L^2}{2r^2}(1 - \frac{2m}{r}) \ . \tag{12.28}$$

The following properties are immediate:

1. When $L = 0$, photons feel no potential at all.

2. $V_{eff}(r = 2m) = 0$.

3. There is one critical point of the potential, at $r = 3m$, with $V_{eff}(r = 3m) = L^2/54m^2$. So there is one unstable circular orbit for photons at $r = 3m$.

Thus the potential has the form sketched in Figure 11.

For energies $E^2 > L^2/27m^2$, photons are captured by the star and will spiral into it. For energies $E^2 < L^2/27m^2$, on the other hand, there will be a turning point, and light rays will be deflected by the star. As this may sound a bit counterintuitive (shouldn't a photon with higher energy be more likely to zoom by the star without being forced to spiral into it?), think about this in the following way. $L = 0$ corresponds to a photon falling radially towards the star, L small corresponds to a slight deviation from radial motion, while L large (thus $\dot{\phi}$ large) means that the photon is travelling along a trajectory that will not bring it very close to the star at all. It is then not surprising
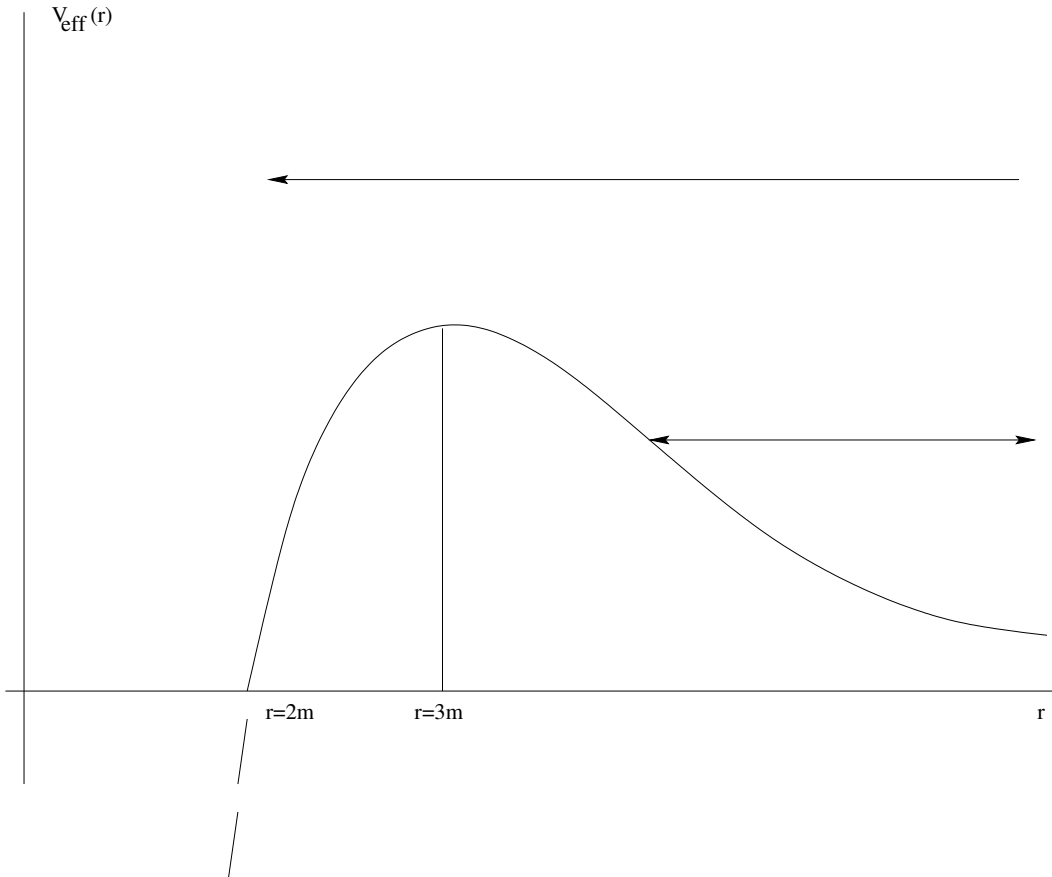
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Figure 11: Effective potential for a massless particle. Displayed is the location of the unstable circular orbit at $r = 3m$. A photon with an energy $E^2 < L^2/27m^2$ will be deflected (lower arrow), photons with $E^2 > L^2/27m^2$ will be captured by the star.

that photons with small L are more likely to be captured by the star (this happens for $L^2 < 27m^2E^2$) than photons with large L which will only be deflected in their path.

We will study this in more detail below. But let us first also consider the opposite situation, that of light from or near the star (and we are of course assuming that $r_0 > r_S$). Then for $r_0 < 3m$ and $E^2 < L^2/27m^2$, the light cannot escape to infinity but falls back to the star, whereas for $E^2 > L^2/27m^2$ light will escape. Thus for a path sufficiently close to radial (L small, because $\dot{\phi}$ is then small) light can always escape as long as $r > 2m$.

<center>THE BENDING OF LIGHT BY A STAR</center>

To study the binding of light by a star, we consider an incoming photon (or light ray) with impact parameter b (see Figure 12) and we need to calculate $\phi(r)$ for a trajectory with turning point at $r = r_1$. At that point we have $\dot{r} = 0$ (the dot now indicates differentiation with respect to the affine parameter σ of the null geodesic, we can but need not choose this to be the coordinate time t) and therefore

$$r_1^2 = \frac{L^2}{E^2}(1 - \frac{2m}{r_1}) \ . \tag{12.29}$$

The first thing we need to establish is the relation between b and the other parameters E and L. Consider the ratio

$$\frac{L}{E} = \frac{r^2\dot{\phi}}{(1 - 2m/r)\dot{t}} \ . \tag{12.30}$$

For large values of r, $r \gg 2m$, this reduces to

$$\frac{L}{E} = r^2\frac{d\phi}{dt} \ . \tag{12.31}$$

On the other hand, for large r we can approximate $b/r = \sin\phi$ by ϕ. Since we also have $dr/dt = -1$ (for an incoming light ray), we deduce

$$\frac{L}{E} = r^2\frac{d}{dt}\frac{b}{r} = b \ . \tag{12.32}$$

Now just as before, the shape of the curve is described by

$$\begin{aligned}
\frac{d\phi}{dr} &= \frac{L}{r^2}[E^2 - \frac{L^2}{r^2}(1 - \frac{2m}{r})]^{-1/2} \\
&= \frac{1}{r^2}[\frac{1}{b^2} - \frac{1}{r^2}(1 - \frac{2m}{r})]^{-1/2} \ .
\end{aligned} \tag{12.33}$$

The angle $\Delta\phi$ is now given by

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{d\phi}{dr}dr \ , \tag{12.34}$$

and the deflection angle (which would be zero in the Newtonian theory) is

$$\delta\phi = \Delta\phi - \pi \ . \tag{12.35}$$

<center>100</center>
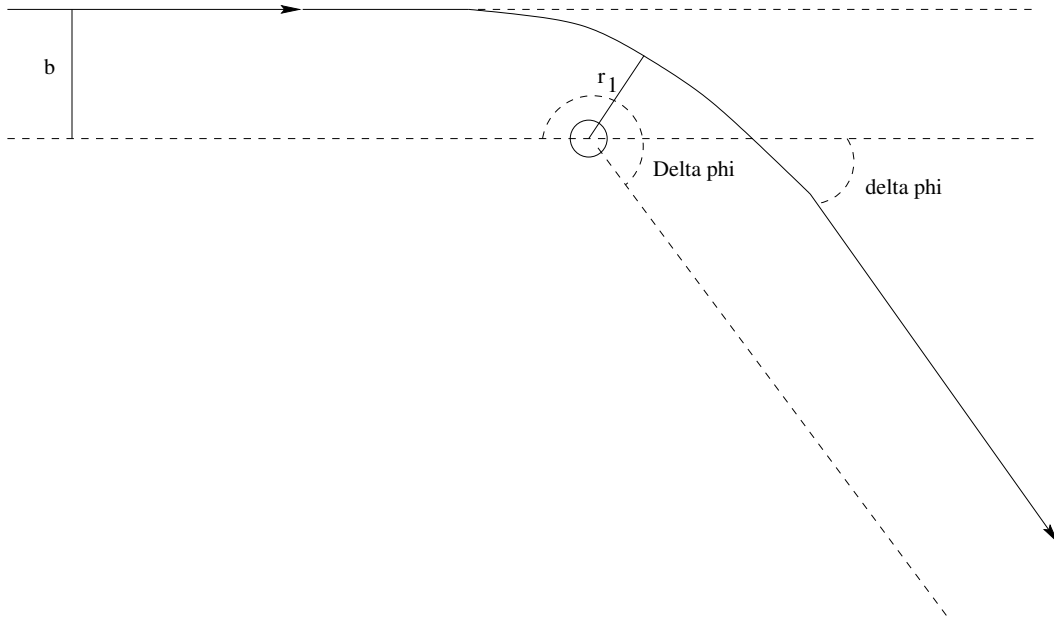
Figure 12: Bending of light by a star. Indicated are the definitions of the impact parameter b, the perihelion r_1, and of the angles $\Delta\phi$ and $\delta\phi$.

With the substitution $x = b/r$, this becomes

$$\Delta\phi = 2\int_0^{x_1} dx\, [1 - x^2 + \frac{2m}{b}x^3]^{-1/2} \ . \tag{12.36}$$

This is once again an elliptic integral that cannot be solved in closed form. But since we evidently have $2m = r_S \ll r_0$ and $r_0 < b$, also $\lambda \equiv m/b \ll 1$. Thus we can try to expand the integral to first order in λ. To zero'th order, one has $r_1 = b$ (no deflection of light in the Newtonian theory) and thus $x_1 = 1$ and

$$\Delta^{(0)}\phi = 2\int_0^1 dx(1 - x^2)^{-1/2} = 2\arcsin 1 = \pi \ , \tag{12.37}$$

which is just the expected Newtonian result $\delta\phi = 0$.

The calculation to the next order in λ is a bit subtle because both the integrand and the range of integration depend on λ but one way to proceed is the following:

1. First of all, it follows from the defining equation (12.29) for r_1 that, to first order in λ, the perihelion r_1 is shifted from its Newtonian value b (no attraction) to $r_1 = b - m$. Therefore

$$x_1 = b/r_1 \sim 1 + \lambda \ . \tag{12.38}$$

2. Thus we are interested in the first order term C in λ of the integral

$$\begin{aligned} I(\lambda) \equiv \Delta\phi &= 2\int_0^{1+\lambda} dx\, [1 - x^2 + 2\lambda x^3]^{-1/2} \\ &\equiv \pi + C\lambda + \mathcal{O}(\lambda^2) \ . \end{aligned} \tag{12.39}$$

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3. C can be obtained as the first order term in the Taylor series expansion of $I(\lambda)$,

$$C = \frac{\partial I(\lambda)}{\partial \lambda}|_{\lambda=0} \quad . \tag{12.40}$$

4. Therefore C is given by

$$
\begin{aligned}
C &= \{2[1 - x^2 + 2\lambda x^3]^{-1/2}|^{x=x_1}\}|_{\lambda=0} + 2\int_0^1 dx\, \{\frac{\partial}{\partial \lambda}[1 - x^2 + 2\lambda x^3]^{-1/2}\}|_{\lambda=0} \\
&= 2(1-x^2)^{-1/2}|^{x=1} - 2\int_0^1 dx\, x^3(1-x^2)^{-3/2} \quad .
\end{aligned}
\tag{12.41}
$$

5. Now obviously the first term is divergent, but it will be cancelled by exactly the same divergence arising from the second term (the integral). In fact, that integral is fortunately elementary. One has

$$\int dx\, x^3(1-x^2)^{-3/2} = (1-x^2)^{-1/2} + (1-x^2)^{1/2} \quad , \tag{12.42}$$

and therefore

$$C = 2(1-x^2)^{-1/2}|^{x=1} - 2(1-x^2)^{-1/2}|^{x=1}_{x=0} - 2(1-x^2)^{1/2}|^{x=1}_{x=0} \quad . \tag{12.43}$$

We see that the divergent first term is cancelled exactly by the upper contribution (at $x = 1$) from the second term, and we are left with

$$C = 2 - 0 + 2 = 4 \quad . \tag{12.44}$$

6. Hence, the leading term in the general relativistic correction of the Newtonian result is $4\lambda = 4m/b$ and one has

$$\Delta\phi = \pi + \frac{4m}{b} + \mathcal{O}(\lambda^2) \quad , \tag{12.45}$$

or

$$\delta\phi \simeq \frac{4m}{b} = \frac{4MG}{bc^2} \quad . \tag{12.46}$$

This effect is physically measurable and was one of the first true tests of Einstein's new theory of gravity. For light just passing the sun the predicted value is

$$\delta\phi \sim 1,75'' \quad . \tag{12.47}$$

Experimentally this is a bit tricky to observe because one needs to look at light from distant stars passing close to the sun. Under ordinary circumstances this would not be observable, but in 1919 a test of this was performed during a total solar eclipse, by observing the effect of the sun on the apparent position of stars in the direction of the sun. The observed value was rather imprecise, yielding $1,5'' < \delta\phi < 2,2''$ which is, if not a confirmation of, at least consistent with General Relativity.

More recently, it has also been possible to measure the deflection of radio waves by the gravitational field of the sun. These measurements rely on the fact that a particular Quasar, known as 3C275, is obscured annually by the sun on October 8th, and the observed result (after correcting for diffraction effects by the corona of the sun) in this case is $\delta = 1,76'' \pm 0,02''$.

The value predicted by General Relativity is, interestingly enough, exactly twice the value that would have been predicted by the Newtonian approximation of the geodesic equation alone (but the Newtonian approximation is not valid anyway because it applies to slowly moving objects, and light certainly fails to satisfy this condition). A calculation leading to this wrong value had first been performed by Soldner in 1801 (!) (by cancelling the mass m out of the Newtonian equations of motion before setting $m = 0$) and also Einstein predicted this wrong result in 1908 (his equivalence principle days, long before he came close to discovering the field equations of General Relativity now carrying his name).

13   APPROACHING AND CROSSING THE SCHWARZSCHILD RADIUS

So far, we have been considering objects of a size larger (in practice much larger) than their Schwarzschild radius, $r_0 > r_S$. We also noted that the effective potential $V_{eff}(r)$ is perfectly well behaved at r_S. We now consider objects with $r_0 < r_S$ and try to unravel some of the bizzarre physics that nevertheless occurs when one aproaches or crosses $r_S = 2m$.

INFINITE GRAVITATIONAL REDSHIFT

One dramatic aspect of what is happening at (or, better, near) the Schwarzschild radius for very (very!) compact objects with $r_S > r_0$ is the following. Recall the formula for the gravitational red-shift, which gave us the ratio between the frequency of light ν_A emitted at the radius r_A and the frequency ν_B received at the radius $r_B > r_A$ in a static spherically symmetric gravitational field. The result was

$$\frac{\nu_B}{\nu_A} = \frac{g_{00}(r_A)^{1/2}}{g_{00}(r_B)^{1/2}} \ . \tag{13.1}$$

In the case of the Schwarzschild metric, this is

$$\frac{\nu_B}{\nu_A} = \frac{(1 - 2m/r_A)^{1/2}}{(1 - 2m/r_B)^{1/2}} \ . \tag{13.2}$$

Now consider a fixed observer at $r_B \gg r_S$, and the emitter at r_A who is gradually approaching the Schwarzschild radius, $r_A \to r_S$. In this limit one finds

$$\frac{\nu_B}{\nu_A} \to 0 \ , \tag{13.3}$$

i.e. there is an *infinite gravitational red-shift*! Note that the far-away observer will never actually see the unfortunate emitter A crossing the Schwarzschild radius: he will see A's signals becoming dimmer and dimmer and arriving at greater and greater intervals, and A will completely disappear from B's sight when A gets very close to r_S. A, on the other hand, does not immediately notice anything special happening as he approaches r_S.

<div align="center">VERTICAL FREE FALL</div>

Our next calculation will show that even though B never sees A crossing the Schwarzschild radius, A reaches r_S in finite proper time. To that end, consider a vertical free fall towards the object with $r_0 < r_S$. 'Vertical' means that $\dot{\phi} = 0$, and therefore there is no angular momentum, $L = 0$. Hence the energy conservation equation (12.12) becomes

$$E^2 - 1 = \dot{r}^2 - \frac{2m}{r} \ . \tag{13.4}$$

In particular, if r_∞ is the point at which the particle (observer) A was initially at rest,

$$\frac{dr}{d\tau}\Big|_{r=r_\infty} = 0 \ , \tag{13.5}$$

we have

$$E^2 - 1 = -\frac{2m}{r_\infty} \ . \tag{13.6}$$

Then we obtain

$$\dot{r}^2 = \frac{2m}{r} - \frac{2m}{r_\infty} \tag{13.7}$$

and upon differentiation

$$\ddot{r} + \frac{m}{r^2} = 0 \ . \tag{13.8}$$

This is just like the Newtonian equation (which should not come as a surprise as V_{eff} coincides with the Newtonian potential for $L = 0$), apart from the fact that r is *not* vertical distance and the familiar $\tau \neq t$. Nevertheless, calculation of the time τ along the path proceeds exactly as in the Newtonian theory. For the proper time required to reach the point with coordinate value $r = r_1$ we obtain

$$\tau = (2m)^{-1/2} \int_{r_\infty}^{r_1} dr \ \left(\frac{r_\infty r}{r_\infty - r}\right)^{1/2} \ . \tag{13.9}$$

In particular, this is finite as $r_1 \to 2m$. Coordinate time, on the other hand, becomes infinite there. This can be understood by looking at the coordinate velocity as a function of r,

$$v(r) = \frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} \ . \tag{13.10}$$

Let us choose $r_\infty = \infty$ for simplicity - other choices will not change our conclusions as we are interested in the behaviour of $v(r)$ as $r \to r_S$. Then we know that

$$\frac{dr}{d\tau} = (2m/r)^{1/2} \tag{13.11}$$

and we know from the definition of E that

$$\frac{d\tau}{dt} = E^{-1}(1 - 2m/r) \quad . \tag{13.12}$$

For $r_\infty = \infty$ we have $E = 1$, and therefore

$$v(r) = (2m)^{1/2}\frac{r - 2m}{r^{3/2}} \quad . \tag{13.13}$$

As a function of r, $v(r)$ reaches a maximum at $r = 6m = 3r_S$, where the velocity is (restoring the velocity of light c)

$$v_{max} = v(r = 6m) = \frac{2c}{3\sqrt{3}} \quad . \tag{13.14}$$

Beyond that point, $v(r)$ decreases again and clearly goes to zero as $r \to 2m$. The fact that the coordinate velocity goes to zero explains why the coordinate time goes to infinity. Somehow, the Schwarzschild coordinates are not suitable for describing the physics at or beyond the Schwarzschild radius because the time coordinate one has chosen is running too fast. This is the crucial insight that will allow us to construct 'better' coordinates in the subsequent sections which are also valid at $r < r_S$.

<h2 style="text-align:center">TORTOISE COORDINATES</h2>

We have now seen in two different ways why the Schwarzschild coordinates are not suitable for exploring the physics in the region $r \le 2m$: in these coordinates the metric becomes singular at $r = 2m$ and the coordinate time becomes infinite. On the other hand, we have seen no indication that the local physics, expressed in terms of covariant quantities like proper time or the geodesic equation, becomes singular as well. So we have good reasons to suspect that the singular behaviour we have found is really just an artefact of a bad choice of coordinates.

To improve our understanding of the Schwarzschild geometry, it is important to study its *causal structure*, i.e. the light cones. Radial null curves satisfy

$$(1 - 2m/r)dt^2 = (1 - 2m/r)^{-1}dr^2 \quad . \tag{13.15}$$

Thus

$$\frac{dt}{dr} = \pm(1 - 2m/r)^{-1} \quad , \tag{13.16}$$

In the (r, t)-diagram of Figure 13, dt/dr represents the slope of the light-cones at a given value of r. Now, as $r \to 2m$, one has

$$\frac{dt}{dr} \stackrel{r \to 2m}{\longrightarrow} \pm\infty \quad , \tag{13.17}$$

so the light clones 'close up' as one approaches the Schwarzschild radius. This is the same statement as before regarding the fact that the coordinate velocity goes to zero at $r = 2m$, but this time for null rather than time-like geodesics.

<div style="text-align:center">105</div>
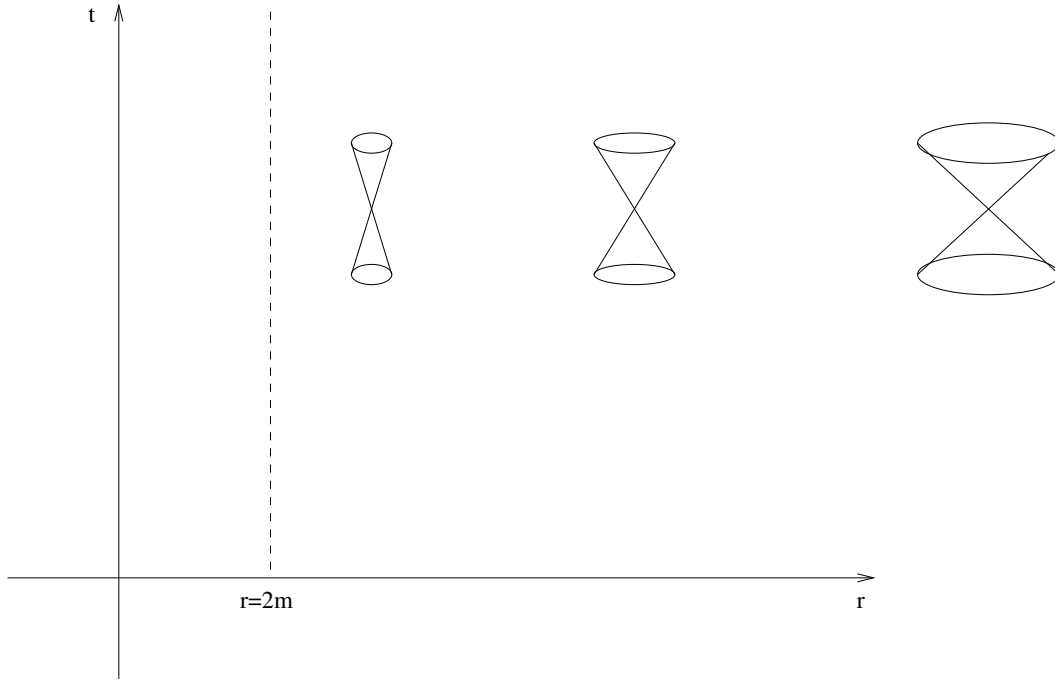
Figure 13: The causal structure of the Schwarzschild geometry in the Schwarzschild coordinates (r, t). As one approaches $r = 2m$, the light cones become narrower and narrower and eventually fold up completely.

As our first step towards introducing coordinates that are more suitable for describing the region around r_S, let us solve equation (13.16). The solution is

$$t = \pm r^* + C \ , \tag{13.18}$$

where the *tortoise coordinate* r^* is

$$r^* = r + 2m \log(r/2m - 1) \ . \tag{13.19}$$

Using the tortoise coordinate r^* instead of r, the Schwarzschild metric reads

$$ds^2 = (1 - 2m/r)(-dt^2 + dr^{*2}) + r^2 d\Omega^2 \ , \tag{13.20}$$

where r is to be thought of as a function of r^*.

We see immediately that we have made some progress. Now the light cones, defined by

$$dt^2 = dr^{*2} \ , \tag{13.21}$$

do not seem to fold up as the light cones have the constant slope $dt/dr^* = \pm 1$ (see Figure 14), and there is no singularity at $r = 2m$. However, r^* is still only defined for $r > 2m$ and the surface $r = 2m$ has been pushed infinitely far away ($r = 2m$ is now at $r^* = -\infty$). Moreover, even though non-singular, the metric components g_{tt} and $g_{r^*r^*}$ vanish at $r = 2m$.

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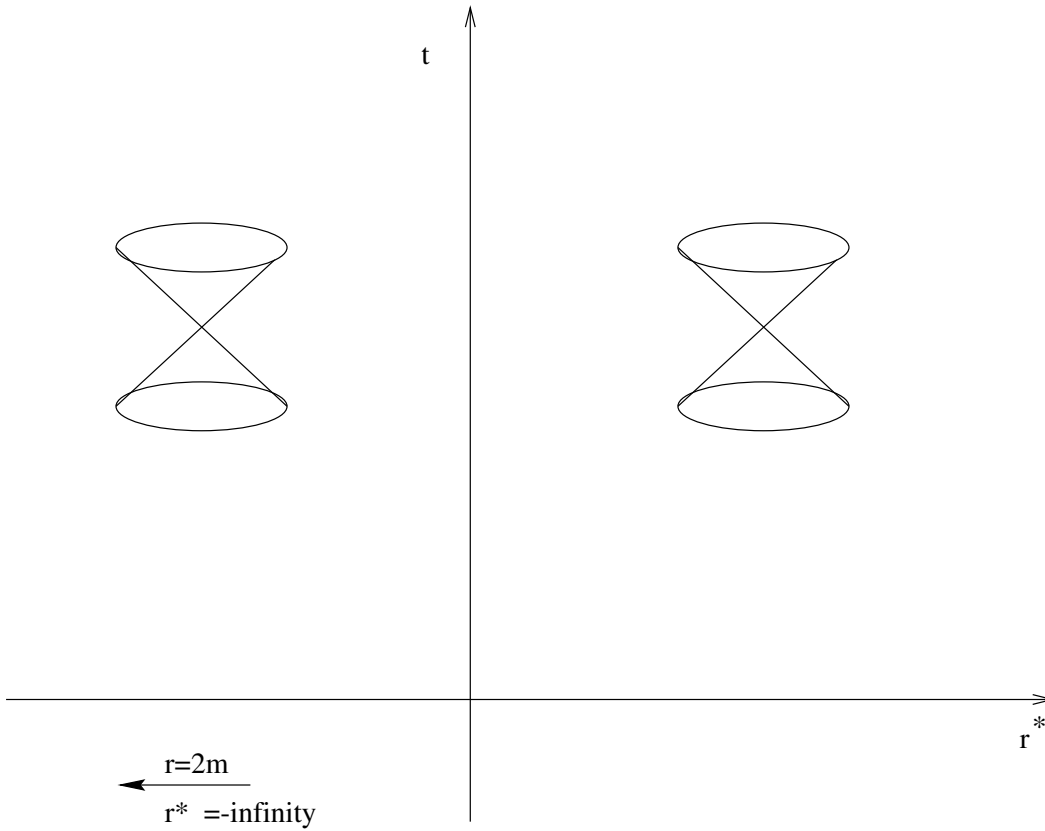
Figure 14: The causal structure of the Schwarzschild geometry in the tortoise coordinates (r^*, t). The light cones look like the light cones in Minkowski space and no longer fold up as $r \to 2m$ (which now sits at $r^* = -\infty$).

Let us introduce coordinates that are naturally adapted to null geodesics, namely

$$
\begin{aligned}
u &= t + r^* \\
v &= t - r^* \ .
\end{aligned}
\tag{13.22}
$$

Then infalling radial null geodesics ($dr^*/dt = -1$) are characterized by $u = const.$ and outgoing radial null geodesics by $v = const.$

Now we pass to the *Eddington-Finkelstein* coordinates (u, r, θ, ϕ) or (v, r, θ, ϕ), in terms of which the Schwarzschild metric reads

$$
\begin{aligned}
ds^2 &= -(1 - 2m/r)du^2 + 2du \ dr + r^2 d\Omega^2 \\
&= -(1 - 2m/r)dv^2 - 2dv \ dr + r^2 d\Omega^2 \ .
\end{aligned}
\tag{13.23}
$$

Even though the metric coefficent g_{uu} or g_{vv} vanishes at $r = 2m$, there is no real degeneracy. The determinant of the metric is

$$
g = r^4 \sin^2 \theta \ ,
\tag{13.24}
$$

which is completely regular at $r = 2m$, and the metric is invertible.

The only real singularity is at $r = 0$. That this is indeed a real singularity can be shown by calculating some invariant of the curvature tensor, like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ which is proportional to r^{-6} (actually, on dimensional grounds, proportional to m^2/r^6). Also, the geodesic deviation equation shows that the force needed to keep neighbouring particles apart is proportional to r^{-3}. Thus the tidal forces within arbitrary objects (solids, atoms, elementary particles) eventually become infinitely big so that these objects will be crushed completely.

In some sense, however, not even the singularity at $r = 0$ is real because the Schwarzschild metric was never meant to be valid there anyway. If there is a very compact star in the interior, then the Schwarzschild metric is not a solution for $r < r_0$, i.e. the interior of the star, and if there is no star (m=0), and one could reach $r = 0$, the solution to the Einstein equations is just Minkowski space.

Thus, even though (for now) the singularity at $r = 0$ is not really worrysome, just being close enough to $r = 0$, without actually reaching that point, is usually more than sufficient to crush any known kind of matter. In that sense again, the physics becomes hopelessly singular even before one reaches $r = 0$ and there seems to be nothing to prevent a collapse of such an object to $r = 0$ and infinite density. Certainly classical mechanics and even current-day quantum field theory are inadequate to describe this situation. If or how a theory of quantum gravity can deal with these matters remains to be seen.

To determine the light cones in the Eddington-Finkelstein coordinates we again look at radial null geodesics which this time are solutions to

$$(1 - 2m/r)du^2 = 2du\ dr\ . \tag{13.25}$$

Thus either $du/dr = 0$ which, as we have seen, describes incoming null geodesics, $u = const.$, or

$$\frac{du}{dr} = 2(1 - 2m/r)^{-1}\ , \tag{13.26}$$

which then describes outgoing null geodesics. Thus the lightcones remain well-behaved (do not fold up) at $r = 2m$, the surface $r = 2m$ is at a finite coordinate distance, namely (to reiterate the obvious) at $r = 2m$, and there is no problem with following geodesics beyond $r = 2m$.

But even though the light cones do not fold up at $r = 2m$, something interesting is certainly hapening there. Whereas, in a (u, r)-diagram (see Figure 15), one side of the light cone always remains horizontal (at $u = const.$), the other side becomes vertical at $r = 2m$ ($du/dr = \infty$) and then tilts over to the other side. In particular, beyond $r = 2m$ all future-directed paths, those within the forward light cone, now have to move in the direction of decreasing r. There is no way to turn back to larger values of r, not on a geodesic but also not on any other path (i.e. not even with a powerful rocket) once one has gone past $r = 2m$.

Thus, even though locally the physics at $r = 2m$ is well behaved, globally the surface $r = 2m$ is very significant as it is a point of no return. Once one has passed the Schwarzschild radius, there is no turning back. Such a surface is known as an *event horizon*. Note that this is a null surface so, in particular, once one has reached the event horizon one has to travel at the speed of light to stay there and not be forced further towards $r = 0$.

In any case, we now encounter no difficulties when entering the region $r < 2m$, e.g. along lines of constant u and this region should be included as part of the physical space-time. Note that because $u = t + r^*$ and $r^* \rightarrow -\infty$ for $r \rightarrow 2m$, we see that decreasing r along lines of constant u amounts to $t \rightarrow \infty$. Thus the new region at $r \leq 2m$ we have discovered is in some sense a future extension of the original Schwarzschild space-time.

Note also that nothing, absolutely nothing, no information, no light ray, no particle, can escape from the region behind the horizon. Thus we have a *Black Hole*, an object that is (classically) completely invisible.

The seeming time-asymmetry we encounter here also shows up in the fact that in the (u, r) coordinate system we can cross the event horizon only on future directed paths, not on past directed ones. The situation is reversed when one uses the coordinates (v, r) instead of (u, r). In that case, the light cones in Figure 15 are flipped (either up-down or left-right), and one can pass through the horizon on past directed curves. The new
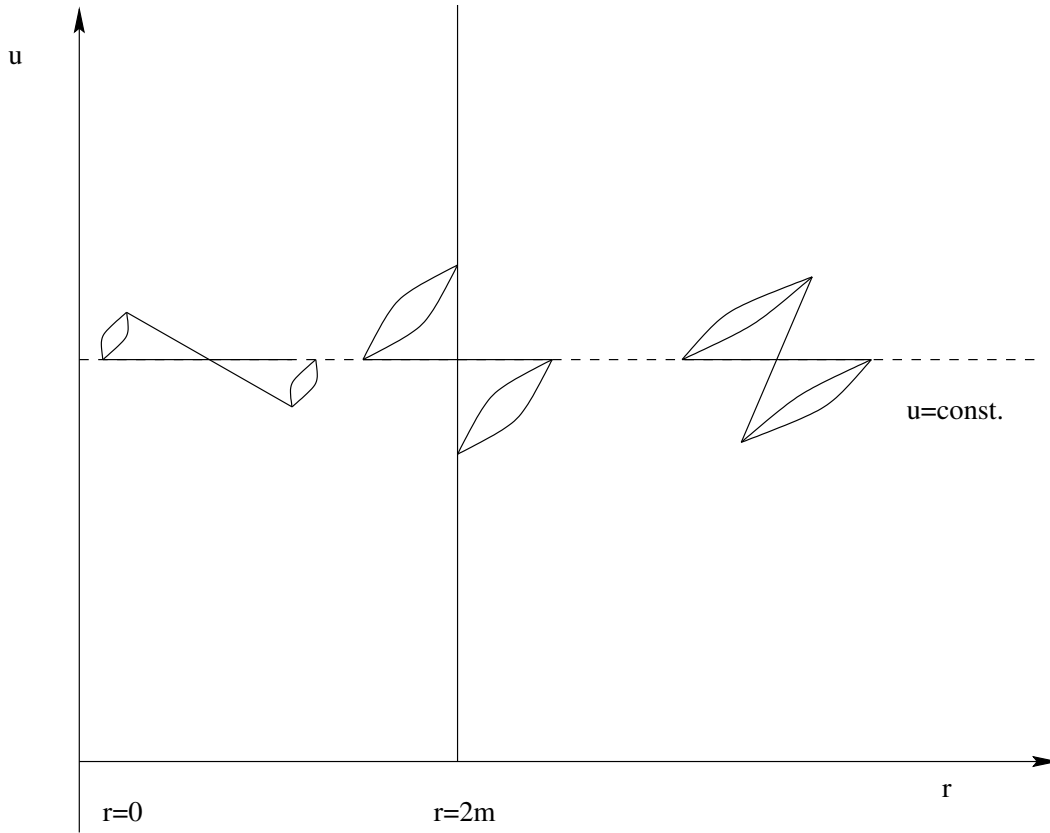
Figure 15: The behaviour of light cones in Eddington-Finkelstein coordinates. Light cones do not fold up at $r = 2m$ but tilt over so that for $r < 2m$ only movement in the direction of decreasing r towards the singularity at $r = 0$ is allowed.
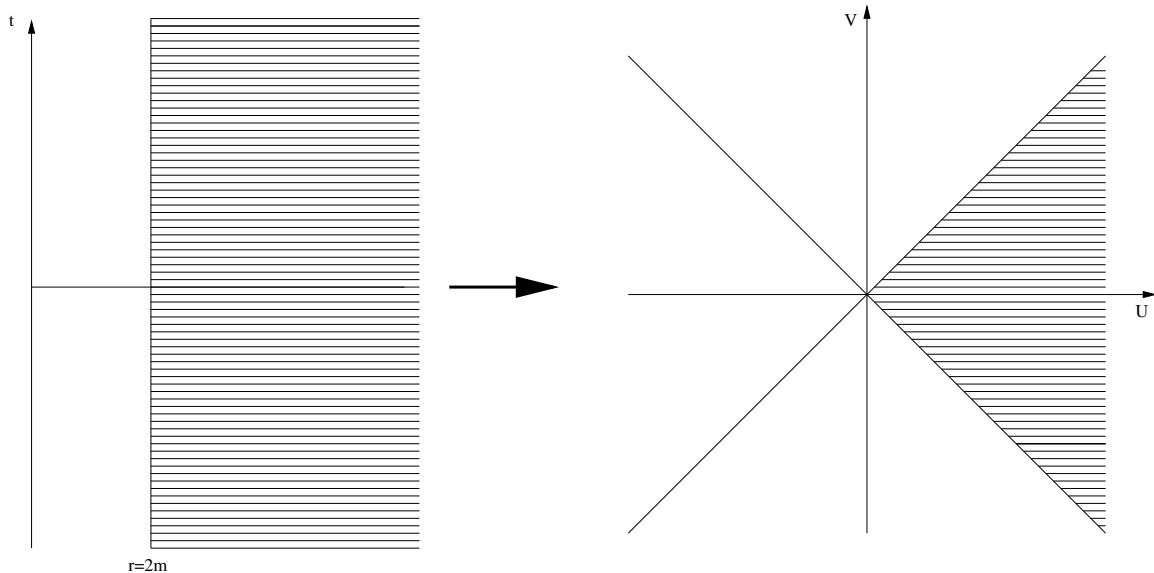
Figure 16: The Schwarzschild patch in the Kruskal metric: the half-plane $r > 2m$ is mapped to the quadrant between the lines $U = \pm V$ in the Kruskal metric.

region of space-time covered by the coordinates (v, r) is definitely different from the new region we uncovered using (u, r) even though both of them lie 'behind' $r = 2m$. In fact, this one is a past extension (beyond $t = -\infty$) of the original Schwarzschild 'patch' of space-time. In this patch, the region behind $r = 2m$ acts like the opposite of a black hole (a white hole) which cannot be entered on any future-directed path.

The Kruskal Metric

Are there still other regions of space-time to be discovered? The answer is yes, and one way to find them would be to study space-like rather than null geodesics. Alternatively, let us try to guess how one might be able to describe the maximal extension of space-time. The first guess might be to use the coordinates u and v simultaneously, instead of r and t. In these coordinates, the metric takes the form

$$ds^2 = -(1 - 2m/r)du\ dv + r^2 d\Omega^2 \ . \tag{13.27}$$

But while this is a good idea, the problem is that in these coordinates the horizon is once again infinitely far away, at $u = -\infty$ or $v = +\infty$. We can rectify this by introducing coordinates u' and v' with

$$
\begin{aligned}
u' &= \ \mathrm{e}^{u/4m} \\
v' &= \ \mathrm{e}^{-v/4m} \ ,
\end{aligned}
\tag{13.28}
$$

so that the horizon is now at either $u' = 0$ or $v' = 0$. And indeed in these coordinates the metric is completely non-singular and regular *everywhere except at $r = 0$*. In fact,

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one has

$$ds^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m}du'dv' + r^2d\Omega^2 \quad . \tag{13.29}$$

Finally, we pass from the null coordinates (u', v') (meaning that $\partial_{u'}$ and $\partial_{v'}$ are null vectors) to more familiar time-like and space-like coordinates (V, U) defined by

$$
\begin{aligned}
U &= \tfrac{1}{2}(u' + v') = (r/2m - 1)^{1/2}\mathrm{e}^{r/4m}\cosh t/4m \\
V &= \tfrac{1}{2}(u' - v') = (r/2m - 1)^{1/2}\mathrm{e}^{r/4m}\sinh t/4m \quad ,
\end{aligned}
\tag{13.30}
$$

in terms of which the metric is

$$ds^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m}(-dV^2 + dU^2) + r^2d\Omega^2 \quad . \tag{13.31}$$

Here r^2 is implicitly given by

$$U^2 - V^2 = (r/2m - 1)\mathrm{e}^{r/2m} \quad . \tag{13.32}$$

As in Minkowski space, null lines are given by $U = \pm V + const.$. The horizon is now at the null surfaces $U = \pm V$. Surfaces of constant r are given by $U^2 - V^2 = const.$ and the singularity at $r = 0$ corresponds to the two sheets of the hyperboloid

$$V^2 - U^2 = 1 \quad . \tag{13.33}$$

We can now let the coordinates (U, V) range over all the values for which the metric is non-singular. This corresponds to $V^2 - U^2 < 1$ with the region between $V^2 - U^2 = 0$ and $V^2 - U^2 = 1$ corresponding to $0 < r < 2m$.

It can be shown that the above represents the maximal analytical extension of the Schwarzschild metric in the sense that every affinely parametrized geodesic can either be continued to infinite values of its parameter or runs into the singularity at $r = 0$ at some finite value of the affine parameter. It was discovered by Kruskal in 1960 and presents us with an amazingly rich and complex picture of what originally appeared to be a rather innocent (and very simple) solution to the Einstein equations.

As Figure 16 shows, the original 'Schwarzschild patch' of the Schwarzschild solution, i.e. the half-plane $r > 2m$ which was the regime of validity of the Schwarzschild coordinates, is mapped to the first quadrant of the Kruskal metric, bounded by the lines $U = \pm V$ which correspond to $r = 2m$. But now that we have the coordinates U and V, and the metric (and thus the physics) is non-singular for all values of (U, V) subject to the constraint $r > 0$ or $V^2 - U^2 < 1$, there is no physical reason to exclude the regions in the other quadrants also satisfying this condition.

By including them, we obtain the *Kruskal diagram* Figure 17. In addition to the Schwarzschild patch, quadrant I, we have three other regions, living in the quadrants
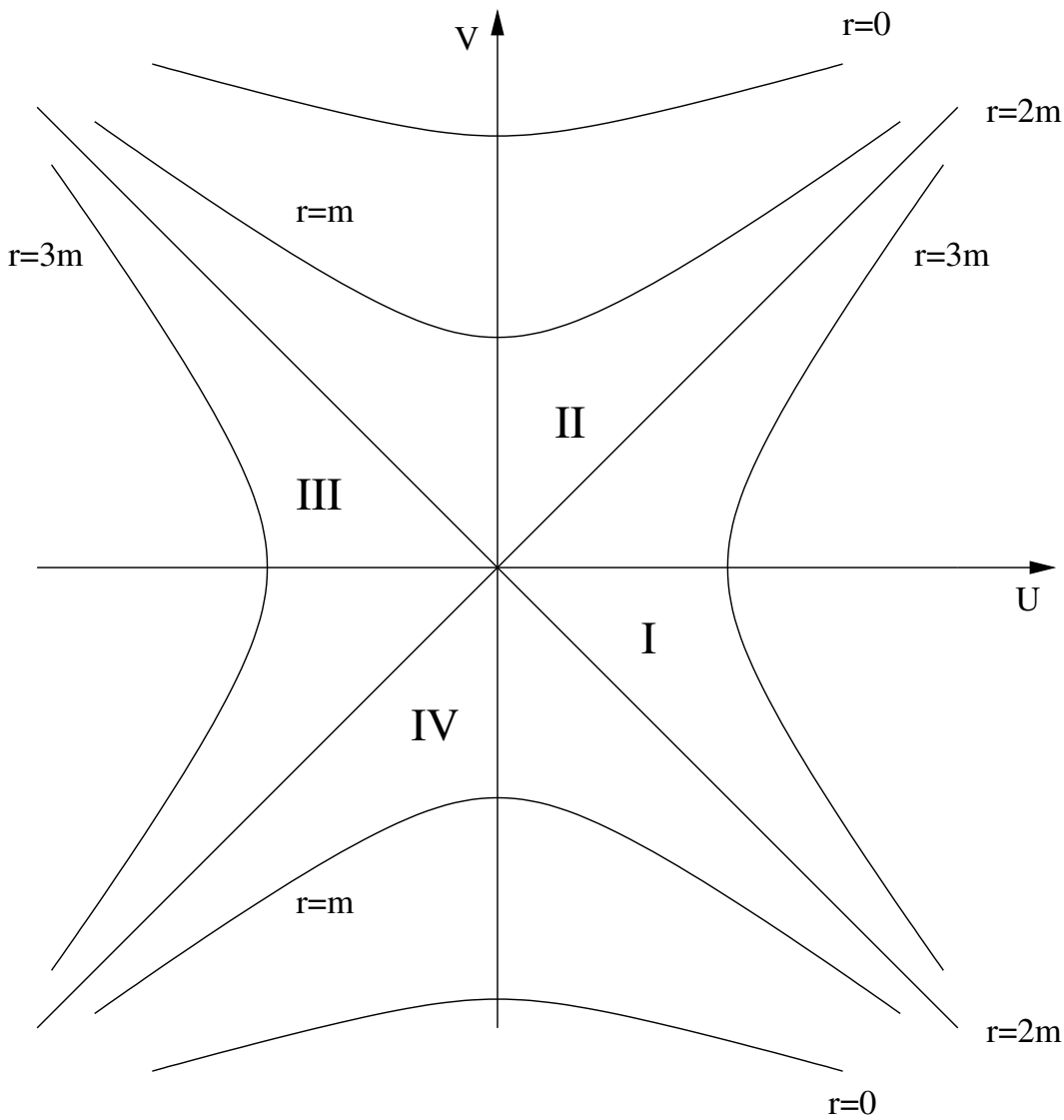
Figure 17: The complete Kruskal universe. Diagonal lines are null, lines of constant r are hyperbolas. Region I is the Schwarzschild patch, seperated by the horizon from regions II and IV. The Eddington-Finkelstein coordinates (u, r) cover regions I and II, (v, r) cover regions I and IV. Regions I and III are filled with lines of constant $r > 2m$. They are causally disconnected. Observers in regions I and III can receive signals from region IV and send signals to region II. An observer in region IV can send signals into both regions I and III (and therefore also to region II) and must have emerged from the singularity at $r = 0$ at a finite proper time in the past. Any observer entering region II will be able to receive signals from regions I and III (and therefore also from IV) and will reach the singularity at $r = 0$ in finite time. Events occuring in region II cannot be observed in any of the other regions.

II, III, and IV, each of them having its own peculiarities. Note that obviously the conversion formulae from $(r,t) \to (U,V)$ in the quadrants II, III and IV differ from those given above for quadrant I. E.g. in region II one has

$$
\begin{aligned}
U &= (1 - r/2m)^{1/2} e^{r/4m} \sinh t/4m \\
V &= (1 - r/2m)^{1/2} e^{r/4m} \cosh t/4m \ .
\end{aligned}
\tag{13.34}
$$

To get acquainted with the Kruskal diagram, let us note the following basic facts:

1. Null lines are diagonals $U = \pm V + const.$, just as in Minkowski space. This greatly facilitates the exploration of the causal structure of the Kruskal metric.

2. In particular, the horizon corresponds to the two lines $U = \pm V$.

3. Lines of constant r are hyperbolas. For $r > 2m$ they fill the quadrants I and III, for $r < 2m$ the other regions II and IV.

4. In particular, the singularity at $r = 0$ is given by the two sheets of the hyperbola $V^2 - U^2 = 1$.

5. Notice in particular also that in regions II and IV worldlines with $r = const.$ are no longer time-like but space-like.

6. The Eddington-Finkelstein coordinates (u, r) cover the regions I and II, the coordinates (v, r) the regions I and IV.

7. Quadrant III is completely new and is seperated from region I by a space-like distance. That is, regions I and III are causally disconnected.

Now let us see what all this tells us about the physics of the Kruskal metric. An observer in region I (the familiar patch) can send signals into region II and receive signals from region IV. The same is true for an observer in the causally disconnected region III. Once an observer enters region II from, say, region I, he cannot escape from it anymore and he will run into the catastrophic region $r = 0$ in finite proper time. As a reward for his or her foolishness, between having crossed the horizon and being crushed to death, our observer will for the first time be able to receive signals and meet observers emerging from the mirror world in region III. Events occurring in region II cannot be observed anywhere outside that region (black hole). Finally, an observer in region IV must have emerged from the (past) singularity at $r = 0$ a finite proper time ago and can send signals and enter into either of the regions I or III.

Another interesting aspect of the Kruskal geometry, but one I wil not go into here, is its dynamical character. This may appear to be a strange thing to say since we explicitly started off with a static metric. But this statement applies only to region I (and its mirror III). An investigation of the behaviour of space-like slices analogous to that we

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performed at the end of section 11 for region I (see Figure 8) reveals a dynamical picture of continuing gravitational collapse in region II. In simple terms, the loss of staticity can be understood by noting that the time-like Killing vector field ∂_t of region I, when expressed in terms of Kruskal coordinates, becomes null on the horizon and space-like in region II. Indeed it is easy to check that in terms of Kruskal coordinates one has

$$\partial_t = (U\partial_V + V\partial_U)/4m \ ,$$ (13.35)

and that this vector field has norm proportional to $V^2 - U^2$. Hence ∂_t is time-like in region I, null on the horizon and space-like in region II. Thus region II has no time-like Killing vector field, therefore cannot possibly be static, but has instead an additional space-like Killing vector field.

* Varia on Black Holes and Gravitational Collapse

Now you may well wonder if all this is for real or just science fiction. Clearly, if an object with $r_0 < 2m$ exists and is described by the Schwarzschild solution, then we will have to accept the conclusions of the previous section. However, this requires the existence of an eternal black hole (in particlar, eternal in the past) in an asymptotically flat space-time, and this is not very realistic. While black holes are believed to exist, they are believed to form as a consequence of the gravitational collapse of a star whose nuclear fuel has been exhausted (and which is so massive that it cannot settle into a less singular final state like a *White Dwarf* or *Neutron Star*).

To see how we could picture the situation of gravitational collapse (without trying to understand why this collapse occurs in the first place), let us estimate the average density ρ of a star whose radius r_0 is equal to its Schwarzschild radius. For a star with mass M we have

$$r_S = \frac{2MG}{c^2}$$ (13.36)

and approximately

$$M = \frac{4\pi r_0^3}{3}\rho \ .$$ (13.37)

Therefore, setting $r_0 = r_S$, we find that

$$\rho = \frac{3c^6}{32\pi G^3 M^2} = 2 \times 10^{16} \text{ g/cm}^3 \left(\frac{M_{\text{sun}}}{M}\right)^2 \ .$$ (13.38)

For stars of a few solar masses, this density is huge, roughly that of nuclear matter. In that case, there will be strong non-gravitational forces and hydrodynamic processes, singificantly complicating the description of the situation. The situation is quite simple, however, when an object of the mass and size of a galaxy ($M \sim 10^{10} M_{\text{sun}}$) collapses. Then the critical density (13.38) is approximately that of air, $\rho \sim 10^{-3}$ g/cm^3, non-gravitational forces can be neglected completely, and the collapse of the object can be approximated by a free fall.

Under these circumstances, a more realistic Kruskal-like space-time diagram of a black hole would be the one depicted in Figure 18. We assume that at time $t = 0$ ($V = 0$) we have a momentarily static mass configuration with radius $R \gg 2m$ and mass M which then starts to collapse in free fall. Neglecting radiation-effects, the mass M of the star (galaxy) remains constant so that the exterior of the star is described by the corresponding subset of region I of the Kruskal metric. Note that regions III and IV no longer exist. Region IV has disappeared because there is no singularity in the past, and region III cannot be reached even on space-like curves because the star is in the way.

The surface of the star can be represented by a time-like geodesic going from $r = R$ at $t = 0$ to $r = 0$. It will reach $r = 0$ after the finite proper time $\tau = \pi (R^3/8M)^{1/2}$ (Exercise). For an object the size of the sun this is approximately one hour! Note that, even if the free fall (geodesic) approximation is no longer justified at some point, once the surface of the star has crossed the Schwarzschild horizon, nothing, no amount of pressure, can stop the catastrophic collapse to $r = 0$ because, whatever happens, points on the surface of the star will have to move within their forward light-cone and will therefore inevitably end up at $r = 0$.

For an observer remaining outside, say at the constant value $r = R$, the situation presents itself in a rather different way. Up to a constant factor $(1 - 2m/R)$, his proper time equals the coordinate time t. As the surface of the collapsing galaxy crosses the horizon at $t = \infty$, strictly speaking the outside observer will never see the black hole form.

We had already encountered a similar phenomenon in our discussion of the infinite gravitational red-shift. However, the gravitational red-shift grows exponentially with time, $\sim \exp t/4m$ for radially emitted photons. The luminosity L of the star decreases exponentially, as a consequence of the gravitational red-shift and the fact that photons emitted at equal time intervals from the surface of the star reach the observer at greater and greater time intervals. It can be shown that

$$L \sim e^{-t/3\sqrt{3}m} \ , \tag{13.39}$$

so that the star becomes very dark very quickly, the characteristic time being of the order of

$$t \sim 3\sqrt{3}m \sim 2,5 \times 10^{-5}\text{s} \ \left(\frac{M}{M_{\text{sun}}}\right) \ . \tag{13.40}$$

Thus, even though for an outside observer the collapsing star never disappears completely, for all practical intents and purposes the star is black and the name 'black hole' is justified.

It is fair to wonder at this point if the above conclusions regarding the collapse to $r = 0$ are only a consequence of the fact that we assumed exact spherical symmetry. Would the singularity be avoided under more general conditions? The answer to this is, somewhat surprisingly and schockingly, a clear 'no'.

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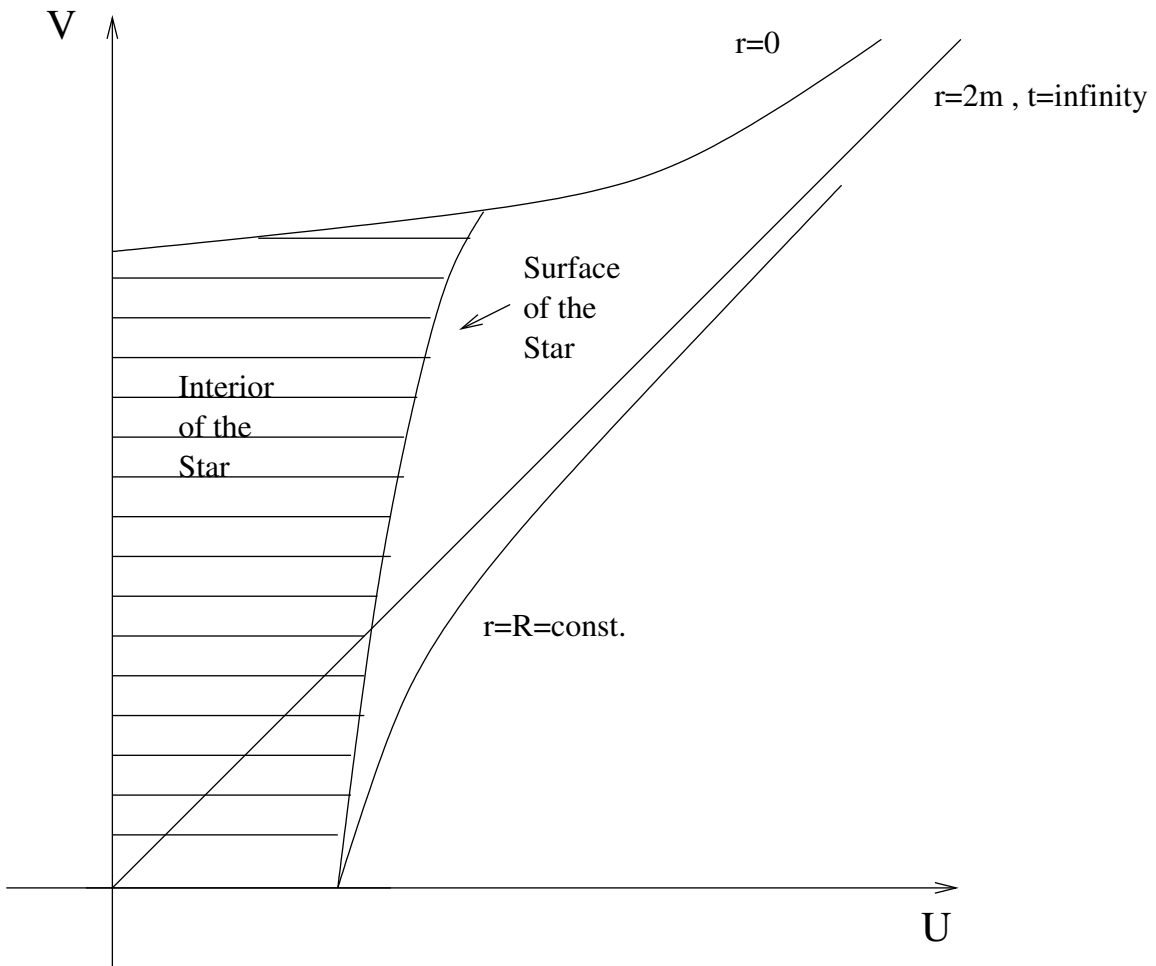
Figure 18: The Kruskal diagram of a gravitational collapse. The surface of the star is represented by a time-like geodesic, modelling a star (or galaxy) in free fall under its own gravitational force. The surface will reach the singularity at $r = 0$ in finite proper time whereas an outside observer will never even see the star collapse beyond its Schwarzschild radius. Howeer, as discussed in the text, even for an outside observer the resulting object is practically 'black'.

On the one hand, there are very general *singularity theorems*, due to Penrose, Hawking and others, which all state in one way or another that if Einstein's equations hold, the energy-momentum tensor satisfies some kind of positivity condition, and there is a regular event horizon, then a singularity will appear. These theorems do not rely on any symmetry assumptions.

On the other hand, it has also been shown that the gravitational field of a static black hole, even without further symmetry assumptions, is necessarily given by the spherically symmetric Schwarzschild metric and is thus characterized by the single parameter M.

Of course, other exact solutions describing isolated systems like a star, meaning that the solution is asymptotically flat, are known. Two important examples are the following:

1. The Reissner-Nordstrøm Metric

   The Reissner-Nordstrøm metric is a solution to the coupled Einstein-Maxwell equations describing the gravitational field of a spherically symmetric electrically charged star. It is characterized by two parameters, its mass M and its charge Q, with $F_{tr} = -Q/r^2$, and the metric is

   $$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \ . \qquad (13.41)$$

   Note that this can be obtained from the Schwarzschild metric by substituting

   $$M \to M - \frac{Q^2}{2r} \ . \qquad (13.42)$$

   The structure of the singularities and event horizons is more complicated now than in the case of the Schwarzschild metric and also depends on the relative size of Q and M.

   If $Q^2 > M^2$ (this is not a very realistic situation), then the metric is non-singular everywhere except, of course, at $r = 0$. In particular, the coordinate t is always time-like and the coordinate r is always space-like. While this may sound quite pleasing, much less insane than what happens for the Schwarzschild metric, this is actually a disaster. The singularity at $r = 0$ is now time-like, and it is not protected by an event-horizon. Such a singularity is known as a *naked singularity*. An observer could travel to the singularity and come back again. Worse, whatever happens at the singularity can influence the future physics away from the singularity, but as there is a singularity this means that the future cannot be predicted/calculated in such a space-time because the laws of physics break down at $r = 0$. There is a famous conjecture, known as the *Cosmic Censorship Conjecture*, which roughly speaking states that the collpase of physically realistic matter configurations will never lead to a naked singularity. In spite of a lot of partial results and circumstantial evidence in favour of this conjecture, it is not known if it holds in General Relativity.

The situation is even more interesting in the somewhat more realistic case $M^2 > Q^2$. In that case, there are two radii

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \qquad (13.43)$$

at which the metric becomes singular. The outer one is just like the event horizon of the Schwarzschild metric, the inner one reverses the role of radius and time once more so that the singularity is time-like and can be avoided by returning to larger values of r. There is much more that can and should be said about this solution butI will not do this here.

2. The Kerr Metric

The Kerr metric describes a rotating black hole and is characterized by its mass M and its angular momentum J. Now one no longer has spherical symmetry (because the axis of rotation picks out a particular direction) but only axial symmetry. The situation is thus a priori much more complicated. A stationary solution (i.e. one with a time-like Killing vector, 'static' is a slightly stronger condition) was found by Kerr only in 1963, almost fifty years after the Schwarzschild and Reissner-Nordstrøm solutions. Its singularity and horizon structure is much more intricate and intriguing than that of the solutions discussed before. One can pass from one universe into a different asymptotically flat universe. The singularity at $r = 0$ has been spread out into a ring; if one enters into the ring, one can not only emerge into a different asymptotically flat space-time but one can also turn back in time (there are closed time-like curves), one can dip into the black hole and emerge with more energy than one had before (at the expense of the angular momentum of the black hole), etc. etc. All this is fun but also rather technical and I will not go into any of this here.

Of course there are also solutions describing a combination of the two above solutions, namely charged rotating black holes (the Kerr-Newman metric). One of the reasons why I mention these solutions is that it can be shown that the most general stationary electrically charged black hole is characterized by just three parameters, namely M, Q and J. This is generally referred to as the fact that *black holes have no hair*, or as the *no-hair theorem*. It roughly states that the only characteristics of a black hole which are not somehow radiated away during the phase of collapse via multipole moments of the gravitational, electro-magnetic, ... fields are those which are protected by some conservation laws, something that in simple cases can be confirmed by an explicit calculation.

Preliminary Remarks

We now turn away from considering isolated systems (stars) to some (admittedly very idealized) description of the universe as a whole. This subject is known as Cosmology. It is certainly one of the most fascinating subjects of theoretical physics, dealing with such issues as the origin and ultimate fate and the large-scale structure of the universe.

Due to the difficulty of performing cosmological experiments and making precise measurements at large distances, many of the most basic questions about the universe are still unanswered today:

1. Is our universe open or closed?

2. Will it keep expanding forever or will it recollapse?

3. Why is the Cosmic Microwave Background radiation so isotropic?

4. What is the mechanism responsible for structure formation in the universe?

5. Where is the 'missing mass'?

6. Why is the cosmological constant so small and what is its value?

Of course, we cannot study any of these questions in detail, in particular because an important role in studying these questions is played by the interaction of cosmology with astronomy, astrophysics and elementary particle physics, each of these subjects deserving at least a course of its own.

Fortunately, however, many of the important features any realistic cosmological model should display are already present in some very simple models, the so-called *Friedmann-Robertson-Walker Models* already studied in the 20's and 30's of this century. They are based on the simplest possible ansatz for the metric compatible with the assumption that on large scales the universe is roughly homogeneous and isotropic (cf. the next section for a more detailed discussion of this *Cosmological Principle*) and have become the 'standard model' of cosmology.

We will see that they already display all the essential features such as

1. a Big Bang

2. expanding universes (Hubble expansion)

3. different long-term behaviour (eternal expansion versus recollapse)

4. and the cosmological red-shift.

Our first aim will be to make maximal use of the symmetries that simple cosmological models should have to find a simple ansatz for the metric. Our guiding principle will be ...

<div align="center">THE COSMOLOGICAL PRINCIPLE</div>

At first, it may sound impossibly difficult to find solutions of the Einstein equations describing the universe as a whole. But: If one looks at the universe at large (very large) scales, in that process averaging over galaxies and even clusters of galaxies, then the situation simplifies a lot in several respects;

1. First of all, at those scales non-gravitational interactions can be completely ignored because they are either short-range (the nuclear forces) or compensate each other at large distances (electro-magnetism).

2. The earth, and our solar system, or even our galaxy, have no privileged position in the universe. This means that at large scales the universe should look the same from any point in the universe. Mathematically this means that there should be translational symmetries from any point of space to any other, in other words, space should be homogeneous.

3. Also, we assume that, at large scales, the universe looks the same in all directions. Thus there should be rotational symmetries and hence space should be isotropic.

It is almost evident from what we already know about Killing vectors, that this implies that the n-dimensional space (of course $n = 3$ for us) has n translational and $n(n-1)/2$ rotational Killing vectors, i.e. the maximal possible number of Killing vectors. Such a space is called *maximally symmetric*. For $n = 3$, we will thus have six Killing vectors, two more than for the Schwarzschild metric, and the ansatz for the metric will simplify accordingly.

Note that since we know from observation that the universe expands, we do not require a maximally symmetric space-time as this would imply that there is also a time-like Killing vector and the resulting model for the universe would be static.

What simplifies life considerably is the fact, we will establish below, that there are essentially only three kinds of maximally symmetric spaces (for any n), namely flat space $\mathbb{R}^n$, the sphere S^n, and its negatively curved counterpart, the n-dimensional pseudosphere or hyperboloid we will call H^n.

Thus, for a space-time metric with maximally symmetric space-like 'slices', the only unknown is the time-dependence of the metric. More concretely, we will see that under the assumptions of maximal symmetry the metric can be chosen to be

$$ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\big) \ , \tag{14.1}$$

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where $k = 0, \pm 1$ corresponds to the three possibilities mentioned above. Thus the metric contains only one unknown function, the 'radius' or cosmic scale factor $a(t)$. This function will be determined by the Einstein equations via the matter content of the universe (we will of couse be dealing with a non-vanishing energy-momentum tensor) and the equation of state for the matter.

<div align="center">HOMOGENEOUS, ISOTROPIC AND MAXIMALLY SYMMETRIC SPACES</div>

We have seen that Killing vectors $K^\mu(x)$ are determined by the values $K^\mu(x_0)$ and $\nabla_\mu K_\nu(x_0)$ at a single point x_0. We will now see how these data are related to translations and rotations.

We define a *homogeneous space* to be such that it has infinitesimal isometries that carry any given point x_0 into any other point in its immediate neighbourhood (this could be stated in more fancy terms!). Thus the metric must admit Killing vectors that, at any given point, can take all possible values. Thus we require the existence of Killing vectors for arbitrary $K_\mu(x_0)$.

We define a space to be *isotropic at a point* x_0 if it has isometries that leave the given point x_0 fixed and such that they can rotate any vector at x_0 into any other vector at x_0. Therefore the metric must admit Killing vectors such that $K_\mu(x_0) = 0$ but such that $\nabla_\mu K_\nu(x_0)$ is an arbitrary antisymmetric matrix (for instance to be thought of as an element of the Lie algebra of $SO(n)$).

Finally, we define a *maximally symmetric space* to be a space with a metric with the maximal number $n(n+1)/2$ of Killing vectors.

Some simple and fairly obvious consequences of these definitions are the following:

1. A homogeneous and isotropic space is maximally symmetric.

2. A space that is isotropic for all x is also homogeneous. (This follows because linear combinations of Killing vectors are again Killing vectors and the difference between two rotational Killing vectors at x and $x + dx$ can be shown to be a translational Killing vector.)

3. (1) and (2) now imply that a space which is isotropic around every point is maximally symmetric.

4. Finally one also has the converse, namely that a maximally symmetric space is homogeneous and isotropic.

In practice the characterization of a maximally symmetric space which is easiest to use is (3) because it only requires consideration of one type of symmetries, namaly rotational symmetries.

On the basis of these simple considerations we can already determine the form of the Riemann curvature tensor of a maximally symmetric space. We will see that maximally symmetric spaces are spaces of constant curvature in the sense that

$$R_{ijkl} = k(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{14.2}$$

for some constant k.

This result could be obtained by making systematic use of the higher order integrability conditions for the existence of a maximal number of Killing vectors. The argument given below is less covariant but more elementary.

Assume for starters that the space is isotropic at x_0 and choose a Riemann normal coordinate system centered at x_0. Thus the metric at x_0 is $g_{ij}(x_0) = \eta_{ij}$ where we may just as well be completely general and assume that

$$\eta_{ij} = \text{diag}(\underbrace{-1, \ldots, -1}_{p \text{ times}} \underbrace{+1, \ldots, +1}_{q \text{ times}}) \ , \tag{14.3}$$

where $p + q = n$ and we only assume $n > 2$.

If the metric is supposed to be isotropic at x_0 then, in particular, the curvature tensor at the origin must be invariant under Lorentz rotations. Now we know that the only invariants of the Lorentz group are the Minkowski metric and products thereof, and the totally antisymmetric epsilon-symbol. Thus the Riemann curvature tensor has to be of the form

$$R_{ijkl}(x_0) = a\eta_{ij}\eta_{kl} + b\eta_{ik}\eta_{jl} + c\eta_{il}\eta_{jk} + d\epsilon_{ijkl} \ , \tag{14.4}$$

where the last term is only possible for $n = 4$. The symmetries of the Riemann tensor imply that $a = d = b + c = 0$, and hence we are left with

$$R_{ijkl}(x_0) = b(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) \ , \tag{14.5}$$

Thus in an arbitrary coordinate system we will have

$$R_{ijkl}(x_0) = b(g_{ik}(x_0)g_{jl}(x_0) - g_{il}(x_0)g_{jk}(x_0)) \ , \tag{14.6}$$

If we now assume that the space is isotropic around every point, then we can deduce that

$$R_{ijkl}(x) = b(x)(g_{ik}(x)g_{jl}(x) - g_{il}(x)g_{jk}(x)) \tag{14.7}$$

for some function $b(x)$. Therefore the Ricci tensor and the Ricci scalar are

$$\begin{aligned} R_{ij}(x) &= (n-1)b(x)g_{ij} \\ R(x) &= n(n-1)b(x) \ . \end{aligned} \tag{14.8}$$

and the Riemann curvature tensor can also be written as

$$R_{ijkl} = \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \ , \tag{14.9}$$

while the Einstein tensor is

$$G_{ij} = b[(n-1)(1-n/2)]g_{ij} \ . \tag{14.10}$$

The contracted Bianchi identity $\nabla^i G_{ij} = 0$ now implies that $b(x)$ has to be a constant, and we have thus established (14.2). Note that we also have

$$R_{ij} = k(n-1)g_{ij} \ , \tag{14.11}$$

so that a maximally symmetric space(-time) is automatically a solution to the vacuum Einstein equations with a cosmological constant. In the physically relevant case $p = 1$ these are known as de Sitter or anti de Sitter space-times. We will come back to them later on. In general, solutions to the equation $R_{ij} = cg_{ij}$ for some constant c are known as *Einstein manifolds* in the mathematics literature.

The Metric of a Maximally Symmetric Space I

We are interested not just in the curvature tensor of a maximally symmetric space but in the metric itself. I will give you two derivations of the metric of a maximally symmetric space, one by directly solving the differential equation

$$R_{ij} = k(n-1)g_{ij} \tag{14.12}$$

for the metric g_{ij}, the other by a direct geometrical construction of the metric which makes the isometries of the metric manifest.

As a maximally symmetric space is in particular spherically symmetric, we already know that we can write its metric in the form

$$ds^2 = B(r)dr^2 + r^2 d\Omega^2_{(n-1)} \ , \tag{14.13}$$

where $d\Omega^2_{(n-1)} = d\theta^2 + \dots$ is the volume-element for the $(n-1)$-dimensional sphere or its counterpart in other signatures. For concreteness, we now fix on $n = 3$, but the argument given below goes through in general.

We have already calculated all the Christoffel symbols for such a metric (set $A(r) = 0$ in the calculations leading to the Schwarzschild metric in section 11), and we also know that $R_{ij} = 0$ for $i \neq j$ and that all the diagonal angular components of the Ricci tensor are determined by $R_{\theta\theta}$ by spherical symmetry. Hence we only need R_{rr} and $R_{\theta\theta}$, which are

$$
\begin{aligned}
R_{rr} &= \frac{1}{r}\frac{B'}{B} \\
R_{\theta\theta} &= -\frac{1}{B} + 1 + \frac{rB'}{2B^2} \ ,
\end{aligned} \tag{14.14}
$$

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and we want to solve the equations

$$
\begin{aligned}
R_{rr} &= 2kg_{rr} = 2kB(r) \\
R_{\theta\theta} &= 2kg_{\theta\theta} = 2kr^2 \ .
\end{aligned}
\tag{14.15}
$$

From the first equation we obtain

$$
B' = 2krB^2 \ ,
\tag{14.16}
$$

and from the second equation we deduce

$$
\begin{aligned}
2kr^2 &= -\frac{1}{B} + 1 + \frac{rB'}{2B^2} \\
&= -\frac{1}{B} + 1 + \frac{2kr^2 B^2}{2B^2} \\
&= -\frac{1}{B} + 1 + kr^2 \ .
\end{aligned}
\tag{14.17}
$$

This is an algebraic equation for B solved by

$$
B = \frac{1}{1 - kr^2}
\tag{14.18}
$$

(and this also solves the first equation). Therefore we have determined the metric of a a maximally symmetric space to be

$$
ds^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2_{(n-1)} \ .
\tag{14.19}
$$

Let us pass back from polar coordinates to Cartesian coordinates, with $r^2 = \vec{x}^2 = \eta_{ij} x^i x^j$. Then we have $r\,dr = \vec{x}.d\vec{x}$ and $dr^2 = (\vec{x}.d\vec{x})^2/\vec{x}^2$. Hence this metric can also be written as

$$
ds^2 = d\vec{x}^2 + \frac{k(\vec{x}.d\vec{x})^2}{1 - k\vec{x}^2} \ .
\tag{14.20}
$$

Clearly, for $k = 0$ this is just the flat metric on $\mathbb{R}^{p,q}$. For $k = 1$, this should also look familiar as the standard metric on the sphere. We will discuss the $k \neq 0$ metrics from this point of view in the next section. This will make the isometries of the metric manifest and will also exclude the possibility, not logically ruled out by the arguments given so far, that the metrics we have found here for $k \neq 0$ are spherically symmetric and have constant Ricci curvature but are not actually maximally symmetric.

The Metric of a Maximally Symmetric Space II

Recall that the standard metric on the n-sphere can be obtained by restricting the flat metric on an ambient $\mathbb{R}^{n+1}$ to the sphere. We will generalize this construction a bit to allow for $k < 0$ and other signatures as well.

Consider a flat auxiliary vector space V of dimension $(n+1)$ with metric

$$ds^2 = d\vec{x}^2 + \frac{1}{k}dz^2 \quad , \tag{14.21}$$

whre $\vec{x} = (x^1, \ldots, x^n)$ and $d\vec{x}^2 = \eta_{ij}dx^i dx^j$. Thus the metric on V has signature $(p, q+1)$ for k positive and $(p+1, q)$ for k negative. The group $G = SO(p, q+1)$ or $G = SO(p+1, q)$ has a natural action on V by isometries of the metric.

Now consider in V the hypersurface Σ defined by

$$k\vec{x}^2 + z^2 = 1 \quad . \tag{14.22}$$

The group G leaves Σ invariant and therefore G will be the group of isometries of the induced metric on Σ. But $\dim G = n(n+1)/2$. Hence the n-dimensional space has $n(n+1)/2$ Killing vectors and is therefore maximally symmetric. In fact, G acts transitively on Σ (thus Σ is homogeneous) and the stabilizer at a given point is isomorphic to $H = SO(p, q)$ (so Σ is isotropic), and therefore Σ can also be described as the homogeneous space

$$\begin{aligned} \Sigma_{k>0} &= SO(p, q+1)/SO(p, q) \\ \Sigma_{k<0} &= SO(p+1, q)/SO(p, q) \quad . \end{aligned} \tag{14.23}$$

The Killing vectors of the induced metric are simply the restriction to Σ of the standard generators of G on the vector space V.

It just remains to determine explicitly this induced metric. For this we start with the defining relation of Σ and differentiate it to find that on Σ one has

$$dz = -\frac{k\vec{x}.d\vec{x}}{z} \quad , \tag{14.24}$$

so that

$$dz^2 = \frac{k^2(\vec{x}.d\vec{x})^2}{1 - k\vec{x}^2} \quad . \tag{14.25}$$

Thus the metric (14.21) restricted to Σ is

$$\begin{aligned} ds^2|_\Sigma &= d\vec{x}^2 + \frac{1}{k}dz^2|_\Sigma \\ &= d\vec{x}^2 + \frac{k(\vec{x}.d\vec{x})^2}{1 - k\vec{x}^2} \quad . \end{aligned} \tag{14.26}$$

This is precisely the same metric as we obtained in the previous section.

For Euclidean signature, these spaces are spheres and hyperspheres (hyperboloids), and in other signatures they are the corresponding generalizations. In particular, for $(p, q) = (1, n-1)$ we obtain *de Sitter space-time* for $k = 1$ and *anti de Sitter space-time* for $k = -1$. They have the topology of $S^{n-1} \times \mathbb{R}$ and $\mathbb{R}^{n-1} \times S^1$ respectively and, as mentioned before, they solve the vacuum Einstein equations with a positive (negative) cosmological constant.

Finally, it will be useful to see the maximally symmetric metrics in some other coordinate systems. For $k = 0$, there is nothing new to say since this is just the flat metric. Thus we focus on $k \neq 0$.

First of all, let us note that essentially only the sign of k matters as $|k|$ only effects the overall size of the space and nothing else (and can therefore be absorbed in the scale factor $a(t)$ of the metric (14.1)). To see this note that by rescaling of r, $r' = |k|^{1/2} r$, the metric (14.19) can be put into the form

$$ds^2 = \frac{1}{|k|} \Big(\frac{dr'^2}{1 \pm r'^2} + r'^2 d\Omega^2_{(n-1)} \Big) \ . \tag{14.27}$$

Thus we will just need to consider the cases $k = \pm 1$.

For $k = +1$, we have

$$ds^2 = \frac{dr^2}{1 - r^2} + r^2 d\Omega^2_{(n-1)} \ . \tag{14.28}$$

Thus, obviously the range of r is restricted to $r \leq 1$ and by the change of variables $r = \sin \psi$, the metric can be put into the standard form of the metric on S^n in polar coordinates,

$$ds^2 = d\psi^2 + \sin^2 \psi d\Omega^2_{(n-1)} \ . \tag{14.29}$$

This makes it clear that the singularity at $r = 1$ is just a coordinate singularity. It would also appear if one wrote the metric on the two-sphere in terms of the radial coordinate $r = \sin \theta$,

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 = \frac{dr^2}{1 - r^2} + r^2 d\phi^2 \ . \tag{14.30}$$

For $k = -1$, on the other hand, we have

$$ds^2 = \frac{dr^2}{1 + r^2} + r^2 d\Omega^2_{(n-1)} \ . \tag{14.31}$$

Thus the range of r is $0 \leq r < \infty$, and we can use the change of variables $r = \sinh \psi$ to write the metric as

$$ds^2 = d\psi^2 + \sinh^2 \psi d\Omega^2_{(n-1)} \ . \tag{14.32}$$

This is the standard metric of a hyperboloid H^n in polar coordinates.

Finally, by making the change of variables

$$r = \bar{r}(1 + k\bar{r}^2/4)^{-1} \ , \tag{14.33}$$

one can put the metric in the form

$$ds^2 = (1 + k\bar{r}^2/4)^{-2}(d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(n-1)}) \ . \tag{14.34}$$

Note that this differs by the *conformal factor* $(1 + k\bar{r}^2/4)^{-2} > 0$ from the flat metric. One says that such a metric is *conformally flat*. Thus what we have shown is that every maximally symmetric space is conformally flat. Note that conformally flat, on the other hand, does not imply maximally symmetric as the conformal factor could also be any function of the radial and angular variables.

<center>THE ROBERTSON-WALKER METRIC</center>

Having determined that the metric of a maximally symmetric space is of the simple form (14.1), we can now deduce that a space-time metric satisfying the Cosmological Principle can be chosen to be of the form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad . \tag{14.35}$$

Here we have used the fact that (as in the ansatz for a spherically symmetric metric) non-trivial g_{tt} and g_{tr} can be removed by a coordinate transformation. This metric is known as the *Friedmann-Robertson-Walker metric* or just the Robertson-Walker metric, and spatial coordinates in which the metric takes this form are called *comoving coordinates*, for reasons that will become apparent below.

First of all, note that, since $g_{tt} = -1$ is a constant, one has

$$\Gamma_{\mu tt} = \frac{1}{2} (2\partial_t g_{\mu t} - \partial_\mu g_{tt}) = 0 \quad . \tag{14.36}$$

Therefore the vector field ∂_t is geodesic, which can be expressed as the statement that

$$\nabla_t \partial_t := \Gamma^\mu_{tt} \partial_\mu = 0 \quad . \tag{14.37}$$

In simpler terms this means that the curves $\vec{x} = const.$,

$$\tau \to (\vec{x}(\tau), t(\tau)) = (\vec{x}_0, \tau) \tag{14.38}$$

are geodesics. Hence, in this coordinate system, observers remaining at fixed values of the spatial coordinates are in free fall. In other words, the coordinate system is falling with them or *comoving*, and the proper time along such geodesics coincides with the coordinate time, $d\tau = dt$. It is these observers of constant $\vec{x}$ or constant (r, θ, ϕ) who all see the same isotropic universe at a given value of t.

This may sound a bit strange but a good way to visualize such a coordinate system is, as in Figure 19, as a mesh of coordinate lines drawn on a balloon that is being inflated or deflated (according to the behaviour of $a(t)$). Draw some dots on that balloon (that will eventually represent galaxies or clusters of galaxies). As the ballon is being inflated or deflated, the dots will move but the coordinate lines will move with them and the dots remain at fixed spatial coordinate values. Thus, as we now know, regardless of the

<center>128</center>
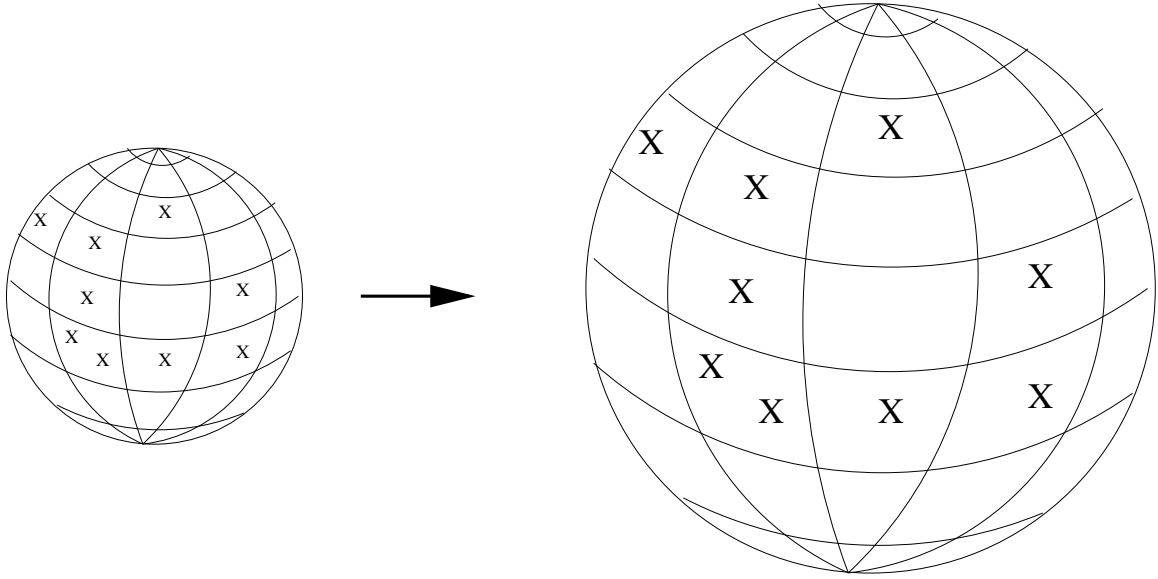
Figure 19: Illustration of a comoving coordinate system: Even though the sphere (universe) expands, the X's (galaxies) remain at the same spatial coordinates. These trajectories are geodesics and hence the X's (galaxies) can be considered to be in free fall. The figure also shows that it is the number density per unit coordinate volume that is conserved, *not* the density per unit proper volume.

behaviour of $a(t)$, these dots follow a geodesic, and we will thus think of galaxies in this description as being in free fall.

Another advantage of this coordinate system is that the six-parameter family of isometries just acts on the spatial part of the metric. Indeed, let $K^i \partial_i$ be a Killing vector of the maximally symmetric spatial metric. Then $K^i \partial_i$ is also a Killing vector of the Robertson-Walker metric. This would not be the case if one had e.g. made an x-dependent coordinate transformation of t or a t-dependent coordinate transformation of the x^i. In those cases there would of course still be six Killing vectors, but they would have a more complicated form.

The metric of the three-space at constant t is

$$g_{ij} = a^2(t)\tilde{g}_{ij} \ , \tag{14.39}$$

where $\tilde{g}_{ij}$ is the maximally symmetric spatial metric. Thus for $k = +1$, $a(t)$ directly gives the size (radius) of the universe. For $k = -1$, space is infinite, so no such interpretation is possible, but nevertheless $a(t)$ still sets the scale for the geometry of the universe, e.g. in the sense that the curvature scalar $R^{(3)}$ of the metric g_{ij} is related to the curvature scalar $\tilde{R}^{(3)}$ of $\tilde{g}_{ij}$ by

$$R^{(3)}(t) = \frac{1}{a^2(t)}\tilde{R}^{(3)} \ . \tag{14.40}$$

Finally, for $k = 0$, three-space is flat and also infinite, but one could replace $\mathbb{R}^3$ by a three-torus T^3 (still maximally symmetric and flat but now compact) and then $a(t)$ would once again be related directly to the size of the universe at constant t. Anyway, in all cases, $a(t)$ plays the role of a (and is known as the) *cosmic scale factor*.

Note that the case $k = +1$ opened up for the very first time the possibility of considering, even conceiving, an unbounded but finite universe! These and other generalizations made possible by a general relativistic approach to cosmology are important as more naive (Newtonian) models of the universe immediately lead to paradoxes or contradictions. One of them we will examine briefly below.

15 Cosmology II: Basics

Olbers' Paradox

One paradox, popularized by Olbers (1826) but noticed before by others is the following. He asked the seemingly innocuous question "Why is the sky dark at night?". According to his calculation, reproduced below, the sky should instead be infinitely bright.

The simplest assumption one could make in cosmology (prior to the discovery of the Hubble expansion) is that the universe is static, infinite and homogeneously filled with stars. In fact, this is probably the naive picture one has in mind when looking at the stars at night, and certainly for a long time astronomers had no reason to believe otherwise.

However, these simple assumptions immediately lead to a paradox, namely the conclusion that the night-sky should be infinitely bright (or at least very bright) whereas, as we know, the sky is actually quite dark at night. This is a nice example of how very simple observations can actually tell us something deep about nature (in this case, the nature of the universe). The argument runs as follows.

1. Assume that there is a star of brightness (luminosity) L at distance r. Then, since the star sends out light into all directions, the apparent luminosity A (neglecting absorption) will be

$$A(r) = L/4\pi r^2 \ . \tag{15.1}$$

2. If the number density ν of stars is constant, then the number of stars at distances between r and $r + dr$ is

$$dN(r) = 4\pi\nu r^2 dr \ . \tag{15.2}$$

Hence the total energy density due to the radiation of all the stars is

$$E = \int_0^\infty A(r)dN(r) = L\nu \int_0^\infty dr = \infty \ . \tag{15.3}$$

3. Therefore the sky should be infinitely bright.

Now what is one to make of this? Clearly some of the assumptions in the above are much too naive. The way out suggested by Olbers is to take into account absorption effects and to postulate some absorbing interstellar medium. But this is also too naive because in an eternal universe we should now be in a stage of thermal equilibrium. Hence the postulated interstellar medium should emit as much energy as it absorbs, so this will not reduce the radiant energy density either.

Of course, the stars themselves are not transparent, so they could block out light completely from distant sources. But if this is to rescue the situation, one would need to postulate so many stars that every line of sight ends on a star, but then the night sky would be bright (though not infinitely bright) and not dark.

Modern cosmological models can resolve this problem in a variety of ways. For instance, the universe could be static but finite (there are such solutions, but this is nevertheless an unlikely scenario) or the universe is not eternal since there was a 'Big Bang' (and this is a more likely scenario).

<center>THE HUBBLE EXPANSION</center>

We have already discussed one of the fundamental inputs of simple cosmological models, namely the cosmological principle. This led us to consider space-times with maximally-symmetric space-like slices. One of the few other things that is definitely known about the universe, and that tells us something about the time-dependence of the universe, is that it expands or, at least that it appears to be expanding.

In fact, in the 1920's and 1930's, the astronomer Edwin Hubble made a remarkable discovery regarding the motion of galaxies. He found that light from distant galaxies is systematically red-shifted (increased in wave-length λ), the increase being proportional to the distance d of the galaxy,

$$z := \frac{\Delta\lambda}{\lambda} \propto d \ . \tag{15.4}$$

Hubble interpreted this red-shift as due to a Doppler effect and therefore ascribed a *recessional velocity* $v = cz$ to the galaxy. While, as we will see, this pure Doppler shift explanation is not tenable, the terminology has stuck, and Hubble's law can be written in the form

$$v = Hd \ , \tag{15.5}$$

where H is *Hubble's constant*. We will see later that in most cosmological models H is actually a function of time, so the H in the above equation should then be interpreted as the value of H today.

Actually, not only in cosmological toy-models but also in experiments, H is a function of time, with current estimates fluctuating rather wildly. It is one of the main goals of

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observational cosmology to determine H as precisely as possible, and the main problem here is naturally a precise determination of the distances of distant galaxies.

Galactic distances are frequently measured in *mega-parsecs* (Mpc). A parsec is the distance from which a star subtends an angle of 2 arc-seconds at the two diametrically opposite ends of the earth's orbit. This unit arose because of the old trigonometric method of measuring stellar distances (a triangle is determined by the length of one side and the two adjacent angles). 1 parsec is approximately 3×10^{18} cm, a little over 3 light-years. The Hubble constant is therefore often expressed in units of km s^{-1} (Mpc)$^{-1}$. We will usually prefer to express it just in terms of inverse units of time. Current estimates for H are in the order of magnitude range of

$$H^{-1} \approx \times 10^{10} \text{ years} \tag{15.6}$$

(whereas Hubble's original estimate was more in the 10^9 year range).

* AREA MEASUREMENTS IN A ROBERTSON-WALKER METRIC AND NUMBER COUNTS

The aim of this and the subsequent sections is to learn as much as possible about the general properties of Robertson-Walker geometries (without using the Einstein equations) with the aim of looking for observational means of distinguishing e.g. among the models with $k = 0, \pm 1$.

To get a feeling for the geometry of the Schwarzschild metric, we studied the properties of areas and lengths in the Schwarzschild geometry. Length measurements are rather obvious in the Robertson-Walker geometry, so here we focus on the properties of areas.

We write the spatial part of the Robertson-Walker metric in polar coordinates as

$$ds^2 = a^2[d\psi^2 + f^2(\psi)d\Omega^2] \ , \tag{15.7}$$

where $f(\psi) = \psi, \sin\psi, \sinh\psi$ for $k = 0, +1, -1$. Now the radius of a surface $\psi = \psi_0$ around the point $\psi = 0$ (or any other point, our space is isotropic and homogeneous) is given by

$$\rho = a \int_0^{\psi_0} d\psi = a\psi_0 \ . \tag{15.8}$$

On the other hand, the area of this surface is determined by the induced metric $a^2 f^2(\psi_0)d\Omega^2$ and is

$$A(\rho) = a^2 f^2(\psi_0) \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi a^2 f^2(\rho/a) \ . \tag{15.9}$$

For $k = 0$, this is just the standard behaviour

$$A(\rho) = 4\pi\rho^2 \ , \tag{15.10}$$

but for $k = \pm 1$ the geometry looks quite different.

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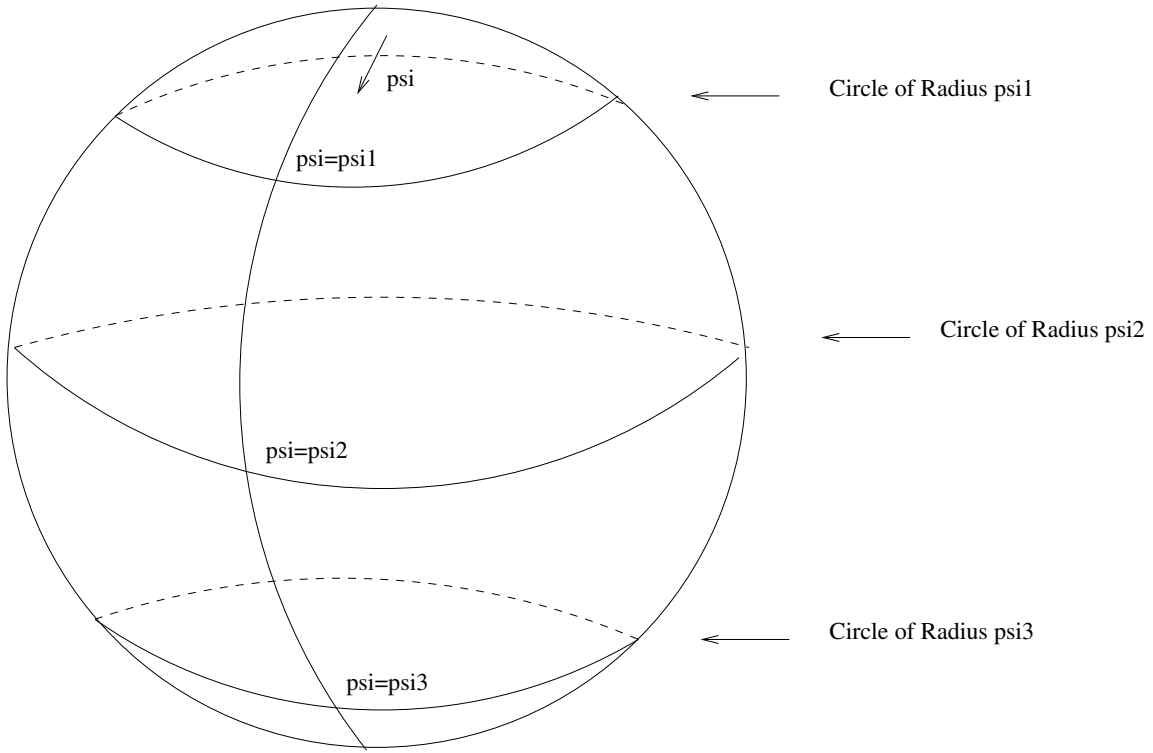
Figure 20: Visualization of the $k = +1$ Robertson-Walker geometry via a two-sphere of unit radius: Circles of radius ψ, measured along the two-sphere, have an area which grows at first, reaches a maximum at $\psi = \pi/2$ and goes to zero when $\psi \to \pi$. E.g. the maximum value of the circumference, at $\psi = \pi/2$, namely 2π, is much smaller than the circumference of a circle with the same radius $\pi/2$ in a flat geometry, namely $2\pi \times \pi/2 = \pi^2$. Only for ψ very small does one approximately see a standard Euclidean geometry.

For $k = +1$, we have

$$A(\rho) = 4\pi a^2 \sin^2(\rho/a) \ . \tag{15.11}$$

Thus the area reaches a maximum for $\rho = \pi a/2$ (or $\psi = \pi/2$), then *decreases* again for larger values of ρ and goes to zero as $\rho \to \pi a$. Already the maximal area, $A_{max} = 4\pi a^2$ is much smaller than the area of a sphere of the same radius in Euclidean space, which would be $4\pi\rho^2 = \pi^3 a^2$.

This behaviour is best visualized by replacing the three-sphere by the two-sphere and looking at the circumference of circles as a function of their distance from the origin (see Figure 20).

For $k = -1$, we have

$$A(\rho) = 4\pi a^2 \sinh^2(\rho/a) \ , \tag{15.12}$$

so in this case the area grows much more rapidly with the radius than in flat space.

In principle, this distinct behaviour of areas in the models with $k = 0, \pm 1$ might allow for an empirical determination of k. For instance, one might make the assumption that there is a homogeneous distribution of the number and brightness of galaxies, and one could try to determine observationally the number of galaxies as a function of their apparent luminosity. As in the discussion of Olbers' paradox, the radiation flux would be proportional to $F \propto 1/\rho^2$. In Euclidean space ($k = 0$), one would expect the number $N(F)$ of galaxies with flux greater than F, i.e. distances less than ρ to behave like ρ^3, so that the expected Euclidean behaviour would be

$$N(F) \propto F^{-3/2} \ . \tag{15.13}$$

Any empirical departure from this behaviour could thus be an indication of a universe with $k \neq 0$, but clearly, to decide this, many other factors (red-shift, evolution of stars, etc.) have to be taken into account and so far it has been impossible to determine the value of k in this way.

<center>The Cosmological Red-Shift</center>

The most important information about the cosmic scale factor $a(t)$ comes from the observation of shifts in the frequency of light emitted by distant sources. To calculate the expected shift in a Robertson-Walker geometry, let us ego- or geocentrically place ourselves at the origin $r = 0$ (remember that because of maximal symmetry this point is as good as any other and in no way privileged). We consider a radially travelling electro-magnetic wave (a light ray) and consider the equation $d\tau^2 = 0$ or

$$dt^2 = a^2(t) \frac{dr^2}{1 - kr^2} \ . \tag{15.14}$$

Let us assume that the wave leaves a galaxy located at $r = r_1$ at the time t_1. Then it will reach us at a time t_0 given by

$$f(r_1) = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = \int_{t_1}^{t_0} \frac{dt}{a(t)} \ . \tag{15.15}$$

As typical galaxies will have constant coordinates, $f(r_1)$ (which can of course be given explicitly, but this is not needed for the present analysis) is time-independent. If the next wave crest leaves the galaxy at r_1 at time $t_1 + \delta t_1$, it will arrive at a time $t_0 + \delta t_0$ determined by

$$f(r_1) = \int_{t_1 + \delta t_1}^{t_0 + \delta t_0} \frac{dt}{a(t)} \ . \tag{15.16}$$

Subtracting these two equations and making the (eminently reasonable) assumption that the cosmic scale factor $a(t)$ does not vary significantly over the period δt given by the frequency of light, we obtain

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \ . \tag{15.17}$$

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Indeed, say that $b(t)$ is the integral of $1/a(t)$. Then we have

$$b(t_0 + \delta t_0) - b(t_1 + \delta t_1) = b(t_0) - b(t_1) \quad, \tag{15.18}$$

and Taylor expanding to first order, we obtain

$$b'(t_0)\delta t_0 = b'(t_1)\delta t_1 \quad, \tag{15.19}$$

which is the same as (15.17). Therefore the observed frequency ν_0 is related to the emitted frequency ν_1 by

$$\frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} \quad. \tag{15.20}$$

Astronomers like to express this in terms of the red-shift parameter (see the discussion of Hubble's law above)

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_0} \quad, \tag{15.21}$$

which in view of the above result we can write as

$$z = \frac{a(t_0)}{a(t_1)} - 1 \quad. \tag{15.22}$$

If $z > 0$ (and thus the universe expands), there is a red-shift, in a contracting universe with $a(t_0) < a(t_1)$ the light of distant glaxies would be blue-shifted.

A few remarks on this result:

1. This cosmological red-shift has nothing to do with the star's own gravitational field - that contribution to the red-shift is completely negligible compared to the effect of the cosmological red-shift.

2. Unlike the gravitational red-shift we discussed before, this cosmological red-shift is symmetric between receiver and emitter, i.e. light sent from the earth to the distant galaxy would likewise be red-shifted if we observe a red-shift of the distant galaxy.

3. This red-shift is a combined effect of gravitational and Doppler red-shifts and it is not very meaningful to interpret this only in terms of, say, a Doppler shift. Nevertheless, as mentioned before, astronomers like to do just that, calling $v = zc$ the recessional velocity.

The Red-Shift Distance Relation (Hubble's Law)

We have seen that there is a cosmological red-shift in Robertson-Walker geometries. Our aim will now be to see if and how these geometries are capable of explaining Hubble's law that the red-shift is approximately proportional to the distance and how the Hubble constant is related to the cosmic scale factor $a(t)$.

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Reliable data for cosmological red-shifts as well as for distance measurements are only available for small values of z, and thus we will consider the case where $t_0 - t_1$ and r_1 are small, i.e. small at cosmic scales. First of all, this allows us to expand $a(t)$ in a Taylor series,

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \tfrac{1}{2}(t - t_0)^2 \ddot{a}(t_0) + \ldots \qquad (15.23)$$

Let us introduce the *Hubble parameter* $H(t)$ and the *deceleration parameter* $q(t)$ by

$$
\begin{aligned}
H(t) &= \frac{\dot{a}(t)}{a(t)} \\
q(t) &= -\frac{a(t)\ddot{a}(t)}{\dot{a}(t)^2} \ ,
\end{aligned}
\qquad (15.24)
$$

and denote their present day values by a subscript zero, i.e. $H_0 = H(t_0)$ and $q_0 = q(t_0)$. $H(t)$ measures the expansion velocity as a function of time while $q(t)$ measures whether the expansion velocity is increasing or decreasing. We will also denote $a_0 = a(t_0)$ and $a(t_1) = a_1$.

In terms of these parameters, the Taylor expansion can be written as

$$a(t) = a_0(1 + H_0(t - t_0) - \tfrac{1}{2}q_0 H_0^2 (t - t_0)^2 + \ldots) \ . \qquad (15.25)$$

This gives us the red-shift parameter z as a power series in the time of flight, namely

$$\frac{1}{1+z} = \frac{a_1}{a_0} = 1 + (t_1 - t_0)H_0 - \tfrac{1}{2}q_0 H_0^2 (t_1 - t_0)^2 + \ldots \qquad (15.26)$$

or

$$z = (t_0 - t_1)H_0 + (1 + \tfrac{1}{2}q_0)H_0^2 (t_0 - t_1)^2 + \ldots \qquad (15.27)$$

For small $H_0(t_0 - t_1)$ this can be inverted,

$$t_0 - t_1 = \frac{1}{H_0}[z - (1 + \tfrac{1}{2}q_0)z^2 + \ldots] \ . \qquad (15.28)$$

We can also use (15.15) to express $(t_0 - t_1)$ in terms of r_1. On the one hand we have

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = r_1 + \mathcal{O}(r_1^3) \ , \qquad (15.29)$$

while expanding $a(t)$ in the denominator we get

$$
\begin{aligned}
\int_{t_1}^{t_0} \frac{dt}{a(t)} &= \frac{1}{a_0} \int_{t_1}^{t_0} \frac{dt}{(1 + (t - t_0)H_0 + \ldots)} \\
&= \frac{1}{a_0} \int_{t_1}^{t_0} dt \, [1 + (t_0 - t)H_0 + \ldots] \\
&= \frac{1}{a_0}[(t_0 - t_1) + t_0(t_0 - t_1)H_0 - \tfrac{1}{2}(t_0^2 - t_1^2)H_0 + \ldots] \\
&= \frac{1}{a_0}[(t_0 - t_1) + \tfrac{1}{2}(t_0 - t_1)^2 H_0 + \ldots] \ . \qquad (15.30)
\end{aligned}
$$

Therefore we obtain

$$r_1 = \frac{1}{a_0}[(t_0 - t_1) + \tfrac{1}{2}(t_0 - t_1)^2 H_0 + \ldots] \quad . \tag{15.31}$$

Using (15.28), we obtain

$$r_1 = \frac{1}{a_0 H_0}[z - \tfrac{1}{2}(1 + q_0)z^2 + \ldots] \quad . \tag{15.32}$$

This clearly indicates to first order a linear dependence of the red-shift on the distance of the galaxy and identifies H_0, the present day value of the Hubble parameter, as being at least proportional to the Hubble constant introduced in (15.5).

However, this is not yet a very useful way of expressing Hubble's law. First of all, the distance $a_0 r_1$ that appears in this expression is not the proper distance (unless $k = 0$), but is at least equal to it in our approximation. Note that $a_0 r_1$ is the present distance to the galaxy, not the distance at the time the light was emitted.

However, even proper distance is not directly measurable and thus, to compare this formula with experiment, one needs to relate r_1 to the measures of distance used by astronomers. One practical way of doing this is the so-called *luminosity distance* d_L. If for some reasons one knows the absolute luminosity of a distant star (for instance because it shows a certain characteristic behaviour known from other stars nearby whose distances can be measured by direct means), then one can compare this absolute luminosity L with the apparent luminosity F. Then one can define the luminosity distance d_L by (cf. (15.1))

$$d_L^2 = \frac{L}{4\pi A} \quad . \tag{15.33}$$

We thus need to relate d_L to the coordinate distance r_1. The key relation is

$$\frac{A}{L} = \frac{1}{4\pi a_0^2 r_1^2} \frac{1}{1 + z} \frac{a_1}{a_0} = \frac{1}{4\pi a_0^2 r_1^2 (1 + z)^2} \quad . \tag{15.34}$$

Here the first factor arises from dividing by the area of the sphere at distance $a_0 r_1$ and would be the only term in a flat geometry (see the discssion of Olbers' paradox). In a Robertson-Walker geometry, however, the photon flux will be diluted. The second factor is due to the fact that each individual photon is being red-shifted. And the third factor (identical to the second) is due to the fact that as a consequence of the expansion of the universe, photons emitted a time δt apart will be measured a time $(1 + z)\delta t$ apart.

Hence the relation between r_1 and d_L is

$$d_L = (L/4\pi A)^{1/2} = r_1 a(t_0)(1 + z) \quad . \tag{15.35}$$

Intuitively, the fact that for z positive d_L is larger than the actual (proper) distance of the galaxy can be understood by noting that the gravitational red-shift makes an object look darker (further away) than it actually is.

This can be inserted into (15.32) to give an expression for the red-shift in terms of d_L, *Hubble's law*

$$\boxed{d_L = H_0^{-1}[z + \tfrac{1}{2}(1 - q_0)z^2 + \ldots]} \ . \tag{15.36}$$

The program would then be to collect as much astronomical information as possible on the relation between d_L and z in order to determine the parameters q_0 and H_0. But this is quite difficult in practice because many other effects need to be taken into account, and as a consequence there are great uncertainties regarding the values for these parameters. For the Hubble 'constant' H_0 one has approximately

$$7.5 \times 10^9 \text{ years } \leq H_0^{-1} \leq 2 \times 10^{10} \text{ years} \tag{15.37}$$

(and currently the larger values appear to be favoured). Even less is known about q_0 and (as far as I know) its value is believed to lie somewhere in the range $0 \leq q_0 \leq 2$, perhaps $q_0 \approx 1$. As we will see below, in the Friedmann-Robertson-Walker models the value of q_0 is strictly correlated with the value of $k = 0, \pm 1$, so it would be very desirable to have more precise information about q_0.

16   Cosmology III: Basics of Friedman-Robertson-Walker Cosmology

So far, we have only used the kinematical framework provided by the Robertson-Walker metrics and we never used the Einstein equations. The benefit of this is that it allows one to deduce relations betweens observed quantities and assumptions about the universe which are valid even if the Einstein equations are not entirely correct, perhaps because of higher derivative or other quantum corrections in the early universe.

Now, on the other hand we will have to be more specific, specify the matter content and solve the Einstein equations for $a(t)$. We will see that a lot about the solutions of the Einstein equations can already be deduced from a purely qualitative analysis of these equations, without having to resort to explicit solutions (Chapter 17). Exact solutions will then be the subject of Chapter 18.

The Ricci Tensor of the Robertson-Walker Metric

Of course, the first thing we need to discuss solutions of the Einstein equations is the Ricci tensor of the Robertson-Walker (RW) metric. Since we already know the curvature tensor of the maximally symmetric spatial metric entering the RW metric (and its contractions), this is not difficult.

1. First of all, we write the RW metric as

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}dx^i dx^j \ . \tag{16.1}$$

From now on, all objects with a tilde, ~, will refer to three-dimensional quantities calculated with the metric $\tilde{g}_{ij}$.

2. One can then calculate the Christoffel symbols in terms of $a(t)$ and $\tilde{\Gamma}^i{}_{jk}$. The non-vanishing components are (we had already established that $\Gamma^\mu{}_{00} = 0$)

$$
\begin{aligned}
\Gamma^i{}_{jk} &= \tilde{\Gamma}^i{}_{jk} \\
\Gamma^i{}_{j0} &= \frac{\dot{a}}{a}\delta^i_j \\
\Gamma^0{}_{ij} &= \dot{a}a\tilde{g}_{ij}
\end{aligned}
$$
(16.2)

3. The relevant components of the Riemann tensor are

$$
\begin{aligned}
R^i{}_{0j0} &= -\frac{\ddot{a}}{a}\delta^i_j \\
R^0{}_{i0j} &= a\ddot{a}\tilde{g}_{ij} \\
R^k{}_{ikj} &= \tilde{R}_{ij} + 2\dot{a}^2\tilde{g}_{ij} \ .
\end{aligned}
$$
(16.3)

4. Now we can use $\tilde{R}_{ij} = 2k\tilde{g}_{ij}$ (i.e. the maximal symmetry of $\tilde{g}_{ij}$) to calculate $R_{\mu\nu}$. The non-zero components are

$$
\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{ij} &= (a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij} \\
&= (\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2})g_{ij} \ .
\end{aligned}
$$
(16.4)

5. Thus the Ricci scalar is

$$
R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \ ,
$$
(16.5)

and

6. the Einstein tensor has the components

$$
\begin{aligned}
G_{00} &= 3(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) \\
G_{0i} &= 0 \\
G_{ij} &= -(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2})g_{ij} \ .
\end{aligned}
$$
(16.6)

THE MATTER CONTENT: A PERFECT FLUID

Next we need to specify the matter content. On physical grounds one might like to argue that in the approximation underlying the cosmological principle galaxies (or clusters) should be treated as non-interacting particles or a perfect fluid. As it turns out, we do

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not need to do this as the symmetries of the metric fix the energy-momentum tensor to be that of a perfect fluid anyway.

Below, I will give a formal argument for this using Killing vectors. But informally we can already deduce this from the structure of the Einstein tensor obtained above. Comparing (16.6) with the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we deduce that the Einstein equations can only have a solution with a Robertson-Walker metric if the energy-momentum tensor is of the form

$$
\begin{aligned}
T_{00} &= \rho(t) \\
T_{0i} &= 0 \\
T_{ij} &= p(t)g_{ij} \ ,
\end{aligned}
\tag{16.7}
$$

where $p(t)$ and $\rho(t)$ are some functions of time.

Here is the formal argument. It is of course a consequence of the Einstein equations that any symmetries of the Ricci (or Einstein) tensor also have to be symmetries of the energy-momentum tensor. Now we know that the metric $\tilde{g}_{ij}$ has six Killing vectors $K^{(a)}$ and that (in the comoving coordinate system) these are also Killing vectors of the RW metric,

$$
L_{K^{(a)}}\tilde{g}_{ij} = 0 \quad \Rightarrow \quad L_{K^{(a)}}g_{\mu\nu} = 0 \ .
\tag{16.8}
$$

Therefore also the Ricci and Einstein tensors have these symmetries,

$$
L_{K^{(a)}}g_{\mu\nu} = 0 \quad \Rightarrow \quad L_{K^{(a)}}G_{\mu\nu} = 0 \ .
\tag{16.9}
$$

Hence the Einstein equations imply that $T_{\mu\nu}$ should have these symmetries,

$$
L_{K^{(a)}}G_{\mu\nu} = 0 \quad \Rightarrow \quad L_{K^{(a)}}T_{\mu\nu} = 0 \ .
\tag{16.10}
$$

Moreover, since the $L_{K^{(a)}}$ act like three-dimensional coordinate transformations, in order to see what these conditions mean we can make a $(3+1)$-decomposition of the energy-momentum tensor. From the three-dimensional point of view, T_{00} transforms like a scalar under coordinate transformations (and Lie derivatives), T_{0i} like a vector, and T_{ij} like a symmetric tensor. Thus we need to determine what are the three-dimensional scalars, vectors and symmetric tensors that are invariant under the full six-parameter group of the three-dimensional isometries.

For scalars ϕ we thus require (calling K now any one of the Killing vectors of $\tilde{g}_{ij}$),

$$
L_K\phi = K^i\partial_i\phi = 0 \ .
\tag{16.11}
$$

But since $K^i(x)$ can take any value in a maximally symmetric space (homogeneity), this implies that ϕ has to be constant (as a function on the three-dimensional space) and therefore T_{00} can only be a function of time,

$$
T_{00} = \rho(t) \ .
\tag{16.12}
$$

For vectors, it is almost obvious that no invariant vectors can exist because any vector would single out a particular direction and therefore spoil isotropy. The formal argument (as a warm up for the argument for tensors) is the following. We have

$$L_K V^i = K^j \tilde{\nabla}_j V^i + V^j \tilde{\nabla}_j K^i \ . \tag{16.13}$$

We now choose the Killing vectors such that $K^i(x) = 0$ but $\tilde{\nabla}_i K_j \equiv K_{ij}$ is an arbitrary antisymmetric matrix. Then the first term disappears and we have

$$L_K V^i = 0 \quad \Rightarrow \quad K_{ij} V^j = 0 \ . \tag{16.14}$$

To make the antisymmetry manifest, we rewrite this as

$$K_{ij} V^j = K_{kj} \delta_i^k V^j = 0 \ . \tag{16.15}$$

If this is to hold for all antisymmetric matrices, we must have

$$\delta_i^k V^j = \delta_i^j V^k \ , \tag{16.16}$$

and by contraction one obtains $nV^j = V^j$, and hence $V_j = 0$. Therefore, as expected, there is no invariant vector field and

$$T_{0i} = 0 \ . \tag{16.17}$$

We now come to symmetric tensors. Once again we choose our Killing vectors to vanish at a given point x and such that K_{ij} is an arbitrary antisymmetric matrix. Then the condition

$$L_K T_{ij} = K^k \tilde{\nabla}_k T_{ij} + \tilde{\nabla}_i K^k T_{kj} + \tilde{\nabla}_j K^k T_{ik} = 0 \tag{16.18}$$

reduces to

$$K_{mn}(\tilde{g}^{mk} \delta_i^n T_{kj} + \tilde{g}^{mk} \delta_j^n T_{ik}) = 0 \ . \tag{16.19}$$

If this is to hold for all antisymmetric matrices K_{mn}, the antisymmetric part of the term in brackets must be zero or, in other words, it must be symmetric in the indices m and n, i.e.

$$\tilde{g}^{mk} \delta_i^n T_{kj} + \tilde{g}^{mk} \delta_j^n T_{ik} = \tilde{g}^{nk} \delta_i^m T_{kj} + \tilde{g}^{nk} \delta_j^m T_{ik} \ . \tag{16.20}$$

Contracting over the indices n and i, one obtains

$$n\tilde{g}^{mk} T_{kj} + \tilde{g}^{mk} T_{jk} = \tilde{g}^{mk} T_{kj} + \delta_j^m T_k^k \ . \tag{16.21}$$

Therefore

$$T_{ij} = \frac{\tilde{g}_{ij}}{n} T_k^k \ . \tag{16.22}$$

But we already know that the scalar T_k^k has to be a constant. Thus we conclude that the only invariant tensor is the metric itself, and therefore the T_{ij}-components of the

energy-momentum tensor can only be a function of t times $\tilde{g}_{ij}$. Writing this function as $p(t)a^2(t)$, we arrive at

$$T_{ij} = p(t)g_{ij} \ . \tag{16.23}$$

We thus see that the energy-momentum tensor is determined by two functions, $\rho(t)$ and $p(t)$. A covariant way of writing this tensor is as

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu} \ , \tag{16.24}$$

where $u^\mu = (1, 0, 0, 0)$ in a comoving coordinate system. This is precisely the energy-momentum tensor of a *perfect fluid*. u_μ is known as the *velocity field* of the fluid, and the comoving coordinates are those with respect to which the fluid is at rest. ρ is the energy-density of the perfect fluid and p is the pressure. In the particular case where there is no pressure, $p(t) = 0$, the matter is referred to as *dust*.

The trace of the energy-momentum tensor is

$$T^\mu_\mu = -\rho + 3p \ . \tag{16.25}$$

For radiation, for example, the energy-momentum tensor is (like that of Maxwell theory) traceless, and hence radiation has the equation of state

$$p = \rho/3 \ , \tag{16.26}$$

while the equation of state for dust is just $p = 0$.

Conservation Laws

The same arguments as above show that a current J^μ in a Robertson-Walker metric has to be of the form $J^\mu = (n(t), 0, 0, 0)$ in comoving coordinates, or

$$J^\mu = n(t)u^\mu \tag{16.27}$$

in covariant form. Here $n(t)$ could be a number density like a galaxy number density. It gives the number density per unit proper volume. The conservation law $\nabla_\mu J^\mu = 0$ is equivalent to

$$\nabla_\mu J^\mu = 0 \quad \Leftrightarrow \quad \partial_t(\sqrt{g}n(t)) = 0 \ . \tag{16.28}$$

Thus we see that $n(t)$ is not constant, but the number density per unit coordinate volume is (as we had already anticipated in the picture of the balloon, Figure 19). For a RW metric, the time-dependent part of $\sqrt{g}$ is $a(t)^3$, and thus the conservation law says

$$n(t)a(t)^3 = const. \tag{16.29}$$

Let us now turn to the conservation laws associated with the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{16.30}$$

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The spatial components of this conservation law,

$$\nabla_\mu T^{\mu i} = 0 \ , \tag{16.31}$$

turn out to be identically satisfied, by virtue of the fact that the u^μ are geodesic and that the functions ρ and p are only functions of time. This could hardly be otherwise because $\nabla_\mu T^{\mu i}$ would have to be an invariant vector, and we know that there are none (nevertheless it is instructive to check this explicitly).

The only interesting conservation law is thus the zero-component

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\nu} T^{\nu 0} + \Gamma^0_{\mu\nu} T^{\mu\nu} = 0 \ , \tag{16.32}$$

which for a perfect fluid becomes

$$\partial_t \rho(t) + \Gamma^\mu_{\mu 0} \rho(t) + \Gamma^0_{00} \rho(t) + \Gamma^0_{ij} T^{ij} = 0 \ . \tag{16.33}$$

Inserting the explicit expressions (16.2) for the Christoffel symbols, this becomes

$$\dot\rho = -3(\rho + p)\frac{\dot a}{a} \ . \tag{16.34}$$

For instance, when the pressure of the cosmic matter is negligible, like in the universe today, and we can treat the galaxies (without disrespect) as dust, then one has

$$\frac{\dot\rho}{\rho} = -3\frac{\dot a}{a} \ , \tag{16.35}$$

and this equation can trivially be integrated to

$$\rho(t)a(t)^3 = const. \tag{16.36}$$

On the other hand, if the universe is dominated by, say, radiation, then one has the equation of state $p = \rho/3$, and the conservation equation reduces to

$$\frac{\dot\rho}{\rho} = -4\frac{\dot a}{a} \ , \tag{16.37}$$

and therefore

$$\rho(t)a(t)^4 = const. \tag{16.38}$$

The reason why the energy density of photons decreases faster with $a(t)$ than that of dust is of course ... the red-shift.

<center>THE EINSTEIN AND FRIEDMANN EQUATIONS</center>

After these preliminaries, we are now prepared to tackle the Einstein equations. We allow for the presence of a cosmological cosntant and thus consider the equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{16.39}$$

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It will be convenient to rewrite these equations in the form

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^\lambda_\lambda) + \Lambda g_{\mu\nu} \quad . \tag{16.40}$$

Because of isotropy, there are only two independent equations, namely the 00-component and any one of the non-zero ij-components. Using (16.4), we find

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) - \Lambda$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} = 4\pi G(\rho - p) + \Lambda \quad . \tag{16.41}$$

We supplement this by the conservation equation

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \quad . \tag{16.42}$$

Using the first equation to eliminate $\ddot{a}$ from the second, one obtains the set of equations

$$\boxed{\begin{array}{rrcc} (F1) & \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} & = & \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \\ (F2) & -3\frac{\ddot{a}}{a} & = & 4\pi G(\rho + 3p) - \Lambda \\ (F3) & \dot{\rho} & = & -3(\rho + p)\frac{\dot{a}}{a} \quad . \end{array}} \tag{16.43}$$

Together, this set of equation is known as the *Friedmann equations*. They govern every aspect of Friedmann-Robertson-Walker cosmology. From now on I will simply refer to them as equations (F1), (F2), (F3) respectively. In terms of the Hubble parameter $H(t)$ and the deceleration parameter $q(t)$, these equations can also be written as

$$\boxed{\begin{array}{rrcc} (F1') & H^2 & = & \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\ (F2') & q & = & \frac{1}{3H^2}[4\pi G(\rho + 3p) - \Lambda] \\ (F3') & \frac{d}{dt}(\rho a^3) & = & -3Hpa^3 \quad . \end{array}} \tag{16.44}$$

Note that because of the Bianchi identities, the Einstein equations and the conservation equations should not be independent, and indeed they are not. It is easy to see that (F1) and (F3) imply the second order equation (F2) so that, a pleasant simplification, in practice one only has to deal with the two first order equations (F1) and (F3). Sometimes, however, (F2) is easier to solve than (F1), because it is linear in $a(t)$, and then (F1) is just used to fix one constant of integration.

17 COSMOLOGY IV: QUALITATIVE ANALYSIS

A lot can be deduced about the solutions of the Friedmann equations, i.e. the evolution of the universe in the Friedmann-Robertson-Walker cosmologies, without solving the equations directly and even without specifying an equation of state, i.e. a relation between p and ρ. In the following we will, in turn, discuss the critical density, Big Bang, age of the universe, and its long term behaviour, from this qualitative point of view.

For starters, let us consider the case $\Lambda = 0$. (F1') can be read as

$$\frac{8\pi G\rho}{3H^2} - 1 = \frac{k}{a^2 H^2} \quad . \tag{17.1}$$

If one defines the *critical density* ρ_c by

$$\rho_c = \frac{3H^2}{8\pi G} \quad , \tag{17.2}$$

and the *density parameter* Ω by

$$\Omega = \frac{\rho}{\rho_c} \quad , \tag{17.3}$$

then (F1') becomes

$$\Omega - 1 = \frac{k}{a^2 H^2} \quad . \tag{17.4}$$

Thus the sign of k is determined by whether the actual energy density ρ in the universe is greater than, equal to, or less than the critical density,

$$
\begin{array}{ccccc}
\rho < \rho_c & \Leftrightarrow & k = -1 & \Leftrightarrow & \text{open} \\
\rho = \rho_c & \Leftrightarrow & k = 0 & \Leftrightarrow & \text{flat} \\
\rho > \rho_c & \Leftrightarrow & k = +1 & \Leftrightarrow & \text{closed}
\end{array}
\tag{17.5}
$$

Therefore the determination of ρ (and ρ_c via the Hubble constant) is very important. As regards Ω, there are good reasons to believe that Ω is at least 0.1, and very little direct observational evidence for the value $\Omega = 1$ which might perhaps have been favoured on grounds of naturalness (and thus theoretical prejudices about how the universe should behave).

THE BIG BANG

One amazing thing about the FRW models is that all of them (provided that the matter content is physical) predict an initial singularity, commonly known as a Big Bang. This is very easy to see.

(F2) shows that, as long as $\rho + 3p$ is positive (and this is the case for all physical matter), one has $q > 0$, i.e. $\ddot{a} < 0$ so that the universe is decelerating due to gravitational attraction. Since $a > 0$ by definition, $\dot{a}(t_0) > 0$ because we observe a red-shift, and $\ddot{a} < 0$ because $\rho + 3p > 0$, it follows that there cannot have been a turning point in the past and $a(t)$ must be concave downwards. Therefore $a(t)$ must have reached $a = 0$ at some *finite* time in the past. We will call this time $t = 0$, $a(0) = 0$.

As ρa^4 is constant for radiation (an appropriate description of earlier periods of the universe), this shows that the energy density grows like $1/a^4$ as $a \to 0$ so this leads to quite a singular situation.

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With the normalization $a(0) = 0$, it is fair to call t_0 the age of the universe. If $\ddot{a}$ had been zero in the past for all $t \leq t_0$, then we would have

$$\ddot{a} = 0 \quad \Rightarrow \quad a(t) = a_0 t/t_0 \ , \tag{17.6}$$

and

$$\dot{a}(t) = a_0/t_0 = \dot{a}_0 \ . \tag{17.7}$$

This would determine the age of the universe to be

$$t_0 = \frac{a_0}{\dot{a}_0} = H_0^{-1} \ , \tag{17.8}$$

where H_0^{-1} is the *Hubble time*. However, as $\ddot{a} < 0$ for $t \leq t_0$, the actual age of the universe must be smaller than this,

$$\ddot{a} < 0 \quad \Rightarrow \quad t_0 < H_0^{-1} \ . \tag{17.9}$$

Thus the Hubble time sets an upper bound on the age of the universe. See Figure 21 for an illustration of this.

LONG TERM BEHAVIOUR

Let us now try to take a look into the future of the universe. Again we will see that it is remarkably simple to extract relevant information from the Friedmann equations without ever having to solve an equation.

For $k = -1$ or $k = 0$, (F1) can be written as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \ . \tag{17.10}$$

The right hand side of this equation is strictly positive. Therefore $\dot{a}$ is never zero and since $\dot{a}_0 > 0$, we must have

$$\dot{a}(t) > 0 \ \forall \ t \ . \tag{17.11}$$

Thus we can immediately conclude that open and flat universes must expand forever, i.e. they are open in space *and* time.

By taking into account the equation (F3') we can even be somewhat more precise about the long term behaviour. (F3') shows that for non-negative pressure, $p \geq 0$ one has

$$\frac{d}{dt}(\rho a^3) \leq 0 \ . \tag{17.12}$$

Thus ρ must decrease with increasing a at least as fast as a^{-3} (we have seen the behaviour a^{-3} for dust and a^{-4} for radiation). In particular,

$$\lim_{\alpha \to \infty} \rho a^2 = 0 \ . \tag{17.13}$$
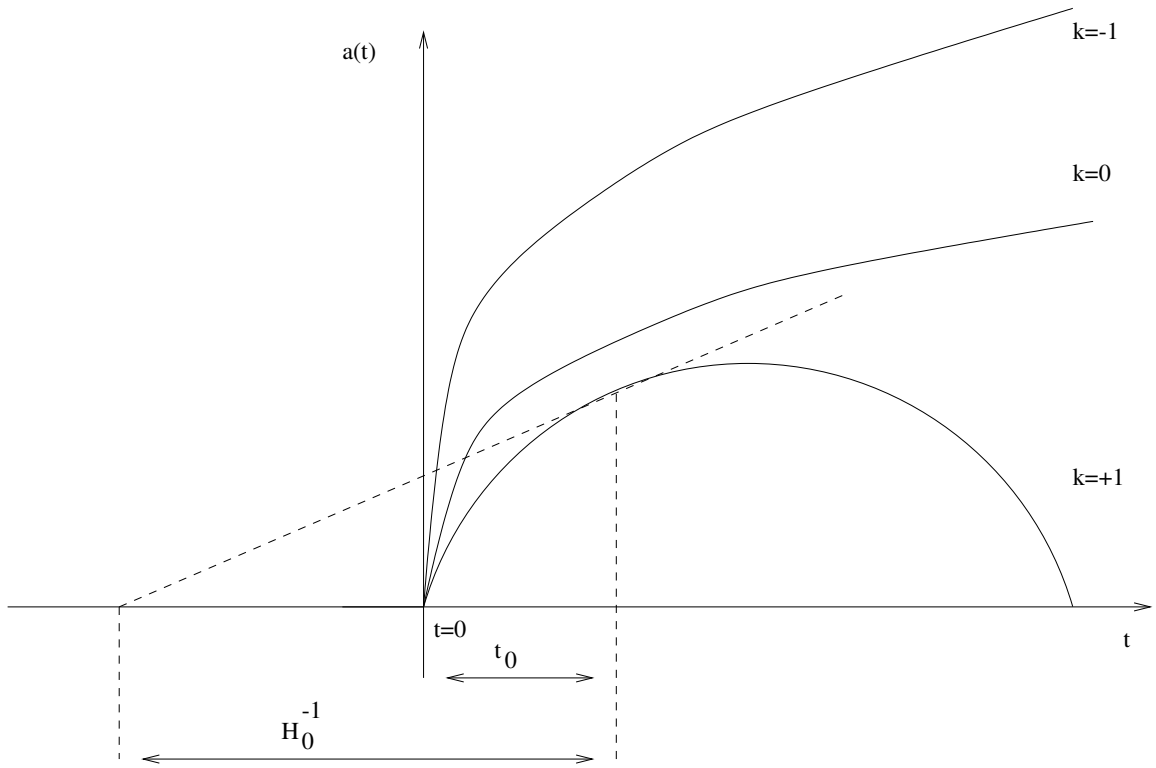
Figure 21: Qualitative behaviour of the Friedmann-Robertson-Walker models for $\Lambda = 0$. All models start with a Big Bang. For $k = +1$ the universe reaches a maximum radius and recollapses after a finite time. For $k = 0$, the universe keeps expanding but the expansion velocity tends to zero for $t \to \infty$ or $a \to \infty$. For $k = -1$, the expansion velocity approaches a non-zero constant value. Also shown is the significance of the Hubble time for the $k = +1$ universe showing clearly that H_0^{-1} gives an upper bound on the age of the universe.

Now let us take another look at (F1),

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \quad . \tag{17.14}$$

For $k = 0$, we learn that

$$k = 0 : \quad \lim_{a \to \infty} \dot{a}^2 = 0 \quad . \tag{17.15}$$

Thus the universe keeps expanding but more and more slowly as time goes on.

By the same reasoning we see that for $k = -1$ we have

$$k = -1 : \quad \lim_{a \to \infty} \dot{a}^2 = 1 \quad . \tag{17.16}$$

Thus the universe keeps expanding, reaching a constant limiting velocity.

For $k = +1$, we would conclude $\dot{a}^2 \to -1$, but this is obviously a contradiction. Therefore we learn that the $k = +1$ universes never reach $a \to \infty$ and that there is therefore

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a maximal radius a_{max}. This maximal radius occurs for $\dot{a} = 0$ and therefore

$$k = +1 : \quad a_{max}^2 = \frac{3}{8\pi G \rho} \quad . \tag{17.17}$$

Note that intuitively this makes sense. For larger ρ or larger G the gravitational attraction is stronger, and therefore the maximal radius of the universe will be smaller. Since we have $\ddot{a} < 0$ also at a_{max}, again there is no turning point and the universe recontracts back to zero size leading to a *Big Crunch*. Therefore, spatially closed universes ($k = +1$) are also closed in time. All of these findings are summarized in Figure 21.

Density and Pressure of the Present Universe

In order to compare the Friedmann-Robertson-Walker models with astronomical observations, we would like to relate the fundamental paramaters H_0 and q_0 to the observed matter content of the universe. The actual numerical values of H_0 and q_0 I use in this and the following sections to illustrate the results should not necessarily be trusted (e.g. the latest observations even seem to suggest a small negative q_0).

Let us recall the equations (F1') and (F2'),

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ q &= \frac{4\pi G(\rho + 3p)}{3H^2} \quad , \end{aligned} \tag{17.18}$$

which we can write as

$$\begin{aligned} \frac{\rho}{\rho_c} - 1 &= \frac{k}{H^2 a^2} \\ q &= \frac{\rho + 3p}{2\rho_c} \quad . \end{aligned} \tag{17.19}$$

From this we can deduce that the present energy-density ρ_0 and pressure p_0 are given by

$$\begin{aligned} \rho_0 &= \rho_c[1 + \frac{k}{H_0^2 a_0^2}] \\ p_0 &= -\frac{1}{8\pi G}[\frac{k}{a_0^2} + H_0^2(1 - 2q_0)] \quad . \end{aligned} \tag{17.20}$$

Since $p_0 \ll \rho_0$, we set $p_0 = 0$. Then one obtains a relation between the spatial curvature k/a_0^2 and the observables H_0 and q_0, namely

$$\frac{k}{a_0^2} = (2q_0 - 1)H_0^2 \quad . \tag{17.21}$$

Together with

$$\frac{\rho_0}{\rho_c} = 2q_0 \quad , \tag{17.22}$$

this allows us to refine our previous statements regarding open versus closed universes (17.5), which gave us a correlation between ρ_0/ρ_c and the value of k. Now we learn that

$$q_0 > 1/2 \quad \Rightarrow \quad k = +1 \;\; , \;\; \rho_0 > \rho_c$$
$$q_0 < 1/2 \quad \Rightarrow \quad k = -1 \;\; , \;\; \rho_0 < \rho_c \;\; . \tag{17.23}$$

If one believes the value $q_0 \approx 1$ obtained from the red-shift distance relation, then one would deduce $\rho_0 \approx 2\rho_c$. Using the approximate value for H_0 quoted there, one finds that

$$\rho_c \approx 10^{-29} \; \mathrm{g/cm}^3 \;\; , \tag{17.24}$$

On the other hand, other observations suggest that the visible matter content of the universe only accounts for about $\rho_0 = 0.1\rho_c$, so something is wrong here. Where is all the mass predicted by the analysis of red-shifts? This is known as the *missing mass problem*. Note also that if one accepts the value $\rho_0 = 0.1\rho_c$, this would predict $q_0 \approx 0.05$ which is almost certainly ruled out by other observations. It could be that a small non-zero cosmological constant is needed to reconcile these (and some other) observations, but this is still a matter of (rather heated) debate.

18   Cosmology V: Exact Solutions

Preliminaries

We have seen that a lot can be learnt about the Friedmann-Robertson-Walker models without ever having to solve a differential equation. On the other hand, more precise information can be obtained by specifying an equation of state for the matter content and solving the Friedmann equations. We will now also reinclude the cosmological constant in our discussion.

In addition to the vacuum energy (and pressure) provided by Λ, there are typically two other kinds of matter which are relevant in our approximation, namely dust and radiation. If we assume that these two do not interact, then one can just add up their contributions in the Friedmann equations. To keep track of these two kinds of matter (and their different qualitative behaviour), it will be convenient to introduce the constants C_m and C_r related to the conserved quantities for matter and radiation respectively,

$$\begin{aligned}
C_m &:= \frac{8\pi G}{3}\rho_m(t)a(t)^3 = \frac{8\pi G}{3}\rho_m(t_0)a_0^3 \\
C_r &:= \frac{8\pi G}{3}\rho_r(t)a(t)^4 = \frac{8\pi G}{3}\rho_r(t_0)a_0^4 \;\; .
\end{aligned} \tag{18.1}$$

In terms of these, the Friedmann equation (F1) takes the more transparent form

$$(F1'') \quad \dot{a}^2 = \frac{C_m}{a} + \frac{C_r}{a^2} - k + \frac{\Lambda}{3}a^2 \;\; , \tag{18.2}$$

illustrating the qualitatively different conntributions to the time-evolution. One can then characterize the different eras in the evolution of the universe by which of the above terms dominates, i.e. gives the leading contribution to the equation of motion for a. This already gives some insight into the physics of the situation.

We will call a universe

1. *matter dominated* if C_m/a dominates

2. *radiation dominated* if C_r/a^2 dominates

3. *curvature dominated* if k dominates

4. *vacuum dominated* if Λa^2 dominates

Our present universe appears to be in the matter dominated era. Certain other things can immediately be deduced from the Friedmann equations:

1. No matter how small C_r is, for sufficiently small values of a that term will dominate and one is in the radiation dominated era. In that case, one finds the characteristic behaviour
$$\dot{a}^2 = \frac{C_r}{a^2} \quad \Rightarrow \quad a(t) = (4C_r)^{1/4} t^{1/2} \ . \tag{18.3}$$

2. On the other hand, if C_m dominates, one has
$$\dot{a}^2 = \frac{C_m}{a} \quad \Rightarrow \quad a(t) = (9C_m/4)^{1/3} t^{2/3} \ . \tag{18.4}$$

3. For sufficiently large a, Λ, if not identically zero, will always dominate, no matter how small the cosmological constant may be, as all the other energy-content of the universe gets more and more diluted.

4. Only for $\Lambda = 0$ does k dominate for large a and one obtains, as we saw before, a constant expansion velocity.

5. Finally, for $\Lambda = 0$ the Friedmann equation can be integrated in terms of elementary functions whereas for $\Lambda \neq 0$ one typically encounters elliptic integrals (unless $\rho = p = 0$).

The Einstein Universe

This particular solution is only of historical interest. Einstein was looking for a static cosmological solution and for this he was forced to introduce the cosmological constant.

Static means that $\dot{a} = 0$. Thus (F3) tells us that $\dot{\rho} = 0$. (F2) tells us that $4\pi G(\rho + 3p) = \Lambda$, where $\rho = \rho_m + \rho_r$. Therefore $p(t)$ also has to be time-independent, $\dot{p} = 0$, and

moreover Λ has to be positive. We see that with $\Lambda = 0$ we would already not be able to satisfy this equation for physical matter content $\rho + 3p > 0$. From (F1") one deduces that

$$k = \frac{C_m}{a} + \frac{C_r}{a^2} + \frac{\Lambda}{3}a^2 \ . \tag{18.5}$$

As all the terms on the right hand side are positive, this means that necessarily $k = +1$. Using the definitions of C_m and C_r, and substituting Λ by $4\pi G(\rho + 3p)$, this becomes a simple algebraic equation for $a(t) = a_0$, namely

$$\begin{aligned}
a^2 &= (8\pi G\rho/3 + 4\pi G(\rho + 3p)/3)^{-1} \\
&= (4\pi G(\rho + p))^{-1} \ .
\end{aligned} \tag{18.6}$$

This is thus a static universe, with topology $\mathbb{R} \times S^3$ in which the gravitational attraction is precisely balanced by the cosmological constant.

<center>THE MATTER DOMINATED ERA</center>

This is somewhat more realistic. In this case we have to solve

$$\dot{a}^2 = \frac{C_m}{a} - k \ . \tag{18.7}$$

For $k = 0$, this is the equation we already discussed above, leading to the solution (18.4). This solution is also known as the Einstein - de Sitter universe.

For $k = +1$, the equation is

$$\dot{a}^2 = \frac{C_m}{a} - 1 \ . \tag{18.8}$$

We recall that in this case we will have a recollapsing universe with $a_{max} = C_m$, which is attained for $\dot{a} = 0$. This can be solved in closed form for t as a function of a, and the solution to

$$\frac{dt}{da} = \left(\frac{a}{a_{max} - a}\right)^{-1/2} \tag{18.9}$$

is

$$t(a) = \frac{a_{max}}{2}\arccos(1 - 2a/a_{max}) - \sqrt{aa_{max} - a^2} \ , \tag{18.10}$$

as can easily be verified.

The universe starts at $t = 0$ with $a(0) = 0$, reaches its maximum $a = a_{max}$ at

$$t_{max} = a_{max}\arccos(-1)/2 = a_{max}\pi/2 \ , \tag{18.11}$$

and ends in a Big Crunch at $t = 2t_{max}$. The curve $a(t)$ is a cycloid, as is most readily seen by writing the solution in parametrized form. For this it is convenient to introduce the time-coordinate u via

$$\frac{du}{dt} = \frac{1}{a(t)} \ . \tag{18.12}$$

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As an aside, note that with this time-coordinate the Robertson-Walker metric (for any k) takes the simple form

$$ds^2 = a^2(u)(-du^2 + d\tilde{s}^2) \ , \tag{18.13}$$

where again a tilde refers to the maximally symmetric spatial metric. In polar coordinates, this becomes

$$ds^2 = a^2(u)(-du^2 + d\psi^2 + f^2(\psi)d\Omega^2) \ . \tag{18.14}$$

Thus radial null lines are determined by $du = \pm d\psi$, as in flat space, and this coordinate system is very convenient for discussing the causal structure of the Friedmann-Robertson-Walker universes.

Anyway, in terms of the parameter u, the solution to the Friedmann equation for $k = +1$ can be written as

$$\begin{aligned} a(u) &= \frac{a_{max}}{2}(1 - \cos u) \\ t(u) &= \frac{a_{max}}{2}(u - \sin u) \ , \end{aligned} \tag{18.15}$$

which makes it transparent that the curve is indeed a cycloid, roughly as indicated in Figure 21.

Analogously, for $k = -1$, the Friedmann equation can be solved in parametrised form, with the trignometric functions replaced by hyperbolic functions,

$$\begin{aligned} a(u) &= \frac{C_m}{2}(\cosh u - 1) \\ t(u) &= \frac{C_m}{2}(\sinh u - u) \ . \end{aligned} \tag{18.16}$$

AGE AND LIFE-TIME OF THE UNIVERSE

It is instructive to express the above solutions in terms of the parameters H_0 and q_0 rather than C_m to see what concrete predictions these models make regarding the age of the universe. Recall that we had derived the relations (17.21,17.22) for $p = 0$ (or the matter dominated era in our new terminology). From these equations it follows that $C_m = 8\pi G\rho_0 a_0^3/3$ can be written as

$$C_m = \frac{2q_0 a_0 k}{2q_0 - 1} \ . \tag{18.17}$$

For $k = +1$, this expresses a_{max} in terms of q_0 and a_0, and one finds for t_{max}

$$t_{max} = \frac{q_0 a_0 \pi}{2q_0 - 1} \ . \tag{18.18}$$

To eliminate a_0 from this, we substitute (once more from (17.21))

$$a_0 = \frac{1}{H_0}(2q_0 - 1)^{1/2} \ , \tag{18.19}$$

and obtain

$$t_{max} = \frac{\pi q_0}{H_0(2q_0 - 1)^{3/2}} \quad . \tag{18.20}$$

The same substitution in (18.17) leads to

$$C_m = \frac{2q_0}{H_0(2q_0 - 1)^{3/2}} \quad . \tag{18.21}$$

The total life-time of the $k = +1$ universe is $2t_{max}$, and with $q_0 \approx 1$ and $H_0^{-1} \approx 1.3 \times 10^{10}$ years, this is

$$t_{max} = 2\pi H_0^{-1} \approx 8 \times 10^{10} \text{ years} \quad . \tag{18.22}$$

To calculate the age t_0 of the universe, we recall the parametrised form of the solution (18.15). We will determine $u(t_0)$ from $a(u)$ and then plug this into $t(u)$ to obtain t_0. We have

$$1 - \cos u(t) = \frac{2a(t)}{C_m} = \frac{2q_0 - 1}{q_0} \frac{a(t)}{a_0} \quad . \tag{18.23}$$

Thus for $t = t_0$ we obtain

$$1 - \cos u(t_0) = \frac{2q_0 - 1}{q_0} \quad . \tag{18.24}$$

With $q_0 \approx 1$, this becomes $\cos u(t_0) \approx 0$ or $u(t_0) \approx \pi/2$. Therefore, from the equation for $t(u)$ we get

$$t_0 = \frac{C_m}{2}(u(t_0) - \sin u(t_0)) = \frac{C_m}{2}(\frac{\pi}{2} - 1) \quad . \tag{18.25}$$

Using (18.21), we obtain

$$t_0 = H_0^{-1}(\frac{\pi}{2} - 1) \approx 7.5 \times 10^9 \text{ years} \quad . \tag{18.26}$$

This value is too small, but then we have already seen that there is very little other observational evidence for a closed $k = +1$ universe anyway. The same calculations for a $k = 0$ or $k = -1$ universe yield more acceptable values for the age of the universe, in the order of ten to twelve billion years.

<div align="center">THE RADIATION DOMINATED ERA</div>

In this case we need to solve

$$a^2 \dot{a}^2 = C_r - ka^2 \quad . \tag{18.27}$$

Because a appears only quadratically, it is convenient to make the change of variables $b = a^2$. Then one obtains

$$\frac{\dot{b}^2}{4} + kb = C_r \quad . \tag{18.28}$$

For $k = 0$ we had already seen the solution in (18.3). For $k = \pm 1$, one necessarily has $b(t) = b_0 + b_1 t + b_2 t^2$. Fixing $b(0) = 0$, one easily finds the solution

$$a(t) = [2C_r^{1/2}t - kt^2]^{1/2} \quad , \tag{18.29}$$

<div align="center">153</div>

so that for $k = +1$

$$a(0) = a(2C_r^{1/2}) = 0 \ . \tag{18.30}$$

Thus already electro-magnetic radiation is sufficient to shrink the universe again and make it recollapse. For $k = -1$, on the other hand, the universe expands forever. All this is of course in agreement with the results of the qualitative discussion given earlier.

<div align="center">THE VACUUM DOMINATED ERA</div>

Even though not very realistic, this is of some interest for two reasons. On the one hand, as we know, Λ is the dominant driving force for a very large. On the other hand, recent cosmological models trying to solve the so-called *horizon problem* use a mechanicsm called *inflation* and postulate a vacuum dominated era during some time in the early universe.

Thus the equation to solve is

$$\dot{a}^2 = -k + \frac{\Lambda}{3}a^2 \ . \tag{18.31}$$

We see immediately that Λ has to be positive for $k = +1$ or $k = 0$, whereas for $k = -1$ both positive and negative Λ are possible.

This is one instance where the solution to the second order equation (F2),

$$\ddot{a} = \frac{\Lambda}{3}a \ , \tag{18.32}$$

is more immediate, namely trigonometric functions for $\Lambda < 0$ (only possible for $k = -1$) and hyperbolic functions for $\Lambda > 0$. The first order equation then fixes the constants of integration according to the value of k.

For $k = 0$, the solution is obviously

$$a_\pm(t) = \sqrt{3/\Lambda}\,\mathrm{e}^{\pm\sqrt{\Lambda/3}t} \ , \tag{18.33}$$

and for $k = +1$, thus $\Lambda > 0$, one has

$$a(t) = \sqrt{3/\Lambda}\cosh\sqrt{\Lambda/3}t \ . \tag{18.34}$$

This is also known as the de Sitter universe. It is a maximally symmetric (in space-time) solution of the Einstein equations with a cosmological constant and thus has a metric of constant curvature (cf. the discussion in section 14). But we know that such a metric is unique. Hence the three solutions with $\Lambda > 0$, for $k = 0, \pm 1$ must all represent the same space-time metric, only in different coordinate systems (and it is a good exercise to check this explicitly). This is interesting because it shows that de Sitter space is so symmetric that it has space-like slicings by three-spheres, by three-hyperboloids and by three-planes.

The solution for $k = -1$ involves $\sin\sqrt{|\Lambda|/3}t$ for $\Lambda < 0$ and $\sinh\sqrt{\Lambda/3}t$ for $\Lambda > 0$, as is easily checked. The former is known as the *anti de Sitter* universe.

This ends our survey of exact cosmological solutions. Once again it is natural to wonder at this point if the singularities predicted by General Relativity in the case of cosmological models are generic or only artefacts of the highly symmetric situations we were considering. And again there are singularity theorems applicable to these situations which state that, under reasonable assumptions about the matter content, singularities will occur independently of assumptions about symmetries.

In previous sections we have dealt with situations in General Relativity in which the gravitational field is strong and the full non-linearity of the Einstein equations comes into play (Black Holes, Cosmology). In most ordinary siutations, however, the gravitational field is weak, very weak, and then it is legitimate to work with a linearization of the Einstein equations. Our first aim will be to derive these linearized equations. As we will see, these turn out to be wave equations and we are thus naturally led to the subject of gravitational waves. These are an important prediction of General Relativity (there are no gravitational waves in Newton's theory). It is therefore important to understand how or under which circumstances they are created and how they can be detected. These, unfortunately, are rather complicated questions in general and I will not enter into this. The things I will cover in the following are much more elementary, both technically and conceptually, than anything else we have done recently.

THE LINEARIZED EINSTEIN EQUATIONS

When we first derived the Einstein equations we checked that we were doing the right thing by deriving the Newtonian theory in the limit where

1. the gravitational field is weak

2. the gravitational field is static

3. test particles move slowly

We will now analyze a less restrictive situation in which we only impose the first condition. This is sufficient to deal with issues like gravitational waves and relativistic test-particles.

We express the weakness of the gravitational field by the condition that the metric be 'close' to that of Minkowski space, i.e. that

$$g_{\mu\nu} = g^{(1)}_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \tag{19.1}$$

with $|h_{\mu\nu}| \ll 1$. This means that we will drop terms which are quadratic or of higher power in $h_{\mu\nu}$. Here and in the following the superscript $^{(1)}$ indicates that we keep only up to linear (first order) terms in $h_{\mu\nu}$. In particular, the inverse metric is

$$g^{(1)\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{19.2}$$

where indices are raised with $\eta^{\mu\nu}$. As one has thus essentially chosen a backgroudn metric, the Minkowski metric, one can think of the linearized version of the Einstein equations (which are field equations for $h_{\mu\nu}$) as a Lorentz-invariant theory of a symmetric tensor field propagating in Minkowski space-time. I won't dwell on this but it is good to keep this in mind. It gives rise to the field theorist's picture of gravity as the theory of an interacting spin-2 field (which I do not subscribe to unconditionally because it is an inherently perturbative and background dependent picture).

It is straightforward to work out the Christoffel symbols and curvature tensors in this approximation. The terms quadratic in the Christoffel symbols do not contribute to the Riemann curvature tensor and one finds

$$
\begin{aligned}
\Gamma^{(1)\mu}_{\nu\lambda} &= \eta^{\mu\rho}\tfrac{1}{2}(\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\nu\lambda}) \\
R^{(1)}_{\mu\nu\rho\sigma} &= \tfrac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\rho\mu}) \ .
\end{aligned}
\tag{19.3}
$$

Hence the linearized Ricci tensor is

$$
R^{(1)}_{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma_{\ \mu} + \partial_\sigma\partial_\mu h^\sigma_{\ \nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \ ,
\tag{19.4}
$$

where $h = h^\mu_{\ \mu}$ is the trace of $h_{\mu\nu}$ and $\Box = \partial^\mu\partial_\mu$. Thus the Ricci scalar is

$$
R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \ ,
\tag{19.5}
$$

and the Einstein tensor is

$$
G^{(1)}_{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma_{\ \mu} + \partial_\sigma\partial_\mu h^\sigma_{\ \nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h) \ ,
\tag{19.6}
$$

Therefore the linearized Einstein equations are

$$
G^{(1)}_{\mu\nu} = 8\pi G T^{(0)}_{\mu\nu} \ .
\tag{19.7}
$$

Note that only the zero'th order term in the h-expansion appears on the right hand side of this equation. This is due to the fact that $T_{\mu\nu}$ must itself already be small in order for the linearized approximation to be valid, i.e. $T^{(0)}_{\mu\nu}$ should be of order $h_{\mu\nu}$. Therefore, any terms in $T_{\mu\nu}$ depending on $h_{\mu\nu}$ would already be of order $(h_{\mu\nu})^2$ and can be dropped.

Therefore the conservation law for the energy-momentum tensor is just

$$
\partial_\mu T^{(0)\mu\nu} = 0 \ ,
\tag{19.8}
$$

and this is indeed compatible with the linearized Bianchi identity

$$
\partial_\mu G^{(1)\mu\nu} = 0 \ ,
\tag{19.9}
$$

which can easily be verified. In fact, one has the stronger statement that

$$
G^{(1)\mu\nu} = \partial_\rho Q^{\rho\mu\nu}
\tag{19.10}
$$

with $Q^{\rho\mu\nu} = -Q^{\mu\rho\nu}$, and this obviously implies the Bianchi identity.

To simplify life, it is now useful to employ the freedom we have in the choice of coordinates. What remains of general coordinate invariance in the linearized approximation are, naturally, linearized general coordinate transformations. Indeed, $h_{\mu\nu}$ and

$$h'_{\mu\nu} = h_{\mu\nu} + L_V \eta_{\mu\nu} \tag{19.11}$$

represent the same physical perturbation because $\eta_{\mu\nu} + L_V \eta_{\mu\nu}$ is just an infinitesimal coordinate transform of the Minkowski metric $\eta_{\mu\nu}$. Therefore linearized gravity has the gauge freedom

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu \ . \tag{19.12}$$

For example, the linearized Riemann tensor $R^{(1)}_{\mu\nu\rho\sigma}$ is, rather obviously, invariant under this transformation (and hence so are the Einstein tensor etc.).

In general, a very useful gauge condition is

$$g^{\mu\nu} \Gamma^\rho_{\mu\nu} = 0 \ . \tag{19.13}$$

It is called the *harmonic gauge condition* (or Fock, or de Donder gauge condition), and the name harmonic derives from the fact that in this gauge the coordinate fuctions x^μ are harmonic:

$$\Box x^\mu \equiv g^{\nu\rho} \nabla_\rho \partial_\nu x^\mu = -g^{\nu\rho} \Gamma^\mu_{\nu\rho} \ , \tag{19.14}$$

and thus

$$\Box x^\mu = 0 \ \Leftrightarrow \ g^{\nu\rho} \Gamma^\mu_{\nu\rho} = 0 \ . \tag{19.15}$$

It is the analogue of the Lorentz gauge $\partial_\mu A^\mu = 0$ in Maxwell theory. Moreover, in flat space Cartesian coordinates are obviously harmonic, and in general harmonic coordinates are (like geodesic coordinates) a nice and useful curved space counterpart of Cartesian coordinates.

In the linearized theory, this gauge condition becomes

$$\partial_\mu h^\mu_\lambda - \tfrac{1}{2} \partial_\lambda h = 0 \ . \tag{19.16}$$

The gauge parameter V_μ which will achieve this is the solution to the equation

$$\Box V_\lambda = -(\partial_\mu h^\mu_\lambda - \tfrac{1}{2} \partial_\lambda h) \ . \tag{19.17}$$

Indeed, with this choice one has

$$\partial_\mu (h^\mu_\lambda + \partial^\mu V_\lambda + \partial_\lambda V^\mu) - \tfrac{1}{2} (\partial_\lambda h + 2 \partial^\mu V_\mu) = 0 \ . \tag{19.18}$$

Note for later that, as in Maxwell theory, this gauge choice does not necessarily fix the gauge completely. Any transformation $x^\mu \rightarrow x^\mu + \xi^\mu$ with $\Box \xi^\mu = 0$ will leave the harmonic gauge condition invariant.

Now let us use this gauge condition in the linearized Einstein equations. In this gauge they simplify somewhat to

$$\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T^{(0)}_{\mu\nu} \quad . \tag{19.19}$$

In particular, the vacuum equations are just

$$T^{(0)}_{\mu\nu} = 0 \quad \Rightarrow \quad \Box h_{\mu\nu} = 0 \quad , \tag{19.20}$$

which is the standarad relativistic wave equation. Together, the equations

$$
\begin{aligned}
\Box h_{\mu\nu} &= 0 \\
\partial_\mu h^\mu_{\ \lambda} - \tfrac{1}{2}\partial_\lambda h &= 0
\end{aligned}
\tag{19.21}
$$

determine the evolution of a disturbance in a gravitational field in vacuum in the harmonic gauge.

It is often convenient to define the *trace reversed perturbation*

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \quad , \tag{19.22}$$

with

$$\bar{h}^\mu_{\ \mu} = -h^\mu_{\ \mu} \quad . \tag{19.23}$$

Note, as an aside, that with this notation and terminology the Einstein tensor is the trace reversed Ricci tensor,

$$\bar{R}_{\mu\nu} = G_{\mu\nu} \quad . \tag{19.24}$$

In terms of $\bar{h}_{\mu\nu}$, the Einstein equations and gauge conditions are just

$$
\begin{aligned}
\Box \bar{h}_{\mu\nu} &= -16\pi G T^{(0)}_{\mu\nu} \\
\partial_\mu \bar{h}^\mu_{\ \nu} &= 0 \quad .
\end{aligned}
\tag{19.25}
$$

In this equation, the $\bar{h}_{\mu\nu}$ are now decoupled. One solution is, of course, the retarded potential

$$\bar{h}_{\mu\nu}(\vec{x}, t) = 4G \int d^3x' \; \frac{T^{(0)}_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} \quad . \tag{19.26}$$

Note that, as a consequence of $\partial_\mu T^{(0)\mu\nu} = 0$, this solution is automatically in the harmonic gauge.

THE POLARIZATION TENSOR

The linearized vacuum Einstein equation in the harmonic gauge,

$$\Box \bar{h}_{\mu\nu} = 0 \ , \tag{19.27}$$

is clearly solved by

$$\bar{h}_{\mu\nu} = C_{\mu\nu} \mathrm{e}^{ik_\alpha x^\alpha} \ , \tag{19.28}$$

where $C_{\mu\nu}$ is a constant, symmetric *polarization tensor* and k^α is a constant *wave vector*, provided that k^α is null, $k^\alpha k_\alpha = 0$. (In order to obtain real metrics one should of course use real solutions.)

Thus plane waves are solutions to the linearized equations of motion and the Einstein equations predict the existence of gravitational waves travelling along null geodesics (at the speed of light). The timelike component of the wave vector is often referred to as the *frequency* ω of the wave, and we can write $k^\mu = (\omega, k^i)$. Plane waves are of course not the most general solutions to the wave equations but any solution can be written as a superposition of plane wave solutions (wave packets).

So far, we have ten parameters $C_{\mu\nu}$ and four parameters k^μ to specify the wave, but many of these are spurious, i.e. can be eliminated by using the freedom to perform linearized coordinate transformations and Lorentz rotations.

First of all, the harmonic gauge condition implies that

$$\partial_\mu \bar{h}^\mu_{\ \nu} = 0 \ \Rightarrow \ k^\mu C_{\mu\nu} = 0 \ . \tag{19.29}$$

Now we can make use of the residual gauge freedom $x^\mu \to x^\mu + \xi^\mu$ with $\Box \xi^\mu = 0$ to impose further conditions on the polarization tensor. Since this is a wave equation for ξ^μ, once we have specified a solution for ξ^μ we will have fixed the gauge completely. Taking this solution to be of the form

$$\xi^\mu = B_\mu \mathrm{e}^{ik_\alpha x^\alpha} \ , \tag{19.30}$$

one can choose the B_μ in such a way that the new polarization tensor satisfies $k^\mu C_{\mu\nu} = 0$ (as before) as well as

$$C_{\mu 0} = C^\mu_{\ \mu} = 0 \ . \tag{19.31}$$

All in all, we appear to have nine conditions on the polarization tensor $C_{\mu\nu}$ but as both (19.29) and the first of (19.31) imply $k^\mu C_{\mu 0} = 0$, only eight of these are independent. Therefore, there are two independent polarizations for a gravitational wave.

For example, we can choose the wave to travel in the x^3-direction. Then

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, k^3) \ , \tag{19.32}$$

and $k^\mu C_{\mu\nu} = 0$ and $C_{0\nu} = 0$ imply $C_{3\nu} = 0$, so that the only independent components are C_{ab} with $a, b = 1, 2$. As C_{ab} is symmetric and traceless, this wave is completely characterized by $C_{11} = -C_{22}$, $C_{12} = C_{21}$ and the frequency ω.

In other words, the metric describing a gravitational wave travelling in the x^3-direction can always be put into the form

$$ds^2 = -dt^2 + (\delta_{ab} + h_{ab})dx^a dx^b + (dx^3)^2 \ , \tag{19.33}$$

with the h_{ab} describing travelling waves in the $t - x^3$-directions (e.g. $h_{ab} = h_{ab}(t - x^3)$).

PHYSICAL EFFECTS OF GRAVITATIONAL WAVES

To determine the physical effect of a gravitational wave racing by, we consider its influence on the relative motion of nearby particles. In other words, we look at the geodesic deviation equation. Consider a family of nearby particles described by the *velocity field* $u^\mu(x)$ and seperation vector $S^\mu(x)$,

$$\frac{D^2}{D\tau^2}S^\mu = R^\mu{}_{\nu\rho\sigma}u^\nu u^\rho S^\sigma \ . \tag{19.34}$$

Now let us take the test particles to move slowly,

$$u^\mu = (1, 0, 0, 0) + \mathcal{O}(h) \ . \tag{19.35}$$

Then, because the Riemann tensor is already of order h, the right hand side of the geodesic deviation equation reduces to

$$R^{(1)}_{\mu 0 0 \sigma} = \tfrac{1}{2}\partial_0\partial_0 h_{\mu\sigma} \tag{19.36}$$

(because $h_{0\mu} = 0$). On the other hand, to lowest order the left hand side is just the ordinary time derivative. Thus the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2}S^\mu = \tfrac{1}{2}S^\sigma\frac{\partial^2}{\partial t^2}h^\mu_\sigma \ . \tag{19.37}$$

In particular, we see immediately that the gravitational wave is *transversally polarized*, i.e. the component S^3 of S^μ in the longitudinal direction of the wave is unaffected and the particles are only disturbed in directions perpendicular to the wave. This gives rise to characteristic oscillating movements of the test particles in the 1-2 plane.

For example, with $C_{12} = 0$ one has

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}S^1 &= \tfrac{1}{2}S^1\frac{\partial^2}{\partial t^2}(C_{11}\mathrm{e}^{ik_\alpha x^\alpha}) \\
\frac{\partial^2}{\partial t^2}S^2 &= -\tfrac{1}{2}S^2\frac{\partial^2}{\partial t^2}(C_{11}\mathrm{e}^{ik_\alpha x^\alpha}) \ .
\end{aligned} \tag{19.38}$$

Thus, to lowest order one has

$$\begin{aligned}
S^1 &= (1 + \tfrac{1}{2}C_{11}\mathrm{e}^{ik_\alpha x^\alpha})S^1(0) \\
S^2 &= (1 - \tfrac{1}{2}C_{11}\mathrm{e}^{ik_\alpha x^\alpha})S^2(0)
\end{aligned}$$

$$\tag{19.39}$$

Figure 22: Effect of a gravitational wave with polarization C_{11} moving in the x^3-direction, on a ring of test particles in the $x^1 - x^2$-plane.

Recalling the interpretation of S^μ as a seperation vector, this means that particles originally seperated in the x^1-direction will oscillate back and forth in the x^1-direction and likewise for x^2. A nice (and classical) way to visualize this (see Figure 22) is to start off with a ring of particles in the $1 - 2$ plane. As the wave passes by the particles will start bouncing in such a way that the ring bounces in the shape of a cross $+$. For this reason, C_{11} is also frequently written as C_+.

If, on the other hand, $C_{11} = 0$ but $C_{12} \neq 0$, then S^1 will be displaced in the direction of S^2 and vice versa,

$$
\begin{aligned}
S^1 &= S^1(0) + \tfrac{1}{2} C_{12} e^{ik_\alpha x^\alpha} S^2(0) \\
S^2 &= S^2(0) + \tfrac{1}{2} C_{12} e^{ik_\alpha x^\alpha} S^1(0) \ ,
\end{aligned}
\tag{19.40}
$$

and the ring of particles will bounce in the shape of a $\times$ ($C_{12} = C_\times$) - see Figure 23.

Of course, one can also construct circularly polarized waves by using

$$
C_{R,L} = \frac{1}{\sqrt{2}}(C_{11} \pm iC_{12}) \ .
\tag{19.41}
$$

These solutions display the characteristic behaviour of *quadrupole radiation*, and this is something that we might have anticipated on general grounds. First of all, we know from Birkhoff's theorem that there can be no monopole (s-wave) radiation. Moreover, dipole radiation is due to oscillations of the center of charge. While this is certainly possible for electric charges, an oscillation of the center of mass would violate momentum conservation and is therefore ruled out. Thus the lowest possible mode of gravitational radiation is quadrupole radiation, just as we have found.
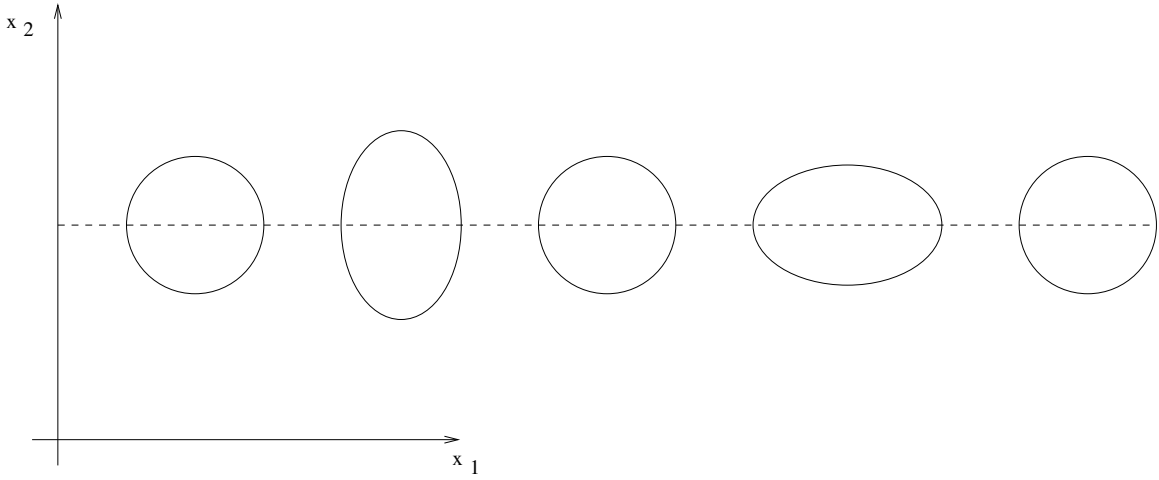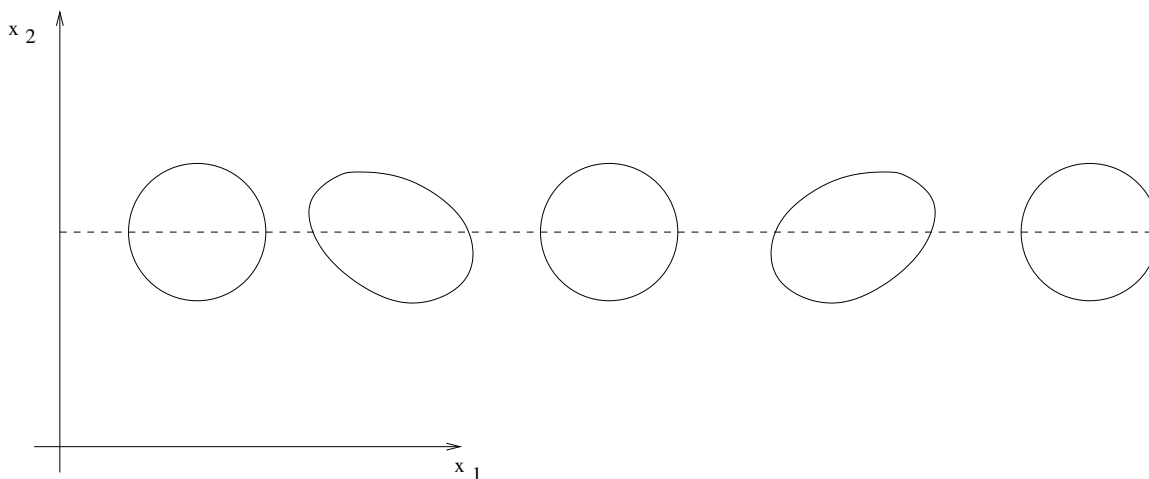
DETECTION OF GRAVITATIONAL WAVES

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Figure 23: Effect of a gravitational wave with polarization C_{12} moving in the x^3-direction, on a ring of test particles in the $x^1 - x^2$-plane.

In principle, now that we have solutions to the vacuum equations, we should include sources and study the production of gravitational waves, characterize the type of radiation that is emitted, estimate the energy etc. I will not do this but just make some general comments on the detection of gravitational waves.

In principle, this ought to be straightforward. For example, one might like to simply try to track the separation of two freely suspended masses. Alternatively, the particles need not be free but could be connected by a solid piece of material. Then gravitational tidal forces will stress the material. If the resonant frequency of this 'antenna' equals the frequency of the gravitational wave, this should lead to a detectable oscillation. This is the principle of the so-called *Weber detectors* (1966-...), but these have not yet, as far as I know, produced completely conclusive results. In a sense this is not surprising as gravitational waves are extremely weak, so weak in fact that the quantum theory of the detectors (huge garbage can size aluminium cylinders, for example) needs to be taken into account.

However, there is indirect (and very compelling) evidence for gravitational waves. According to the theory (we have not developed), a binary system of stars rotating around its common center of mass should radiate gravitational waves (much like electromagnetic synchroton radiation). For two stars of equal mass M at distance $2r$ from each other, the prediction of General Relativity is that the power radiated by the binary system is

$$P = \frac{2}{5} \frac{G^4 M^5}{r^5} \quad . \tag{19.42}$$

This energy loss has actually been observed. In 1974, Hulse and Taylor discovered a binary system, affectionately known as PSR1913+16, in which both stars are very small and one of them is a *pulsar*, a rapidly spinning neutron star. The period of the orbit is

only eight hours, and the fact that one of the stars is a pulsar provides a highly accurate clock with respect to which a change in the period as the binary loses energy can be measured. The observed value is in good agreement with the theoretical prediction for loss of energy by gravitational radiation and Hulse and Taylor were rewarded for these discoveries with the 1993 Nobel Prize.

Other situations in which gravitational waves might be either detected directly or inferred indirectly are extreme situations like gravitational collapse (supernovae) or matter orbiting black holes.

20   Kaluza-Klein Theory I

Motivation

Looking at the Einstein equations and the variational principle, we see that gravity is nicely geometrized while the matter part has to be added by hand and is completely non-geometric. This may be perfectly acceptable for phenomenological Lagrangians (like that for a perfect fluid in Cosmology), but it would clearly be desirable to have a unified description of all the fundamental forces of nature.

Today, the fundamental forces of nature are described by two very different concepts. On the one hand, we have - as we have seen - gravity, in which forces are replaced by geometry, and on the other hand there are the gauge theories of the electroweak and strong interactions (the standard model) or their (grand unified, ...) generalizations.

Thus, if one wants to unify these forces with gravity, there are two possibilities:

1. One can try to realize gravity as a gauge theory (and thus geometry as a consequence of the gauge principle).

2. Or one can try to realize gauge theories as gravity (and hence make them purely geometric).

The first is certainly an attractive idea and has attracted a lot of attention. It is also quite natural since, in a broad sense, gravity is already a gauge theory in the sense that it has a local invariance (under general coordinate transformations or, actively, diffeomorphisms). Also, the behaviour of Christoffel symbols under general coordinate transformations is analogous to the transformation behaviour of non-Abelian gauge fields under gauge transformations, and the whole formalism of covariant derivatives and curvatures is reminiscent of that of non-Abelian gauge theories.

At first sight, equating the Christoffel symbols with gauge fields (potentials) may appear to be a bit puzzling because we originally introduced the metric as the potential

of the gravitational field and the Christoffel symbol as the corresponding field strength (representing the gravitational force). However, as we know, the concept of 'force' is itself a gauge (coordinate) dependent concept in General Relativity, and therefore these 'field strengths' behave more like gauge potentials themselves, with their curvature, the Riemann curvature tensor, encoding the gauge covariant information about the gravitational field. This fact, which reflects deep properties of gravity not shared by other forces, is just one of many which suggest that an honest gauge theory interpretation of gravity may be hard to come by. But let us proceed in this direction for a little while anyway.

Clearly, the gauge group should now not be some 'internal' symmetry group like $U(1)$ or $SU(3)$, but rather a space-time symmetry group itself. Among the gauge groups that have been suggested in this context, one finds

1. the translation group (this is natural because, as we have seen, the generators of coordinate transformations are infinitesimal translations)

2. the Lorentz group (this is natural if one wants to view the Christoffel symbols as the analogues of the gauge fields of gravity)

3. and the Poincaré group (a combination of the two).

However, what - by and large - these investigations have shown is that the more one tries to make a gauge theory look like Einstein gravity the less it looks like a standard gauge theory and vice versa.

The main source of difference between gauge theory and gravity is the fact that in the case of Yang-Mills theory the internal indices bear no relation to the space-time indices whereas in gravity these are the same - contrast $F_{\mu\nu}^a$ with $(F_\sigma^\lambda)_{\mu\nu} = R^\lambda{}_{\sigma\mu\nu}$.

In particular, in gravity one can contract the 'internal' with the space-time indices to obtain a scalar Lagrangian, R, linear in the curvature tensor. This is fortunate because, from the point of view of the metric, this is already a two-derivative object.

For Yang-Mills theory, on the other hand, this is not possible, and in order to construct a Lagrangian which is a singlet under the gauge group one needs to contract the space-time and internal indices separately, i.e. one has a Lagrangian quadratic in the field stregths. This gives the usual two-derivative action for the gauge potentials.

In spite of these and other differences and difficulties, this approach has not been completely abandoned and the gauge theory point of view is still very fruitful and useful provided that one appreciates the crucial features that set gravity apart from standard gauge theories.

The second possibility alluded to above, to realize gauge theories as gravity, is much more radical. But how on earth is one supposed to achieve this? The crucial idea

has been known since 1919/20 (T. Kaluza), with important contributions by O. Klein (1926). So what is this idea?

In the early parts of this century, the only other fundamental force that was known, in addition to gravity, was electro-magnetism, In 1919, Kaluza submitted a paper (to Einstein) in which he made a number of remarkable observations.

First of all, he stressed the similarity between Christoffel symbols and the Maxwell field strength tensor,

$$
\begin{aligned}
\Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(\partial_\nu g_{\mu\lambda} - \partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu}) \\
F_{\nu\mu} &= \partial_\nu A_\mu - \partial_\mu A_\nu \ .
\end{aligned}
\tag{20.1}
$$

He then noted that $F_{\mu\nu}$ looks like a truncated Christoffel symbol and proposed, in order to make this more manifest, to introduce a *fifth dimension* with a metric such that $\Gamma_{\mu\nu 5} \sim F_{\mu\nu}$. This is inded possible. If one makes the identification

$$
A_\mu = g_{\mu 5} \ , \tag{20.2}
$$

and the assumption that $g_{\mu 5}$ is independent of the fifth coordinate x^5, then one finds, using the standard formula for the Christoffel symbols, now extended to five dimensions, that

$$
\begin{aligned}
\Gamma_{\mu\nu 5} &= \tfrac{1}{2}(\partial_5 g_{\mu\nu} + \partial_\nu g_{\mu 5} - \partial_\mu g_{\nu 5}) \\
&= \tfrac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) = \tfrac{1}{2} F_{\nu\mu} \ .
\end{aligned}
\tag{20.3}
$$

But much more than this is true. Kaluza went on to show that when one postulates a five-dimensional metric of the form (hatted quantities will from now on refer to five dimensional quantities)

$$
d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + (dx^5 + A_\mu dx^\mu)^2 \ , \tag{20.4}
$$

and calculates the five-dimensional Einstein-Hilbert Lagrangian $\hat{R}$, one finds *precisely the four-dimensional Einstein-Maxwell Lagrangian*

$$
\hat{R} = R - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{20.5}
$$

This fact is affectionately known as the *Kaluza-Klein Miracle*! Moreover, the five-dimensional geodesic equation turns into the four-dimensional Lorentz force equation for a charged particle, and in this sense gravity and Maxwell theory have really been unified in five-dimensional gravity.

However, although this is very nice, rather amazing in fact, and is clearly trying to tell us something deep, there are numerous problems with this and it is not really clear what has been achieved:

1. Should the fifth direction be treated as real or as a mere mathematical device?

2. If it is to be treated as real, why should one make the assumption that the fields are independent of x^5? But if one does not make this assumption, one will not get Einstein-Maxwell theory.

3. Moreover, if the fifth dimension is to be taken seriously, why are we justified in setting $g_{55} = 1$? If we do not do this, we will not get Einstein-Maxwell theory.

4. If the fifth dimension is real, why have we not discovered it yet?

In spite of all this and other questions, related to non-Abelian gauge symmetries or the quantum behaviour of these theories, Kaluza's idea has remained popular ever since or, rather, has periodically created psychological epidemics of frantic activity, interrupted by dormant phases. Today, Kaluza's idea, with its many reincarnations and variations, is an indispensable and fundamental ingredient in the modern theories of theoretical high energy physics (supergravity and string theories) and many of the questions/problems mentioned above have been addressed, understood and overcome.

<center>THE KALUZA-KLEIN MIRACLE</center>

We now consider a five-dimensional space-time with coordinates $\widehat{x}^M = (x^\mu, x^5)$ and a metric of the form (20.4). For later convenience, we will introduce a parameter λ into the metric (even though we will set $\lambda = 1$ for the time being) and write it as

$$d\widehat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu + (dx^5 + \lambda A_\mu dx^\mu)^2 \ . \tag{20.6}$$

More explictly, we therefore have

$$\begin{aligned}
\widehat{g}_{\mu\nu} &= g_{\mu\nu} + A_\mu A_\nu \\
\widehat{g}_{\mu 5} &= A_\mu \\
\widehat{g}_{55} &= 1 \ .
\end{aligned} \tag{20.7}$$

The determinant of the metric is $\widehat{g} = g$, and the inverse metric has components

$$\begin{aligned}
\widehat{g}^{\mu\nu} &= g^{\mu\nu} \\
\widehat{g}^{\mu 5} &= -A^\mu \\
\widehat{g}^{55} &= 1 + A_\mu A^\mu \ .
\end{aligned} \tag{20.8}$$

We will (for now) assume that nothing depends on x^5 (in the old Kaluza-Klein literature this assumption is known as the *cylindricity condition*).

Introducing the notation

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
B_{\mu\nu} &= \partial_\mu A_\nu + \partial_\nu A_\mu \ ,
\end{aligned} \tag{20.9}$$

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the Christoffel symbols are readily found to be

$$
\begin{aligned}
\widehat{\Gamma}^\mu_{\nu\lambda} &= \Gamma^\mu_{\nu\lambda} - \tfrac{1}{2}(F^\mu_{\ \nu}A_\lambda + F^\mu_{\ \lambda}A_\nu) \\
\widehat{\Gamma}^5_{\nu\lambda} &= \tfrac{1}{2}B_{\nu\lambda} - \tfrac{1}{2}A^\mu(F_{\nu\mu}A_\lambda + F_{\lambda\mu}A_\nu) - A^\mu\Gamma_{\mu\nu\lambda} \\
\widehat{\Gamma}^\mu_{5\lambda} &= -\tfrac{1}{2}F^\mu_{\ \lambda} \\
\widehat{\Gamma}^5_{5\mu} &= -\tfrac{1}{2}F_{\mu\nu}A^\nu \\
\widehat{\Gamma}^\mu_{55} &= \widehat{\Gamma}^5_{55} = 0 \ .
\end{aligned}
\tag{20.10}
$$

This does not look particularly encouraging, in particular because of the presence of the $B_{\mu\nu}$ term, but Kaluza was not discouraged and proceeded to calculate the Riemann tensor. I will spare you all the components of the Riemann tensor, but the Ricci tensor we need:

$$
\begin{aligned}
\widehat{R}_{\mu\nu} &= R_{\mu\nu} + \tfrac{1}{2}F_\mu^{\ \rho}F_{\rho\nu} + \tfrac{1}{4}F^{\lambda\rho}F_{\lambda\rho}A_\mu A_\nu + \tfrac{1}{2}(A_\nu\nabla_\rho F_\mu^{\ \rho} + A_\mu\nabla_\rho F_\nu^{\ \rho}) \\
\widehat{R}_{5\mu} &= +\tfrac{1}{2}\nabla_\nu F_\mu^{\ \nu} + \tfrac{1}{4}A_\mu F_{\nu\lambda}F^{\nu\lambda} \\
\widehat{R}_{55} &= \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ .
\end{aligned}
\tag{20.11}
$$

This looks a bit more attractive and covariant but still not very promising. But now the miracle happens. Calculating the curvature scalar, all the annoying terms drop out and one finds

$$
\widehat{R} = R - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,
\tag{20.12}
$$

i.e. the Lagrangian of Einstein-Maxwell theory. For $\lambda \neq 1$, the second term would have been multiplied by λ^2. We now consider the five-dimensional pure gravity Einstein-Hilbert action

$$
\widehat{S} = \frac{1}{8\pi\widehat{G}} \int \sqrt{\widehat{g}}d^5x \ \widehat{R} \ .
\tag{20.13}
$$

In order for the integral over x^5 to converge we assume that the x^5-direction is a circle with radius L and we obtain

$$
\widehat{S} = \frac{2\pi L}{8\pi\widehat{G}} \int \sqrt{g}d^4x \ (R - \tfrac{1}{4}\lambda^2 F_{\mu\nu}F^{\mu\nu}) \ .
\tag{20.14}
$$

Therefore, if we make the identifications

$$
\begin{aligned}
G &= \widehat{G}/2\pi L \\
\lambda^2 &= 8\pi G \ ,
\end{aligned}
\tag{20.15}
$$

we obtain

$$
\widehat{S} = \frac{1}{8\pi G} \int \sqrt{g}d^4x \ R - \frac{1}{4} \int \sqrt{g}d^4x \ F_{\mu\nu}F^{\mu\nu} \ ,
\tag{20.16}
$$

i.e. precisely the four-dimensional Einstein-Maxwell Lagrangian! This amazing fact, that coupled gravity gauge theory systems can arise from higher-dimensional pure gravity, is certainly trying to tell us something.

In physics, at least, miracles require a rational explanation. So let us try to understand on a priori grounds why the Kaluza-Klein miracle occurs. For this, let us recall Kaluza's ansatz for the line element (20.4),

$$ d\widehat{s}^2_{KK} = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu + (dx^5 + A_\mu(x^\lambda)dx^\mu)^2 \quad . \tag{20.17} $$

and contrast this with the most general form of the line element in five dimensions, namely

$$
\begin{aligned}
d\widehat{s}^2 &= \widehat{g}_{MN}(x^L)dx^M dx^N \\
&= \widehat{g}_{\mu\nu}(x^\lambda, x^5)dx^\mu dx^\nu + 2\widehat{g}_{\mu5}(x^\lambda, x^5)dx^\mu dx^5 + \widehat{g}_{55}(x^\mu, x^5)(dx^5)^2 \quad . \tag{20.18}
\end{aligned}
$$

Clearly, the form of the general five-dimensional line element (20.18) is invariant under arbitrary five-dimensional general coordinate transformations $x^M \to \xi^{M'}(x^N)$. This is not true, however, for the Kaluza-Klein ansatz (20.17), as a general x^5-dependent coordinate transformation would destroy the x^5-independence of $\widehat{g}_{\mu\nu} = g_{\mu\nu}$ and $\widehat{g}_{\mu5} = A_\mu$ and would also not leave $\widehat{g}_{55} = 1$ invariant.

The form of the Kaluza-Klein line element is, however, invariant under the following two classes of coordinate transformations:

1. There are four-dimensional coordinate transformations

$$
\begin{aligned}
x^5 &\to x^5 \\
x^\mu &\to \xi^{\nu'}(x^\mu) \tag{20.19}
\end{aligned}
$$

   Under these transformations, as we know, $g_{\mu\nu}$ transforms in such a way that $g_{\mu\nu}dx^\mu dx^\nu$ is invariant, $A_\mu = \widehat{g}_{\mu5}$ transforms as a four-dimensional covector, thus $A_\mu dx^\mu$ is invariant, and the whole metric is invariant.

2. There is also another remnant of five-dimensional general covariance, namely

$$
\begin{aligned}
x^5 &\to \xi^5(x^\mu, x^5) = x^5 + f(x^\mu) \\
x^\mu &\to \xi^\mu(x^\nu) = x^\mu \quad . \tag{20.20}
\end{aligned}
$$

   Under this transformation, $g_{\mu\nu}$ and g_{55} are invariant, but $A_\mu = g_{\mu5}$ changes as

$$
\begin{aligned}
A'_\mu &= \widehat{g}'_{\mu5} = \frac{\partial x^M}{\partial \xi^\mu}\frac{\partial x^N}{\partial \xi^5}\widehat{g}_{MN} \\
&= \frac{\partial x^M}{\partial x^\mu}g_{\mu5} \\
&= g_{\mu5} - \frac{\partial f}{\partial x^\mu}g_{55} \\
&= A_\mu - \partial_\mu f \quad . \tag{20.21}
\end{aligned}
$$

In other words, the Kaluza-Klein line element is invariant under the shift $x^5 \to x^5 + f(x^\mu)$ accompanied by $A_\mu \to A_\mu - \partial_\mu f$ (and this can of course also be read off directly from the metric).

But this is precisely a *gauge transformation* of the vector potential A_μ and we see that in the present context *gauge transformations arise as remnants of five-dimensional general covariance*!

But now it is clear that we are guaranteed to get Einstein-Maxwell theory in four dimensions: First of all, upon integration over x^5, the shift in x^5 is irrelevant and starting with the five-dimensional Einstein-Hilbert action we are bound to end up with an action in four dimensions, depending on $g_{\mu\nu}$ and A_μ, which is (a) generally covariant (in the four-dimensional sense), (b) second order in derivatives, and (c) invariant under gauge transformations of A_μ. But then the only possibility is the Einstein-Maxwell action.

A fruitful way of looking at the origin of this gauge invariance is as a consequence of the fact that constant shifts in x^5 are isometries of the metric, i.e. that $\partial/\partial x^5$ is a Killing vector of the metric (20.17). Then the isometry group of the 'internal' circle in the x^5-direction, namely $SO(2)$, becomes the gauge group $U(1) = SO(2)$ of the four-dimensional theory.

From this point of view, the gauge transformation of the vector potential arises from the Lie derivative of $\widehat{g}_{\mu 5}$ along the vector field $f(x^\mu)\partial_5$:

$$
\begin{aligned}
Y = f(x^\mu)\partial_5 \quad &\Rightarrow \quad Y^\mu = 0 \\
&\qquad\;\; Y^5 = f \\
&\Rightarrow \quad Y_\mu = A_\mu f \\
&\qquad\;\; Y_5 = f \quad . \qquad\qquad\qquad\qquad (20.22)\\
(L_Y \widehat{g})_{\mu 5} \;=\; &\; \widehat{\nabla}_\mu Y_5 + \widehat{\nabla}_5 Y_\mu \\
=\; &\; \partial_\mu Y_5 - 2\widehat{\Gamma}^\mu_{5M} Y^M \\
=\; &\; \partial_\mu f + F^\nu_{\;\mu} Y_\nu + F_{\mu\nu} A^\nu Y_5 \\
=\; &\; \partial_\mu f \\
\Leftrightarrow\; &\; \delta A_\mu = -\partial_\mu f \quad . \qquad\qquad\qquad\qquad (20.23)
\end{aligned}
$$

This point of view becomes particularly useful when one wants to obtain non-Abelian gauge symmetries in this way (via a Kaluza-Klein reduction): One starts with a higher-dimensional internal space with isometry group G and makes an analogous ansatz for the metric. Then among the remnants of the higher-dimensional general coordinate transformations there are, in particular, x^μ-dependent 'isometries' of the internal metric. These act like non-Abelian gauge transformations on the off-block-diagonal components of the metric and, upon integration over the internal space, one is guaranteed to

get, perhaps among other things, the four-dimensional Einstein-Hilbert and Yang-Mills actions.

<center>GEODESICS</center>

There is something else that works very beautifully in this context, namely the description of the motion of charged particles in four dimensions moving under the combined influence of a gravitational and an electro-magnetic field. As we will see, also these two effects are unfied from a five-dimensional Kaluza-Klein point of view.

Let us consider the five-dimensional geodesic equation

$$\ddot{x}^M + \widehat{\Gamma}^M_{NL}\dot{x}^N\dot{x}^L = 0 \ . \tag{20.24}$$

Either because the metric (and hence the Lagrangian) does not depend on x^5, or because we know that $V = \partial_5$ is a Killing vector of the metric, we know that we have a conserved quantity

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^5} \sim V_M\dot{x}^M = \dot{x}^5 + A_\mu\dot{x}^\mu \ , \tag{20.25}$$

along the geodesic world lines. We will see in a moment what this quantity corresponds to. The remaining x^μ-component of the geodesic equation is

$$
\begin{aligned}
\ddot{x}^\mu + \widehat{\Gamma}^\mu_{NL}\dot{x}^N\dot{x}^L &= \ddot{x}^\mu + \widehat{\Gamma}^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda \\
&+ 2\widehat{\Gamma}^\mu_{\nu 5}\dot{x}^\nu\dot{x}^5 + 2\widehat{\Gamma}^\mu_{55}\dot{x}^5\dot{x}^5 \\
&= \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda - F^\mu_\nu A_\lambda\dot{x}^\nu\dot{x}^\lambda - F^\mu_\nu\dot{x}^\nu\dot{x}^5 \\
&= \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda - F^\mu_\nu\dot{x}^\nu(A_\lambda\dot{x}^\lambda + \dot{x}^5) \ .
\end{aligned}
\tag{20.26}
$$

Therefore this component of the geodesic equation is equivalent to

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = (A_\lambda\dot{x}^\lambda + \dot{x}^5)F^\mu_\nu\dot{x}^\nu \ . \tag{20.27}$$

This is precisely the Lorentz law if one identifies the constant of motion with the ratio of the charge and the mass of the particle,

$$\dot{x}^5 + A_\mu\dot{x}^\mu = \frac{e}{m} \ . \tag{20.28}$$

Hence electro-magnetic and gravitational forces are indeed unified. The fact that charged particles take a different trajectory from neutral ones is not a violation of the equivalence principle but only reflects the fact that they started out with a different velocity in the x^5-direction!

<center>FIRST PROBLEMS: THE EQUATIONS OF MOTION</center>

The equations of motion of the four-dimensional Einstein-Hilbert-Maxwell action will of course give us the coupled Einstein-Maxwell equations

$$
\begin{aligned}
R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R &= 8\pi G T_{\mu\nu} \\
\nabla_\mu F^{\mu\nu} &= 0 \ .
\end{aligned}
\tag{20.29}
$$

But now let us take a look at the equations of motion following from the five-dimensional Einstein-Hilbert action. These are, as we are looking at the vacuum equations, just the Ricci-flatness equations $\widehat{R}_{MN} = 0$. But looking back at (20.11) we see that these are clearly *not* equivalent to the Einstein-Maxwell equations. In particular, $\widehat{R}_{55} = 0$ imposes the constraint

$$
\widehat{R}_{55} = 0 \ \Rightarrow \ F_{\mu\nu} F^{\mu\nu} = 0 \ ,
\tag{20.30}
$$

and only then do the remaining equations $\widehat{R}_{\mu\nu} = 0$, $\widehat{R}_{\mu 5} = 0$ become equivalent to the Einstein-Maxwell equations (20.29).

What happened? Well, for one, taking variations and making a particular ansatz for the field configurations in the variational principle are two operations that in general do not commute. In particular, the Kaluza-Klein ansatz is special because it imposes the condition $g_{55} = 1$. Thus in four dimensions there is no equation of motion corresponding to $\widehat{g}_{55}$ whereas $\widehat{R}_{55} = 0$, the additional constraint, is just that, the equation arising from varying $\widehat{g}_{55}$. Thus Einstein-Maxwell theory is not a consistent truncation of five-dimensional General Relativity.

But now we really have to ask ourselves what we have actually achieved. We would like to claim that the five-dimensional Einstein-Hilbert action unifies the four-dimensional Einstein-Hilbert and Maxwell actions, but on the other hand we want to reject the five-dimensional Einstein equations? But then we are not ascribing any dynamics to the fifth dimension and are treating the Kaluza-Klein miracle as a mere kinematical, or mathematical, or bookkeeping device for the four-dimensional fields. This is clearly rather artificial and unsatisfactory.

There are some other unsatisfactory features as well in the theory we have developed so far. For instance we demanded that there be no dependence on x^5, which again makes the five-dimensional point of view look rather artificial. If one wants to take the fifth dimension seriously, one has to allow for an x^5-dependence of all the fields (and then explain later, perhaps, why we have not yet discovered the fifth dimension in every-day or high energy experiments).

21 Kaluza-Klein Theory II

With these issues in mind, we will now revisit the Kaluza-Klein ansatz, regarding the fifth dimension as real and exploring the consequences of this. Instead of considering

directly the effect of a full (i.e. not restricted by any special ansatz for the metric) five-dimensional metric on four-dimensional physics, we will start with the simpler case of a free massless scalar field in five dimensions.

<div align="center">MASSES FROM SCALAR FIELDS IN FIVE DIMENIONS</div>

Let us assume that we have a five-dimensional space-time of the form $M_5 = M_4 \times S^1$ where we will at first assume that M_4 is Minkowski space and the metric is simply

$$d\widehat{s}^2 = -dt^2 + d\vec{x}^2 + (dx^5)^2 \ , \tag{21.1}$$

with x^5 a coordinate on a circle with radius L. Now consider a massless scalar field $\widehat{\phi}$ on M_5, satisfying the five-dimensional massless Klein-Gordon equation

$$\Box\widehat{\phi}(x^\mu, x^5) = \widehat{\eta}^{MN}\partial_M\partial_N\widehat{\phi}(x^\mu, x^5) = 0 \ . \tag{21.2}$$

As x^5 is periodic with period $2\pi L$, we can make a Fourier expansion of $\widehat{\phi}$ to make the x^5-dependence more explicit,

$$\widehat{\phi}(x^\mu, x^5) = \sum_n \phi_n(x^\mu)\mathrm{e}^{inx^5/L} \ . \tag{21.3}$$

Plugging this expansion into the five-dimensional Klein-Gordon equation, we find that this turns into an infinite number of decoupled equations, one for each Fourier mode of ϕ_n of $\widehat{\phi}$, namely

$$(\Box - m_n^2)\phi_n = 0 \ . \tag{21.4}$$

Here $\Box$ of course now refers to the four-dimensional d'Alembertian, and the mass term

$$m_n^2 = \frac{n^2}{L^2} \tag{21.5}$$

arises from the x^5-derivative ∂_5^2 in $\widehat{\Box}$.

Thus we see that, from a four-dimensional perspective, a massless scalar field in five dimensions give rise to one massless scalar field in four dimensions (the harmonic or constant mode on the internal space) and an infinite number of massive fields. The masses of these fields, known as the Kaluza-Klein modes, have the behaviour $m_n \sim n/L$. In general, this behaviour mass $\sim 1/$ length scale is characteristic of massive fields arising from dimensional reduction from some higher dimensional space.

<div align="center">CHARGES FROM SCALAR FIELDS IN FIVE DIMENIONS</div>

Now, instead of looking at a scalar field on Minkowski space times a circle with the product metric, let us consider the Kaluza-Klein metric,

$$d\widehat{s}^2 = -dt^2 + d\vec{x}^2 + (dx^5 + \lambda A_\mu dx^\mu)^2 \ , \tag{21.6}$$

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and the corresponding Klein-Gordon equation

$$\Box\widehat{\phi}(x^\mu, x^5) = \widehat{g}^{MN}\widehat{\nabla}_M\partial_N\widehat{\phi}(x^\mu, x^5) = 0 \ . \tag{21.7}$$

Rather than spelling this out in terms of Christoffel symbols, it is more convenient to use (4.33) and recall that $\sqrt{\widehat{g}} = \sqrt{g} = 1$ to write this as

$$\begin{aligned}
\Box &= \partial_M(\widehat{g}^{MN}\partial_N) \\
&= \partial_\mu\widehat{g}^{\mu\nu}\partial_\nu + \partial_5\widehat{g}^{5\mu}\partial_\mu + \partial_\mu\widehat{g}^{\mu5}\partial_5 + \partial_5\widehat{g}^{55}\partial_5 \\
&= \eta^{\mu\nu}\partial_\mu\partial_\nu + \partial_5(-\lambda A^\mu\partial_\mu) + \partial_\mu(-\lambda A^\mu\partial_5) + (1 + \lambda^2 A_\mu A^\mu)\partial_5\partial_5 \\
&= \eta^{\mu\nu}(\partial_\mu - \lambda A_\mu\partial_5)(\partial_\nu - \lambda A_\nu\partial_5) + (\partial_5)^2 \ .
\end{aligned} \tag{21.8}$$

Acting with this operator on the Fourier decomposition of $\widehat{\phi}$, we evidently again get an infinte number of decoupled equations, one for each Fourier mode ϕ_n of $\widehat{\phi}$, namely

$$\left[\eta^{\mu\nu}(\partial_\mu - i\frac{\lambda n}{L}A_\mu)(\partial_\nu - i\frac{\lambda n}{L}A_\nu) - m_n^2\right]\phi_n = 0 \ . \tag{21.9}$$

This shows that the non-constant ($n \neq 0$) modes are not only massive but also charged under the gauge field A_μ. Comparing the operator

$$\partial_\mu - i\frac{\lambda n}{L}A_\mu \tag{21.10}$$

with the standard form of the minimal coupling,

$$\frac{\hbar}{i}\partial_\mu - eA_\mu \ , \tag{21.11}$$

we learn that the electric charge e_n of the n'th mode is given by

$$\frac{e_n}{\hbar} = \frac{n\lambda}{L} \ . \tag{21.12}$$

In particular, these charges are all integer multiples of a basic charge, $e_n = ne$, with

$$e = \frac{\hbar\lambda}{L} = \frac{\sqrt{8\pi G}\hbar}{L} \ . \tag{21.13}$$

Thus we get a formula for L, the radius of the fifth dimension,

$$L^2 = \frac{8\pi G\hbar^2}{e^2} = \frac{8\pi G\hbar}{e^2/\hbar} \ . \tag{21.14}$$

Restoring the velocity of light in this formula, and identifying the present $U(1)$ gauge symmetry with the standard gauge symmetry, we recognize here the fine structure constant

$$\alpha = e^2/4\pi\hbar c \approx 1/137 \ , \tag{21.15}$$

and the *Planck length*

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}\mathrm{cm} \ . \tag{21.16}$$

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Thus

$$L^2 = \frac{2\ell_P^2}{\alpha} \approx 274\ell_P^2 \quad . \tag{21.17}$$

This is very small indeed, and it is therefore no surprise that this fifth dimension, if it is the origin of the $U(1)$ gauge invariance of the world we live in, has not yet been seen.

Another way of saying this is that the fact that L is so tiny implies that the masses m_n are huge, not far from the *Planck mass*

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 10^{-5}\mathrm{g} \approx 10^{19}\,\mathrm{GeV} \quad . \tag{21.18}$$

These would never have been spotted in present-day accelerators. Thus the massive modes are completely irrelevant for low-energy physics, the non-constant modes can be dropped, and this provides a justification for neglecting the x^5-dependence. However, this also means that the charged particles we know (electrons, protons, ...) cannot possibly be identified with these Kaluza-Klein modes.

The way modern Kaluza-Klein theories address this problem is by identifying the light charged particles we observe with the massless Kaluza-Klein modes. One then requires the standard spontaenous symmetry breaking mechanism to equip them with the small masses required by observation. This still leaves the question of how these particles should pick up a charge (as the zero modes are not only masless but also not charged). This is solved by going to higher dimensions, with non-Abelian gauge groups, for which masless particles are no longer necessarily singlets of the gauge group (they could e.g. live in the adjoint).

KINEMATICS OF DIMENSIONAL REDUCTION

We have seen above that a massless scalar field in five dimensions gives rise to a massless scalar field plus an infinite tower of massive scalar fields in four dimensions. What happens for other fields (after all, we are ultimately interested in what happens to the five-dimensional metric)?

Consider, for example, a five-dimensional vector potential (covector field) $\widehat{B}_M(x^N)$. From a four-dimensional vantage point this looks like a four-dimensional vector field $B_\mu(x^\nu, x^5)$ and a scalar $\phi(x^\mu, x^5) = B_5(x^\mu, x^5)$. Fourier expanding, one will then obtain in four dimensions:

1. one massless Abelian gauge field $B_\mu(x^\nu)$

2. an infinite tower of massive charged vector fields

3. one massless scalar field $\phi(x^\mu) = B_5(x^\mu)$

4. an infinite tower of massive charged scalar fields

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Retaining, for the same reasons as before, only the massless, i.e. x^5-independent, modes we therefore obtain a theory involving one scalar field and one Abelian vector field from pure Maxwell theory in five dimensions. The Lagrangian for these fields would be (dropping all x^5-derivatives)

$$
\begin{aligned}
F_{MN}F^{MN} &= F_{\mu\nu}F^{\mu\nu} + 2F_{\mu 5}F^{\mu 5} \\
&\to F_{\mu\nu}F^{\mu\nu} + 2(\partial_\mu\phi)(\partial^\mu\phi) \ .
\end{aligned}
\tag{21.19}
$$

This procedure of obtaining Lagrangians in lower dimensions from Lagrangians in higher dimensions by simply dropping the dependence on the 'internal' coordinates is known as dimensional reduction or Kaluza-Klein reduction. But the terminology is not uniform here - sometimes the latter term is used to indicate the reduction including all the massive modes. Also, in general 'massless' is not the same as 'x^5-independent', and then Kaluza-Klein reduction may refer to keeping the massless modes rather than the x^5-independent modes one retains in dimensional reduction.

Likewise, we can now consider what happens to the five-dimensional metric $\widehat{g}_{MN}(x^L)$. From a four-dimensional perspective, this splits into three different kinds of fields, namely a symmetric tensor $\widehat{g}_{\mu\nu}$, a covector $A_\mu = \widehat{g}_{\mu 5}$ and a scalar $\phi = \widehat{g}_{55}$. As before, these will each give rise to a massless field in four dimensions (which we interpret as the metric, a vector potential and a scalar field) as well as an infinite number of massive fields.

We see that, in addition to the massless fields we considered before, in the old Kaluza-Klein ansatz, we obtain one more massless field, namely the scalar field ϕ. Thus, even if we may be justified in dropping all the massive modes, we should keep this massless field in the ansatz for the metric and the action. With this in mind we now return to the Kaluza-Klein ansatz.

The Kaluza-Klein Ansatz Revisited

Let us once again consider pure gravity in five dimensions, i.e. the Einstein-Hilbert action

$$
\widehat{S} = \frac{1}{8\pi\widehat{G}} \int \sqrt{\widehat{g}}d^5 x \ \widehat{R} \ .
\tag{21.20}
$$

Let us now parametrize the full five-dimensional metric as

$$
d\widehat{s}^2 = \phi^{-1/3}[g_{\mu\nu}dx^\mu dx^\mu + \phi(dx^5 + \lambda A_\mu dx^\mu)^2] \ ,
\tag{21.21}
$$

where all the fields depend on all the coordinates x^μ, x^5. Any five-dimensional metric can be written in this way and we can simply think of this as a change of variables

$$
\widehat{g}_{MN} \to (g_{\mu\nu}, A_\mu, \phi) \ .
\tag{21.22}
$$

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In matrix form, this metric reads

$$(\widehat{g}_{MN}) = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} + \lambda^2 \phi A_\mu A_\nu & \lambda\phi A_\nu \\ \lambda\phi A_\mu & \phi \end{pmatrix} \qquad (21.23)$$

For a variety of reasons, this particular parametrization is useful. In particular, it reduces to the Kaluza-Klein ansatz when $\phi = 1$ and all the fields are independent of x^5 and the ϕ's in the off-diagonal component ensure that the determinant of the metric is independent of the A_μ.

The only thing that may require some explanation is the strange overall power of ϕ. To see why this is a good choice, assume that the overall power is ϕ^a for some a. Then for $\sqrt{\widehat{g}}$ one finds

$$\sqrt{\widehat{g}} = \phi^{5a/2}\phi^{1/2}\sqrt{g} = \phi^{(5a+1)/2}\sqrt{g} \ . \qquad (21.24)$$

On the other hand, for the Ricci tensor one has, schematically,

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \dots \ , \qquad (21.25)$$

and therefore

$$\begin{aligned} \hat{R} &= \widehat{g}^{\mu\nu} R_{\mu\nu} + \dots \\ &= \phi^{-a} g^{\mu\nu} R_{\mu\nu} + \dots \\ &= \phi^{-a} R + \dots \ . \end{aligned} \qquad (21.26)$$

Hence the five-dimensional Einstein-Hilbert action reduces to

$$\begin{aligned} \sqrt{\widehat{g}}\hat{R} &\sim \phi^{(5a+1)/2}\phi^{-a}\sqrt{g}R + \dots \\ &= \phi^{(3a+1)/2}\sqrt{g}R + \dots \ . \end{aligned} \qquad (21.27)$$

Thus, if one wants the five-dimensional Einstein-Hilbert action to reduce to the standard four-dimensional Einstein-Hilbert action (plus other things), without any non-minimal coupling of the scalar field ϕ to the metric, one needs to choose $a = -1/3$ which is the choice made in (21.21,21.23).

Making a Fourier-mode expansion of all the fields, plugging this into the Einstein-Hilbert action

$$\frac{1}{8\pi\widehat{G}} \int \sqrt{\widehat{g}}d^5x \ \hat{R} \ , \qquad (21.28)$$

integrating over x^5 and retaining only the constant modes $g_{(0)\mu\nu}$, $A_{(0)\mu}$ and $\phi_{(0)}$, one obtains the action

$$S = \int \sqrt{g}d^4x \ \left[\frac{1}{8\pi G}R(g_{(0)\mu\nu}) - \frac{1}{4}\phi_{(0)}F_{(0)\mu\nu}F^{\mu\nu}_{(0)} - \frac{1}{48\pi G}\phi^{-2}_{(0)}g^{\mu\nu}_{(0)}\partial_\mu\phi_{(0)}\partial_\nu\phi_{(0)} \right] \ . \qquad (21.29)$$

Here we have once again made the identifications (20.15). This action may not look as nice as before, but it is what it is. It is at least generally covariant and gauge invariant,

as expected. We also see very clearly that it is inconsistenst with the equations of motion for $\phi_{(0)}$,

$$\square \log \phi_{(0)} = \frac{3}{4} 8\pi G \phi_{(0)} F_{(0)\mu\nu} F_{(0)}^{\mu\nu} \ , \tag{21.30}$$

to set $\phi_{(0)} = 1$ as this would imply $F_{(0)\mu\nu} F_{(0)}^{\mu\nu} = 0$, in agreement with our earlier observations regarding $\hat{R}_{55} = 0$.

However, the configuration $g_{(0)\mu\nu} = \eta_{\mu\nu}$, $A_{(0)\mu} = 0$, $\phi_{(0)} = 1$ is a solution to the equations of motion and defines the 'vacuum' or ground state of the theory. From this point of view the zero mode metric, (21.23) with the fields replaced by their zero modes, i.e. the Kaluza-Klein ansatz with the inclusion of ϕ, has the following interpretation: as usual in quantum theory, once one has chosen a vacuum, one can consider fluctuations around that vacuum. The fields $g_{(0)\mu\nu}, A_{(0)\mu}, \phi_{(0)}$ are then the massless fluctuations around the vacuum and are the fields of the low-energy action. The full classical or quantum theory will also contain all the massive and charged Kaluza-Klein modes.

Non-Abelian Generalization and Outlook

Even though in certain respects the Abelian theory we have discussed above is atypical, it is rather straightforward to generalize the previous considerations from Maxwell theory to Yang-Mills theory for an arbitrary non-Abelian gauge group. Of course, to achieve that, one needs to consider higher-dimensional internal spaces, i.e. gravity in $4 + d$ dimensions, with a space-time of the form $M_4 \times M_d$. The crucial observation is that gauge symmetries in four dimensions arise from isometries (Killing vectors) of the metric on M_d.

Let the coordinates on M_d be x^a, denote by g_{ab} the metric on M_d, and let K_i^a, $i = 1, \ldots, n$ denote the n linearly independent Killing vectors of the metric g_{ab}. These generate the Lie algebra of the isometry group G via the Lie bracket

$$[K_i, K_j]^a \equiv K_i^b \partial_b K_j^a - K_j^b \partial_b K_i^a = f_{ij}^k K_k^a \ . \tag{21.31}$$

M_d could for example be the group manifold of the Lie group G itself, or a homogeneous space G/H for some subgroup $H \subset G$.

Now consider the following Kaluza-Klein ansatz for the metric,

$$d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ab}(dx^a + K_i^a A_\mu^i dx^\mu)(dx^b + K_j^b A_\nu^j dx^\nu) \ . \tag{21.32}$$

Note the appearance of fields with the correct index structure to act as non-Abelian gauge fields for the gauge group G, namely the A_μ^i. Again these should be thought of as fluctuations of the metric around its 'ground state', $M_4 \times M_d$ with its product metric $(g_{\mu\nu}, g_{ab})$.

Now consider an infinitesimal coordinate transformation generated by the vector field

$$V^a(x^\mu, x^b) = f^i(x^\mu) K_i^a(x^b) \; , \tag{21.33}$$

i.e.

$$\delta x^a = f^i(x^\mu) K_i^a(x^b) \; . \tag{21.34}$$

This leaves the form of the metric invariant, and

$$\delta \widehat{g}_{\mu a} = L_V \widehat{g}_{\mu a} \tag{21.35}$$

can be seen to imply

$$\delta A_\mu^i = D_\mu f^i \equiv \partial_\mu f^i - f_{jk}^i A_\mu^j f^k \; , \tag{21.36}$$

i.e. precisely an infinitesimal non-Abelian gauge transformation. The easiest way to see this is to use the form of the Lie derivative not in its covariant form,

$$L_V \widehat{g}_{\mu a} = \widehat{\nabla}_\mu V_a + \widehat{\nabla}_a V_\mu \tag{21.37}$$

(which requires knowledge of the Christoffel symbols) but in the form

$$L_V \widehat{g}_{\mu a} = V^c \partial_c \widehat{g}_{\mu a} + \partial_\mu V^c \widehat{g}_{ca} + \partial_a V^c \widehat{g}_{\mu c} \; . \tag{21.38}$$

Inserting the definitions of $\widehat{g}_{\mu a}$ and V^a, using the fact that the K_i^a are Killing vectors of the metric g_{ab} and the relation (21.31), one finds

$$L_V \widehat{g}_{\mu a} = g_{ab} K_i^b D_\mu f^i \; , \tag{21.39}$$

and hence (21.36).

One is then assured to find a Yang-Mills like term

$$L_{YM} \sim F_{\mu\nu}^i F^{j\,\mu\nu} K_i^a K_j^b g_{ab} \tag{21.40}$$

in the reduction of the Lagrangian from $4 + d$ to 4 dimensions.

The problem with this scenario (already prior to worrying about the inclusion of scalar fields, of which there will be plenty in this case, one for each component of g_{ab}) is that the four-dimensional space-time cannot be chosen to be flat. Rather, it must have a huge cosmological constant. This arises because the dimensional reduction of the $(4 + d)$-dimensional Einstein-Hilbert Lagrangian $\widehat{R}$ will also include a contribution from the scalar curvature R_d of the metric on M_d. For a compact internal space with non-Abelian isometries this scalar curvature is non-zero and will therefore lead to an effective cosmological constant in the four-dimensional action. This cosmological constant could be cancelled 'by hand' by introducing an appropriate cosmological constant of the opposite sign into the $(d + 4)$-dimensional Einstein-Hilbert action, but this looks rather contrived and artificial.

Nevertheless, this and other problems have not stopped people from looking for 'realistic' Kaluza-Klein theories giving rise to the standard model gauge group in four dimension. Of course, in order to get the standard model action or something resembling it, fermions need to be added to the $(d+4)$-dimensional action.

An interesting observation in this regard is that the lowest possible dimension for a homogenous space with isometry group $G = SU(3) \times SU(2) \times U(1)$ is seven, so that the dimension of space-time is eleven. This arises because the maximal compact subgroup H of G, giving rise to the smallest dimensional homogeneous space G/H of G, is $SU(2) \times U(1) \times U(1)$. As the dimension of G is $8 + 3 + 1 = 12$ and that of H is $3 + 1 + 1 = 5$, the dimension of G/H is $12 - 5 = 7$. This is intriguing because eleven is also the highest dimension in which supergravity exists (in higher dimensions, supersymmetry would require the existence of spin > 2 particles). That, plus the hope that supergravity would have a better quantum behaviour than ordinary gravity, led to an enourmous amount of activity on Kaluza-Klein supergravity in the early 80's.

Unfortunately, it turned out that not only was supergravity sick at the quantum level as well but also that it is impossible to get a chiral fermion spectrum in four dimensions from pure gravity plus spinors in $(4+d)$ dimensions. One way around the latter problem is to include explicit Yang-Mills fields already in $(d+4)$-dimensions, but that appeared to defy the purpose of the whole Kaluza-Klein idea.

Today, the picture has changed and supergravity is regarded as a low-energy approximation to string theory which is believed to give a consistent description of quantum gravity. These string theories typically live in ten dimensions, and thus one needs to 'compactify' the theory on a small internal six-dimensional space, much as in the Kaluza-Klein idea. Even though non-Abelian gauge fields now typically do not arise from Kaluza-Klein reduction but rather from explicit gauge fields in ten dimensions, in all other respects Kaluza's old idea is alive, doing very well, and an indispensable part of the toolkit of modern theoretical high energy physics.

THE END