Torsion, Localization and Applications

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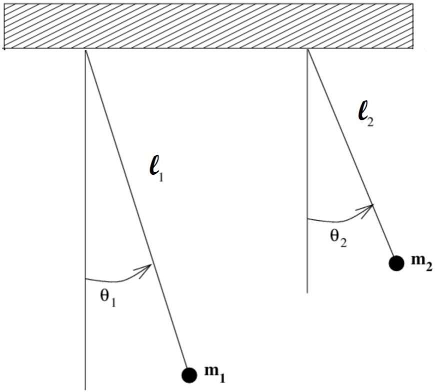
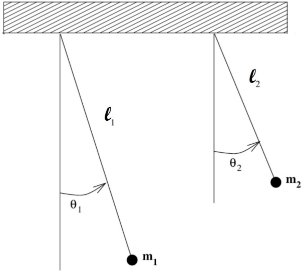


The system of OD equations $\begin{cases} \ddot{x} + \ell_1 \ddot{\theta}_1 + g\theta_1 = 0 \\ \ddot{x} + \ell_2 \ddot{\theta}_2 + g\theta_2 = 0 \end{cases}$ is rewritten in the new variables $x_1 = x + \ell_1\theta_1, x_2 = x + \ell_2\theta_2, u = x$ in the matrix form.



$$\begin{bmatrix} d_t^2 + \frac{g}{\ell_1} & 0 & -\frac{g}{\ell_1} \\ 0 & d_t^2 + \frac{g}{\ell_2} & -\frac{g}{\ell_2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notably: this system does not depend on the masses m_i!

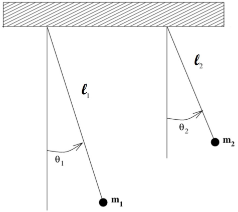
Since g is the gravitarional constant and ℓ_1, ℓ_2 are fixed, we treat them as *parameters*, i.e. our ground field is $\mathbb{Q}(g, \ell_1, \ell_2)$.

Since nothing depends on time t, we have a system of ODEs with *constant coefficients*, rewritten as a matrix over the algebra of operators $\mathbb{Q}(g, \ell_1, \ell_2)[d_t]$.

Computations show that the system is (strongly) controllable; and there's the left inverse matrix of the so-called image representation of the system:

$$\begin{bmatrix} \frac{\ell_1}{g^2(\ell_1-\ell_2)} & -\frac{\ell_2}{g^2(\ell_1-\ell_2)} & 0 \end{bmatrix}$$

We see that the result is valid, provided $\ell_1 \neq \ell_2$!



Let us rerun the computation for the case $\ell_1 = \ell_2 =: \ell$!

Over system, written as a matrix over the algebra of operators $\mathbb{Q}(g, \ell)[d_t]$.

$$\begin{bmatrix} d_t^2 + \frac{g}{\ell} & 0 & -\frac{g}{\ell} \\ 0 & d_t^2 + \frac{g}{\ell} & -\frac{g}{\ell} \end{bmatrix}$$

is not controllable anymore: there is a nonzero **torsion submodule**!

It is annihilated by the ideal $\langle \ell d_t^2 + g \rangle$ meaning there are **autonomous elements** like the difference $x_1 - x_2$ of the positions of the pendula (relative to the bar).

Recall some results from Algebra

Let R be a ring (associative, with 1).

- A **free module** R^m: set of (column) vectors of length m with entries from R $v = [r_1, \dots, r_m]^T = \sum r_i e_i$, where $\{e_i\}$ form a basis of R

  analogy with vector spaces: R^m is closed under addition and under the multiplication by "scalar" elements from R.

- A **submodule** of R^m is an analogue to the subspace: if it is generated by the finite set of vectors; put them into a matrix U.

  Pass to the factor-module $M = R^m/U$, which is *finitely generated* by $[e_1], \dots, [e_m]$ with $[e] = e + U$.

If $R = K$ is a field, we are back in the course of *linear algebra*

Every nonzero submodule is free; every nonzero factor-module is free.

Over a general ring R:

Free $\Rightarrow$ stably free $\Rightarrow$ projective $\Rightarrow$ flat $\Rightarrow$ torsion-free.

The first appearance of torsion

Let R be a PID (principal ideal domain) like $\mathbb{Z}$ or $K[x]$: there are no zero-divisors except 0.

Main Theorem on finitely generated modules over PID

$\exists k \in \mathbb{N}_0, \exists t \in \mathbb{N}_0 \quad \exists a_1, \ldots, a_k \neq 0$ such that $a_1 \mid a_2 \mid \ldots \mid a_k$

$$M \cong R/\langle a_1 \rangle \times \cdots \times R/\langle a_k \rangle \times R^t$$

Let $D := \begin{bmatrix} a_1 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & a_k \end{bmatrix}$, then $M \cong R^k/D \oplus R^t$ and $t(M) := R^k/D$.

For a domain R, $m \in M$ is a **torsion element**, if $\exists 0 \neq r \in R : r \cdot m = 0_M$. The **torsion submodule** $t(M)$ of M consists of torsion elements.

Dichotomy

M is called **torsion module**, if $t(M) = M$, and **torsion-free module**, if $t(M) = \{0\}$. (In System theory reflected by controllable/autonomous.)

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Localization of commutative domains

Convention

From now on: let R be a domain (no zero-divisors).

Definition

A subset S of R is called a **multiplicative set** if

- $0 \notin S$,
- $1 \in S$ and
- S is **multiplicatively closed**, that is, $\forall s, t \in S : s \cdot t \in S$.

Notation: $[S] :=$ the smallest multiplicative superset of S.

Construction

Theorem (Classical)

Let S be a multiplicative set in a commutative domain R. Then

$$S^{-1}R := \left\{ \frac{r}{s} \mid s \in S, r \in R \right\} = \{ s^{-1}r \mid s \in S, r \in R \}$$

is a commutative domain, where

- $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ *if and only if* $s_1 r_2 = s_2 r_1$,
- $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$,
- $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$.

Example

$R = \mathbb{Z}, S = \mathbb{Z} \setminus \{0\} \quad \Rightarrow \quad S^{-1}R = \left\{ \frac{r}{s} \mid r, s \in \mathbb{Z}, s \neq 0 \right\} = \mathbb{Q}$

Commutative examples I

Let R be a commutative domain and K a field.

Quotient fields

$S = R \setminus \{0\} \Rightarrow \mathrm{Quot}(R) := S^{-1}R = \left\{ \frac{r}{s} \mid r, s \in R, s \neq 0 \right\}$ is a field.

- $\mathrm{Quot}(\mathbb{Z}) = \mathbb{Q}$
- $\mathrm{Quot}(K[x]) = K(x)$
- Quot({holomorphic functions}) = {meromorphic functions}

Origin of the name: algebraic geometry

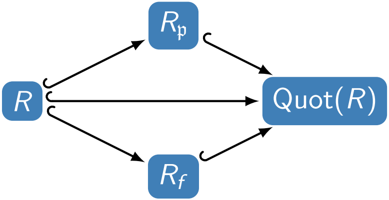
Let $a \in K^n$ and $\mathfrak{m} := \langle x_1 - a_1, \ldots, x_n - a_n \rangle \subseteq K[x_1, \ldots, x_n] =: P$. Then $S := R \setminus \mathfrak{m}$ is a multiplicative set in P and $P_{\mathfrak{m}} := S^{-1}P$ describes the "local" behavior near a.

Laurent polynomials

For $[x] = \left\{ x^k \mid k \in \mathbb{N}_0 \right\}$: $\quad [x]^{-1}K[x] = K[x, x^{-1}] \subsetneq K(x)$

Commutative examples II

$R_{\mathfrak{p}} := \left\{ \frac{p}{q} \mid p, q \in R, q \notin \mathfrak{p} \right\}$, $\mathfrak{p} \subseteq R$ prime ideal
$\Rightarrow R_{\mathfrak{p}} = S^{-1}R$, where $S = R \setminus \mathfrak{p}$
Example: $K[x]_{\langle x \rangle} = \left\{ \frac{f}{g} \in K(x) \mid g(0) \neq 0 \right\}$



$\text{Quot}(R) := \left\{ \frac{p}{q} \mid p, q \in R, q \neq 0 \right\}$
$\Rightarrow \text{Quot}(R) = S^{-1}R$,
where $S = R \setminus \{0\}$
Example: $\text{Quot}(K[x]) = K(x)$

$R_f := \left\{ \frac{p}{f^k} \mid p \in R, k \in \mathbb{N}_0 \right\}$, $f \in R \setminus \{0\}$
$\Rightarrow R_f = S^{-1}R$, where $S = [f]$
Example: $K[x]_x = K[x, x^{-1}]$

Basic properties

Let S be a multiplicative set in a commutative domain R.

Lemma

(a) $1_{S^{-1}R} = \frac{1}{1} = \frac{s}{s}$ *for all* $s \in S$.

(b) $0_{S^{-1}R} = \frac{0}{1} = \frac{0}{s}$ *for all* $s \in S$.

(c) $\frac{r}{s} = 1$ *if and only if* $s = r$.

(d) $\frac{r}{s} = 0$ *if and only if* $r = 0$.

(e) $\frac{r}{s} = \frac{tr}{ts}$ *for all* $t \in R$ *such that* $ts \in S$.

(f) $-\frac{r}{s} = \frac{-r}{s}$.

(g) $R \to S^{-1}R, r \mapsto \frac{r}{1}$ *is an injective homomorphism.*

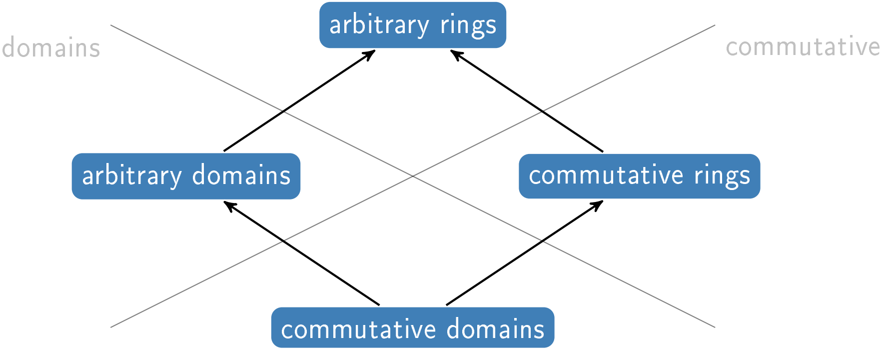
(h) $S^{-1}R$ *is a domain.*

(i) *Every ideal in* $S^{-1}R$ *is the extension of an ideal in* R.

(j) *If* R *is Noetherian/Artinian/PID, so is* $S^{-1}R$.

(k) $\{$*prime ideals in* $S^{-1}R\} \stackrel{1:1}{\longleftrightarrow} \{$*prime ideals in* R *which do not meet* $S\}$

The hierarchy of Ore localizations: localization of...



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Axiomatic definition of left Ore localization

Definition

Let S be a multiplicative set in R. A ring R_S and an injective homomorphism $\varphi : R \to R_S$ are a **left Ore localization** of R at S if:

(1) For all $s \in S$, $\varphi(s)$ is invertible in $S^{-1}R$.

(2) For all $x \in R_S$, there exist $s \in S$ and $r \in R$ such that $x = \varphi(s)^{-1}\varphi(r)$.

Theorem

Let S be a multiplicative set in R. If a left Ore localization of R at S exists, then it is unique up to isomorphism.

"Working" with non-commutative fractions

Let S be a multiplicative set in R such that the left Ore localization of R at S exists. For brevity we write a "left fraction" $\varphi(s)^{-1}\varphi(r)$ simply as $s^{-1}r$.

Multiplication

Take two left fractions $s_1^{-1}r_1, s_2^{-1}r_2 \in R_S$. Their product $s_1^{-1}r_1 \cdot s_2^{-1}r_2$ must again be writable as a left fraction, thus there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$r_1 s_2^{-1} = \tilde{s}^{-1}\tilde{r} \quad \Leftrightarrow \quad \tilde{s}r_1 = \tilde{r}s_2,$$

then we get

$$s_1^{-1}r_1 s_2^{-1}r_2 = s_1^{-1}\tilde{s}^{-1}\tilde{r}r_2 = (\tilde{s}s_1)^{-1}\tilde{r}r_2.$$

Corollary

If the left Ore localization of R at S exists, then S is a left Ore set in R.

Left Ore sets

Definition

Let S be a subset of R.

- S satisfies the **left Ore condition** in R if

$$\forall\ s \in S, r \in R \qquad \exists\ \tilde{s} \in S, \tilde{r} \in R : \qquad \tilde{s} r = \tilde{r} s.$$

  Equivalently: $\forall\ s \in S, r \in R :\ Sr \cap Rs \neq \emptyset$.
- **left Ore set** := multiplicative set + left Ore condition

Consequences of the left Ore condition on S in R

- Any right fraction rs^{-1} can be rewritten as a left fraction $\tilde{s}^{-1}\tilde{r}$.
- Finitely many elements have a **common left multiple** in S.

Construction of the left Ore localization I

Theorem (Ore 1931)

Let S be a left Ore set in R.

(a) *The following is an equivalence relation on $S \times R$:*

$$(s_1, r_1) \sim (s_2, r_2) \;\Leftrightarrow\; \exists\, \tilde{s} \in S, \tilde{r} \in R : \;\tilde{s}s_2 = \tilde{r}s_1 \text{ and } \tilde{s}r_2 = \tilde{r}r_1$$

Write the class of (s, r) wrt. to $\sim$ again as (s, r) or as $s^{-1}r$.

(b) *$S^{-1}R := ((S \times R)/\sim, +, \cdot)$ is a ring with the operations*

$$+ : S^{-1}R \times S^{-1}R \to S^{-1}R,\; (s_1, r_1) + (s_2, r_2) := (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$, and

$$\cdot : S^{-1}R \times S^{-1}R \to S^{-1}R,\; (s_1, r_1) \cdot (s_2, r_2) := (\tilde{s}s_1, \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$.

Construction of the left Ore localization II

Definition

The map

$$\rho_{S,R} : R \to S^{-1}R, \quad r \mapsto (1, r),$$

is called **structural homomorphism** or **localization map** of $S^{-1}R$.

Lemma

The pair $(S^{-1}R, \rho_{S,R})$ is the left Ore localization of R at S.

Corollary

Let S be a multiplicative subset of R. The following are equivalent:

(1) *The left Ore localization of R at S exists.*

(2) *S is a left Ore set in R.*

Basic properties

Lemma

(a) $1_{S^{-1}R} = (1,1) = (s,s)$ *for all* $s \in S$.

(b) $0_{S^{-1}R} = (1,0) = (s,0)$ *for all* $s \in S$.

(c) $(s,r) = 1$ *if and only if* $s = r$.

(d) $(s,r) = 0$ *if and only if* $r = 0$.

(e) $(s,r) = (ts,tr)$ *for all* $t \in R$ *such that* $ts \in S$.

(f) $-(s,r) = (s,-r)$.

(g) $R \to S^{-1}R, r \mapsto (1,r)$ *is an injective homomorphism.*

(h) $S^{-1}R$ *is a domain.*

(i) *Every left ideal in* $S^{-1}R$ *is the extension of a left ideal in* R.

(j) *If* R *is left Noetherian/Artinian/PID, so is* $S^{-1}R$.

The old good Weyl algebra

The 1st polynomial Weyl algebra

$$A_1(K) \;= K\langle x, \partial \mid \partial x = x\partial + 1\rangle$$

or, stressing that we work over the ring of *polynomial coefficients*

$$K[x]\langle \partial \mid \partial x = x\partial + 1\rangle \;=\; K[x][\partial; 1, \tfrac{\partial}{\partial x}]$$

where the latter is the formulation via **Ore extension**.

Note that A_1 is a Noetherian domain.

The 1st rational Weyl algebra

$$B_1(K) = K(x)\langle \partial \mid \partial x = x\partial + 1\rangle \;=\; K(x)[\partial; 1, \tfrac{\partial}{\partial x}]$$

It is the **Ore localization** of A_1 at the Ore set $S = K[x] \setminus \{0\}$, and thus $B_1 \cong S^{-1} A_1$.

The old good Weyl algebra I

Lemma

The following are left Ore sets in A_1:

- $S = K[x] \setminus \{0\}$ *and* $K[\partial] \setminus \{0\}$

  $\Rightarrow S^{-1}A_1 = B_1 := K(x)\langle \partial \mid \partial f = f\partial + \frac{df}{dx} \text{ for all } f \in K(x) \rangle$

- $[x]$ *and* $[\partial]$

  $\Rightarrow [x]^{-1}A_1 \cong K\langle x, x^{-1}, \partial \mid \partial x = x\partial + 1, \partial x^{-1} = x^{-1}\partial + x^{-2} \rangle$

  (the first "Laurent Weyl algebra")

- $V := [x, \partial] = [[x] \cup [\partial]]$

Dimension of the space of holomorphic solutions

Theorem (Cauchy-Kowalewska-Kashiwara)

Let $K = \mathbb{C}$, $D = A_n(\mathbb{C})$ the n-th Weyl algebra, $\mathcal{I} \subset D$ a left ideal such that $D/\mathcal{I}$ is a holonomic D-module (i. e. $\mathrm{GKdim} D/\mathcal{I} = n$). Moreover, let $\mathrm{Sing}(\mathcal{I})$ be the singular locus of $\mathcal{I}$ and U a simply connected domain in $\mathbb{C}^n \setminus \mathrm{Sing}(\mathcal{I})$. Consider the system of differential equations $\{\mathfrak{d} \bullet f = 0 \mid \mathfrak{d} \in I\}$ for holomorphic functions f on U. Then the dimension of the complex vector space of solutions to this system is equal to the holonomic rank of $D/\mathcal{I}$.

... where the **holonomic rank** of $D/\mathcal{I}$ (or of a fin. pres. D-module) is nothing else but

$$\dim_{K(x)} S^{-1}D/S^{-1}\mathcal{I} = \dim_{K(x)} B_n/B_n\mathcal{I}$$

for $S = K[x] \setminus \{0\}$. This value is computable as well as $\mathrm{Sing}(\mathcal{I})$.

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Multiplicative inverses

Let $(s, r) \in S^{-1}R$, then its additive inverse is given by $(s, -r)$.

What about multiplication?
How can we describe $U(S^{-1}R) := \{a \in S^{-1}R \mid a \text{ invertible/unit}\}$?
Some immediate sufficient conditions:

- If $r \in S$, then (s, r) is invertible with $(s, r)^{-1} = (r, s)$.
- If r is a unit in R, then (s, r) is invertible with $(s, r)^{-1} = (1, r^{-1}s)$.

But $S^{-1}R$ may contain more units:

Example

Let K be a field. Consider $\frac{x}{1}$ in $[x^2]^{-1}K[x]$. Now $x \notin [x^2]$ and x is not a unit in $K[x]$, but $\frac{x}{1}$ is invertible with $\left(\frac{x}{1}\right)^{-1} = \frac{x}{x^2}$.

Left saturation closure

Definition

Let P be a subset of R.

- P is called **left saturated** if for all $a, b \in R : ab \in P \Rightarrow b \in P$.
- The **left saturation closure** of P in R is

$$\mathrm{LSat}(P) := \{r \in R \mid \exists w \in R : wr \in P\} \supseteq P.$$

Lemma

(a) *P is left saturated* $\Leftrightarrow P = \mathrm{LSat}(P)$.

(b) *If $P \neq \emptyset$: $U(R) \subseteq \mathrm{LSat}(P)$.*

The old good Weyl algebra II

Consider the first polynomial Weyl algebra $A_1 = K\langle x, \partial \mid \partial x = x\partial + 1\rangle$. Note that $U(A_1) = K \setminus \{0\}$ is contained in any of the following closures, but is sometimes omitted for brevity.

Example

$\mathrm{LSat}([x^n]) = \mathrm{LSat}([x]) = [x]$ and $\mathrm{LSat}(K[x] \setminus \{0\}) = K[x] \setminus \{0\}$.

Definition

The **Euler operator** in A_1 is $\theta := x\partial = \partial x - 1$.

Example

Let $V := [x, \partial]$ and $\Theta := [\theta + \mathbb{Z}] = [\{x\partial + z \mid z \in \mathbb{Z}\}]$.

(a) V and Θ are left Ore sets in A_1. (non-trivial)

(b) $\mathrm{LSat}(V) = \mathrm{LSat}(\Theta)$. (easy)

(c) $\mathrm{LSat}(V) = [(\theta + \mathbb{Z}) \cup \{x, \partial\}]$. ($\subseteq$ highly non-trivial)

The units of the localized ring

Note that $(s, r) = (s, 1) \cdot (1, r)$.

Theorem

Let $(s, r) \in S^{-1}R$. The following are equivalent:

(1) $(s, r) \in U(S^{-1}R)$.

(2) $(1, r) \in U(S^{-1}R) \quad \Leftrightarrow \quad r \in \rho^{-1}(U(S^{-1}R))$.

(3) $r \in \mathrm{LSat}(S) \quad \Leftrightarrow \quad \exists w \in R : wr \in S$.

(4) $Rr \cap S \neq \emptyset$.

$\Rightarrow$ $\mathrm{LSat}(S)$ is the set of all elements of R that become invertible in the localization $S^{-1}R$

Localization at left saturation

Reminder

$$\mathrm{LSat}(S) := \{r \in R \mid \exists w \in R : wr \in S\}$$

Lemma

If S is a left Ore set in R, then $\mathrm{LSat}(S)$ *is a saturated left Ore set in R.*

Theorem

$S^{-1}R \cong \mathrm{LSat}(S)^{-1}R$ *as rings (and K-algebras, if applicable) via*

$$S^{-1}R \to \mathrm{LSat}(S)^{-1}R, \quad (s, r) \mapsto (s, r).$$

$\Rightarrow$ $\mathrm{LSat}(S)$ is the **canonical representative** of the localization at S

The old good Weyl algebra III

Definition

The skew field of fraction of the Weyl algebra is

$$D_1 = (A_1 \setminus \{0\})^{-1} A_1 = \left\{ \frac{p}{q} \mid p \in A_1,\ q \in A_1 \setminus \{0\} \right\}.$$

Theorem (Makar-Limanov 1983)

D_1 contains a free algebra generated by $(\partial x, 1)$ and $(\partial x, 1) \cdot (1 - \partial, 1)$.

The two generators are also contained in $\mathrm{LSat}(S)^{-1} A_1$, where

$$S := [\Theta \cup \{\partial - 1\}] = [(\theta + \mathbb{Z}) \cup \{\partial - 1\}] = [(x\partial + \mathbb{Z}) \cup \{\partial - 1\}].$$

For all $i \in \mathbb{Z}$ we have

$$(\theta + i + 1)(x\partial^2 - x\partial + (i+2)\partial - i) = (\partial - 1)(\theta + i)(\theta + i + 1) \in S,$$

thus $\mathrm{LSat}(S)$ contains the (irreducible) element $x\partial^2 - x\partial + (i+2)\partial - i$.

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Local closure

Definition

Let S be a left Ore set in R and L a left ideal in R. Then

$$S^{-1}L := \{(s,r) \in S^{-1}R \mid r \in L\}$$

is the **localization** of L at S.

Definition

The **local closure** or $\boldsymbol{S}$**-closure** of a left ideal L in R is

$$L^S := \rho^{-1}(S^{-1}L).$$

Lemma

$$L^S = \{r \in R \mid \exists s \in S : sr \in L\} =: \mathrm{LSat}_S(L).$$

Application of local closure

Weyl algebra vs. differential equations

$$r := \sum_{i=0}^{n} p_i \partial^i \in A_1 \text{ with } p_i \in K[x] \quad \leftrightsquigarrow \quad \text{ODE } \sum_{i=0}^{n} p_i f^{(i)}(x) = 0$$

A solution f of such an ODE can only have a singularity at roots of p_n. But: there can be roots of p_n where no solution is singular (**apparent singularities**).

Desingularization

Find $t \in \langle r \rangle^{K[x] \setminus \{0\}}$ such that as many apparent singularities of r as possible are no longer apparent singularities of t.

Example (Barkatou, Maddah 2015)

$$r = x\partial^2 - (x+2)\partial + 2 \in A_1(\mathbb{Q}) \quad \Rightarrow \quad \langle r \rangle^{K[x] \setminus \{0\}} = \langle r, \partial^4 - \partial^3 \rangle$$

Partial classification of Ore localizations

Most common types of Ore localizations

Let K be a field, R a K-algebra and S a left Ore set in R. The set S (and $S^{-1}R$) is called... if...

Monoidal: S is generated as a monoid by countably many elements

Example: $[x+1] \subseteq K[x]$, $\Theta = [\theta + \mathbb{Z}] \in A_1$.

Geometric: Let $K[\underline{x}] = K[x_1, \ldots, x_n]$, $\mathfrak{p} \subseteq K[\underline{x}]$ prime, $S = K[\underline{x}] \setminus \mathfrak{p}$

Example: $K[x] \setminus \langle x \rangle \subseteq K[x] \subseteq A_1$.

Rational: $T \subseteq R$ is a K-subalgebra, $S = T \setminus \{0\}$

Special case: R is generated by $\underline{x} = \{x_1, \ldots, x_n\}$ and T is generated by a subset of $\underline{x} \Rightarrow S$ is **essential** rational

Example: $K[x] \setminus \{0\} \subseteq K[x, y] \subseteq A_2$.

Example: local closure at Θ in A_1

Task

Let L be a left ideal in A_1. Determine L^{Θ}, where $\Theta = [\theta + \mathbb{Z}] = [x\partial + \mathbb{Z}]$.

Lemma

Let S be a left Ore set in R and I a left ideal in R, then

$$I^S = I^{\mathrm{LSat}(S)}.$$

$\Rightarrow L^{\Theta} = L^{[x,\partial]}$ since $\mathrm{LSat}([x,\partial]) = \mathrm{LSat}(\Theta)$.

New Task

Let L be a left ideal in A_1. Determine L^V, where $V = [x, \partial]$.

Thank you for your attention!

The latest version of SINGULAR (including olga.lib) is available at:
http://www.singular.uni-kl.de

Example: local closure at Θ in A_1

Task

Let L be a left ideal in A_1. Determine L^V, where $V = [x, \partial]$.

Knowledge from D-module theory

Determining $L^{[x]}$ is algorithmic (Oaku, Takayama, Walther 1999).

Together with the **Fourier automorphism** $\mathcal{F} : A_1 \to A_1$ induced by $x \mapsto -\partial$ and $\partial \mapsto x$, determining $L^{[\partial]}$ is also algorithmic via

$$L^{[\partial]} = \mathcal{F}^{-1}(\mathcal{F}(L)^{[x]}).$$

Goal

Reduce the computation of L^V to computations of the form $L^{[x]}$ and $L^{[\partial]}$.

Example: local closure at Θ in A_1

Lemma

Let S and T be left Ore sets in R and L a left ideal in R. Then $[S \cup T]$ is a left Ore set in R and

$$L = L^{[S \cup T]} \quad \Leftrightarrow \quad L = L^S \text{ and } L = L^T.$$

Corollary

Let R be Noetherian, then in the chain of left ideals

$$L \subseteq L^S \subseteq (L^S)^T \subseteq ((L^S)^T)^S \subseteq \ldots \subseteq L^{[S \cup T]}.$$

there can only be finitely many strict inclusions. By the lemma above, at the first non-strict inclusion we have already reached $L^{[S \cup T]}$.

Contents

Localization: the paradox of theory vs practice

In theory, localization makes life easier:

- a localized ring is bigger and contains more invertible elements than the original ring, thus less proper ideals
- a localized ring is deeply connected to the original ring via $\rho_{S,R}$
- the structure of the category of $S^{-1}R$-modules is much easier than the structure of the category of R-modules

In practice (i.e. computer algebra) manipulations with objects in the localization $S^{-1}R$ are generally **much more complicated** than with objects in R.

Proof by bad example.

What is $3+5$? What is $\frac{1}{3}+\frac{1}{5}$? See. □

G-algebras (PBW algebras, algebras of solvable type)

Definition

For a field K, $n \in \mathbb{N}$ and $1 \leq i < j \leq n$ consider the constants $c_{i,j} \in K^*$ and polynomials $d_{i,j} \in K[x_1, \ldots, x_n]$. The K-algebra

$$A := K\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{i,j} x_i x_j + d_{i,j} : 1 \leq i < j \leq n\}\rangle$$

is called a ***G*-algebra**, if:

(1) there exists a monomial total well-ordering $<$ on $K[x_1, \ldots, x_n]$ such that for any $1 \leq i < j \leq n$ either $d_{i,j} = 0$ or the leading monomial of $d_{i,j}$ with respect to $<$ is smaller than $x_i x_j$.

(2) $\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}_0\}$ is a K-basis of A.

Remark

- G-algebras are Noetherian domains.
- There exists a Gröbner basis theory for G-algebras plus implementation (most extensive in SINGULAR:PLURAL).

Examples of G-algebras

- Weyl algebras ($K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \forall i : \partial_i x_i = x_i \partial_i + 1\rangle$)
- Shift algebras ($K\langle x_1, \ldots, x_n, s_1, \ldots, s_n \mid \forall i : s_i x_i = (x_i + 1) s_i\rangle$)
- q-Weyl algebras ($K\langle \underline{x}, \underline{\partial} \mid \forall i \exists q_i \in K^* : \partial_i x_i = q_i x_i \partial_i + 1\rangle$)
- q-Shift algebras ($K\langle \underline{x}, \underline{s} \mid \forall i \exists q_i \in K^* : s_i x_i = q_i x_i s_i\rangle$)
- Integration algebras ($K\langle \underline{x}, \underline{I} \mid \forall i : I_i x_i = x_i I_i + I_i^2\rangle$)
- Universal enveloping algebras of finite-dimensional Lie algebras
- Many quantum groups
- Tensor products of G-algebras over the common ground field
- ...

Recent results (Heinle, Levandovskyy, Bell)

Factorization in G-algebras is possible (finitely many cases) and implemented in `ncfactor.lib` in SINGULAR:PLURAL.

Types of **computable** Ore localizations

At the moment we can deal with the following situations:

Let K be a field and R a G-algebra over K, S a left Ore set in R.

Monoidal: S is generated as a monoid by finitely many elements contained in a commutative polynomial subring of R generated by a subset of the variables
Examples: $[x]^{-1}A_1$, $[\partial - 1]^{-1}A_1$, **not** $[x, \partial]^{-1}A_1$

Geometric: Let $T = K[x_1, \ldots, x_n] \subseteq R$, $\mathfrak{p} \subseteq T$ prime, $S = T \setminus \mathfrak{p}$
Example: $(K[x] \setminus \langle x - 42 \rangle)^{-1}A_1$

Rational: $T \subseteq R$ is a sub-G-algebra generated by a subset of the variables, $S = T \setminus \{0\}$
Example: $(K[x] \setminus \{0\})^{-1}A_1$

Algorithmic framework for algebras of operators

OLGA = Ore-localized G-algebra

olga.lib for SINGULAR:

```
locStatus(int, def)
testLocData(int, def)
isInS(poly, int, def)
fracStatus(vector, int, def)
testFraction(vector, int, def)
leftOre(poly, poly, int, def)
rightOre(poly, poly, int, def)
convertRightToLeftFraction(vector, int, def)
convertLeftToRightFraction(vector, int, def)
addLeftFractions(vector, vector, int, def)
multiplyLeftFractions(vector, vector, int, def)
areEqualLeftFractions(vector, vector, int, def)
isInvertibleLeftFraction(vector, int, def)
invertLeftFraction(vector, int, def)
```

Available as part of the SINGULAR distribution.

Examples I

Left-to-right conversion in the second rational q-shift algebra A:

```
LIB "olga.lib";
ring Q = (0,q),(x,y,Qx,Qy),dp;   // commutative polynomial ring
matrix C[4][4] = UpOneMatrix(4); // defines a matrix of
C[1,3] = q; C[2,4] = q;          //   non-commutative relations
def A = nc_algebra(C,0);         // creates A from Q
setring A;
intvec v = 1,2;                  // rational localization at K[x,y]\{0}
poly f = Qx+Qy; poly g = x^2+1;
vector frac = [g,f,0,0];
vector result = convertLeftToRightFraction(frac,2,v);
print(result);
-> [x^2+1,Qx+Qy,(q^4)*x^2*Qx+x^2*Qy+(q^2)*Qx+(q^2)*Qy,x^4+(q^2+1)*x^2+(q^2)]
```

Now `result` contains the left representation $(x^2+1)^{-1}(Q_x+Q_y)$ of `frac` as well as its newly computed right representation $(q^4x^2Q_x+x^2Q_y+q^2Q_y)\cdot(x^4+(q^2+1)x^2+q^2)^{-1}$. Plausibility check:

```
f * result[4] == g * result[3];
-> 1
isInS(result[4],2,v);
-> 1
```

Examples II

Basic arithmetic with two left fractions in a monoidal localization of the second Weyl algebra A_2:

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;    // commutative polynomial ring
def A2 = Weyl();  setring A2; // creates A2 from R
poly g1 = x+3; poly g2 = x*y+y; list L = g1,g2;
vector frac1 = [g1,Dx,0,0]; vector frac2 = [g2,Dy,0,0];
vector result = addLeftFractions(frac1, frac2, 0, L); print(result);
-> [x^2*y+4*x*y+3*y,x*y*Dx+y*Dx+x*Dy+3*Dy]
```

Thus, the sum is $(x^2y + 4xy + 3y)^{-1}(xy\partial_x + y\partial_x + x\partial_y + 3\partial_y)$.

```
result = multiplyLeftFractions(frac1, frac2, 0, L); print(result);
-> [x^3*y^2+5*x^2*y^2+7*x*y^2+3*y^2,x*y*Dx*Dy+y*Dx*Dy-y*Dy]
```

This product is $(x^3y^2 + 5x^2y^2 + 7xy^2 + 3y^2)^{-1}(xy\partial_x\partial_y + y\partial_x\partial_y - y\partial_y)$.

```
result = multiplyLeftFractions(frac2, frac1, 0, L); print(result);
-> [x^2*y+4*x*y+3*y,Dx*Dy]
```

In this order, the product is $(x^2y + 4xy + 3y)^{-1}(\partial_x\partial_y)$.

The latest version of SINGULAR (including `olga.lib`) is available at:
`http://www.singular.uni-kl.de`