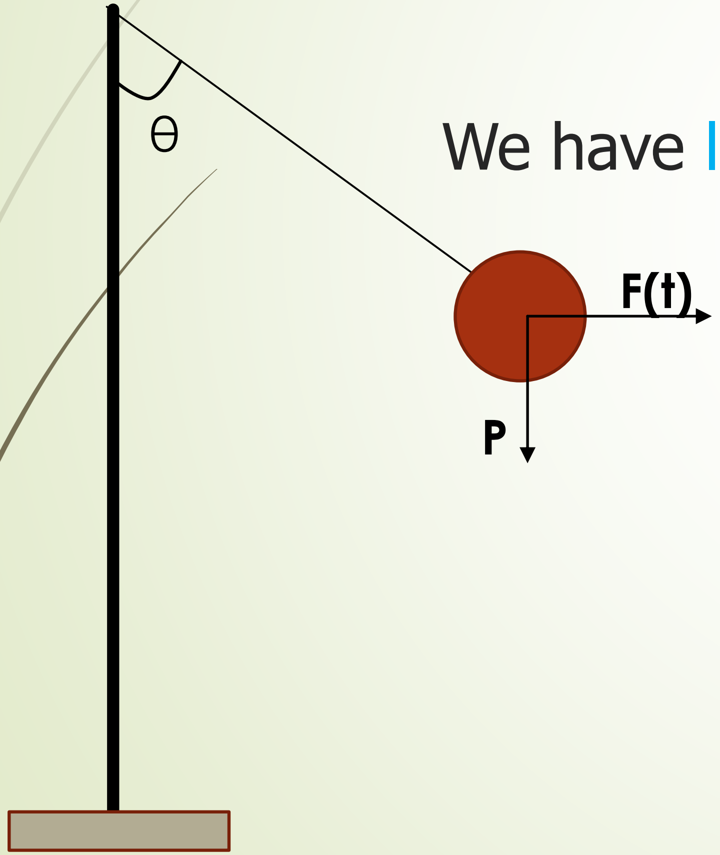
Short course on chaos theory

Consider the case of pendulum



The equation of motion :

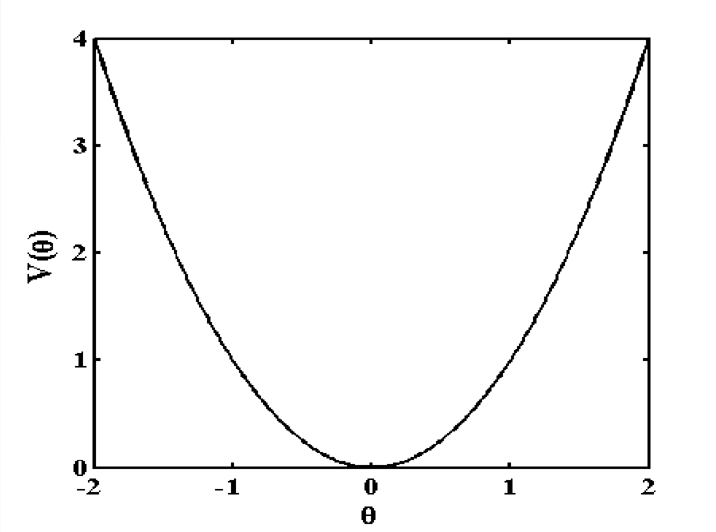
$$\ddot{\theta} + \omega^2 \sin\theta = F(t)$$

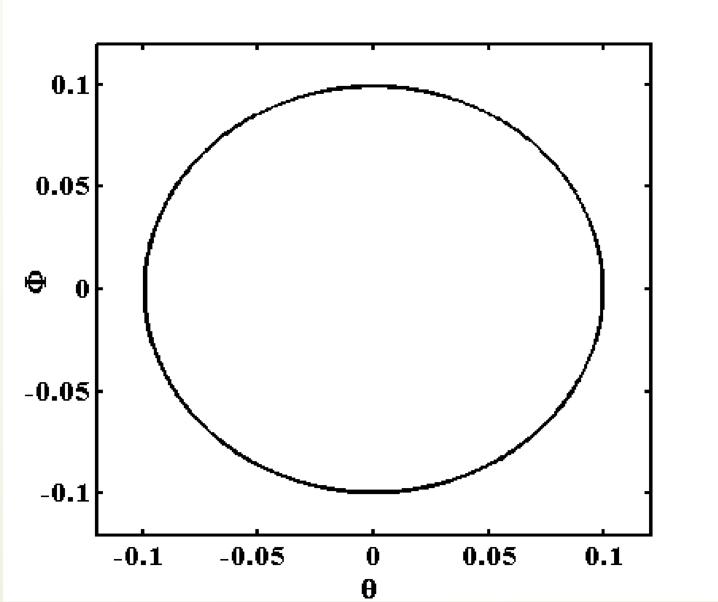
We have linear and non-linear case

Linear case:

$$\ddot{\theta} + \omega^2\theta = F(t)$$

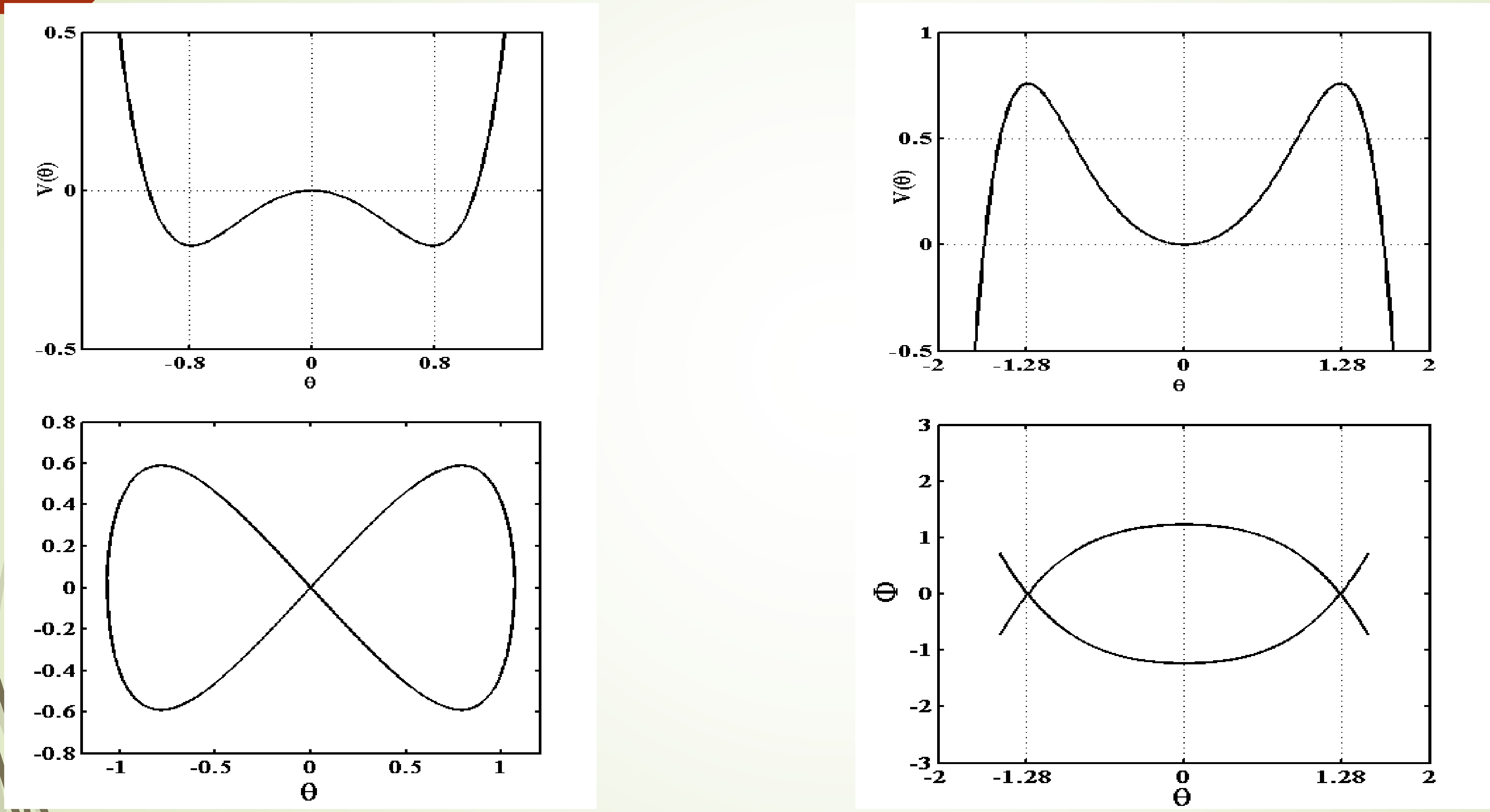
Linear case:





Non-linear case:

$$\ddot{\theta}+\omega^{2}\left(\theta-\frac{1}{3}\theta^{3}+\frac{1}{6}\theta^{5}\right)+0\left(\varepsilon^{6}\right)=F\left(t\right)$$



-Periodic ω
- Bi-periodic 2ω
-Quasi periodic 3ω-4ω
- Chaotic 5ω an so on

-The presence of non-linearity gives some news information on the dynamics of the system

- $$\ddot{\theta} + \delta\dot{\theta} + a\theta + b\theta^3 = F(t)$$

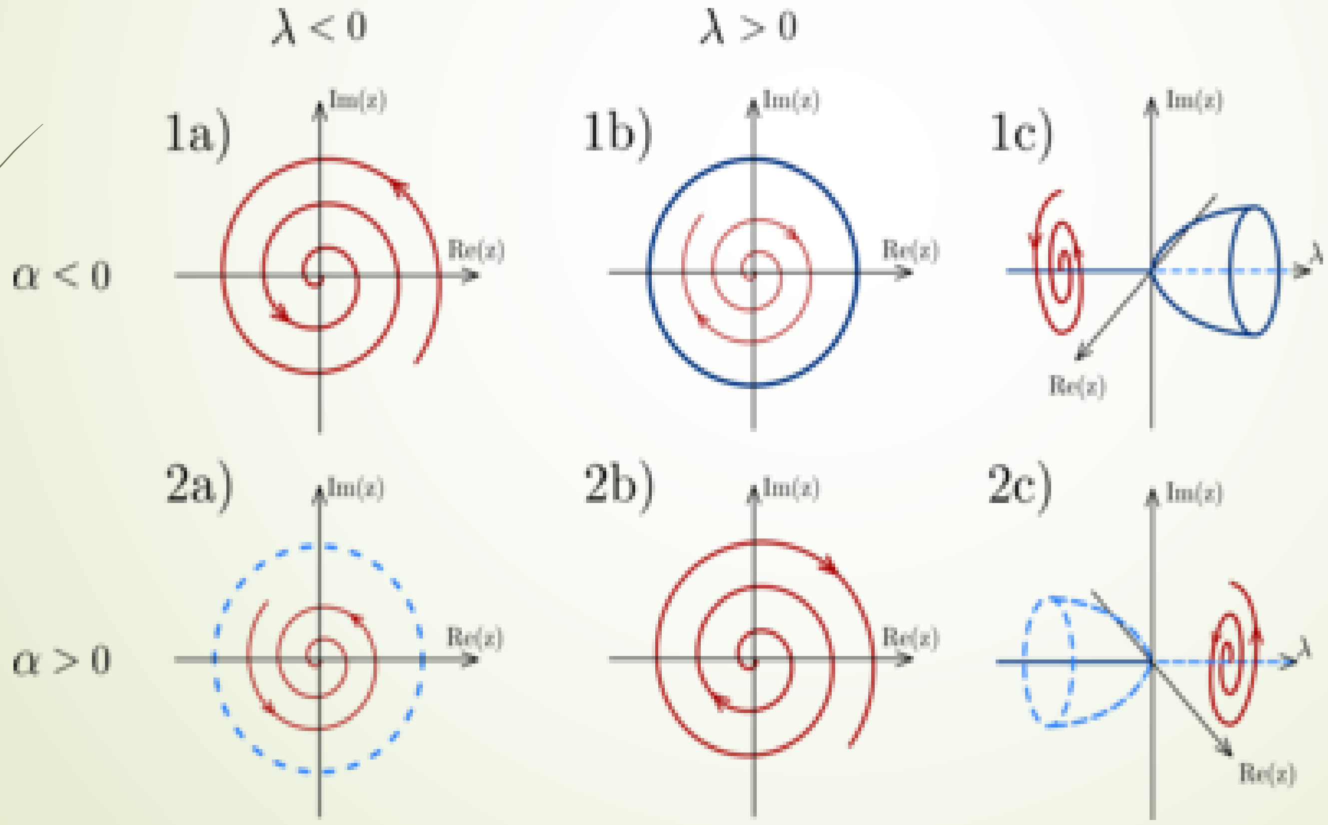
$$\begin{cases} \dot{\theta} = z \\ \dot{z} = -a\theta - b\theta^3 - \delta z + F(t) \end{cases}$$

- Fixed points $f\left(\dot{\theta}, \dot{z}\right) = (0,0)$

Attractor of fixed points

Autonomous system

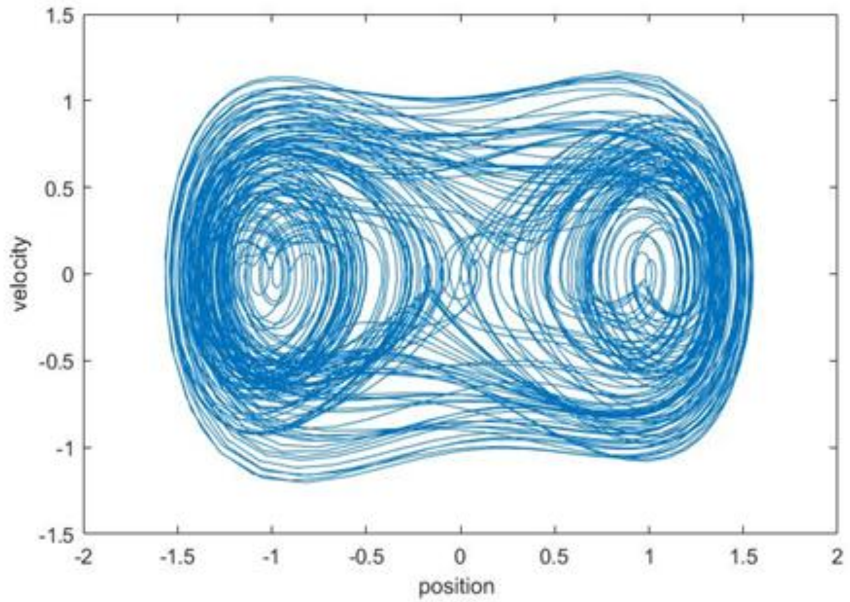
F(t)=0

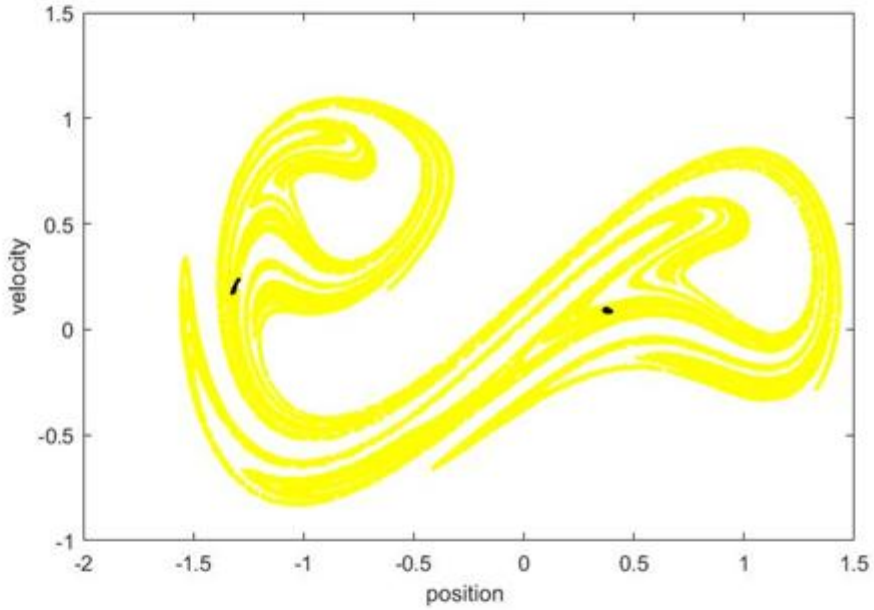


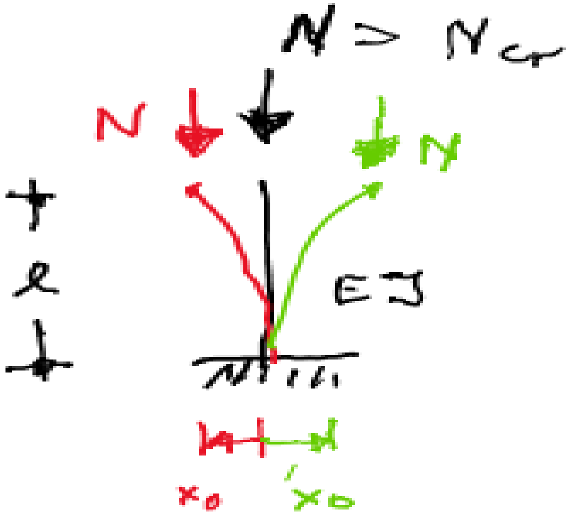
According to mechanical Structures the best situation is to be able to prevent chaotic motion.

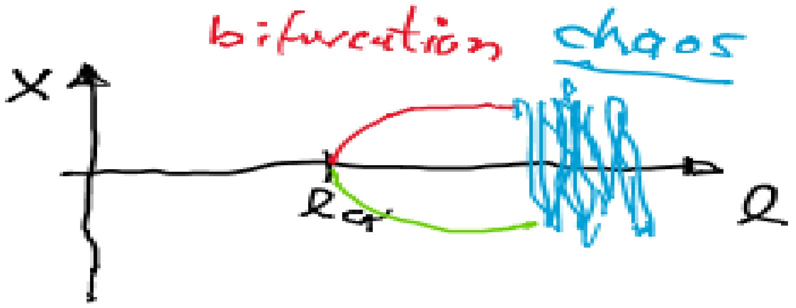
Non-Autonomous system

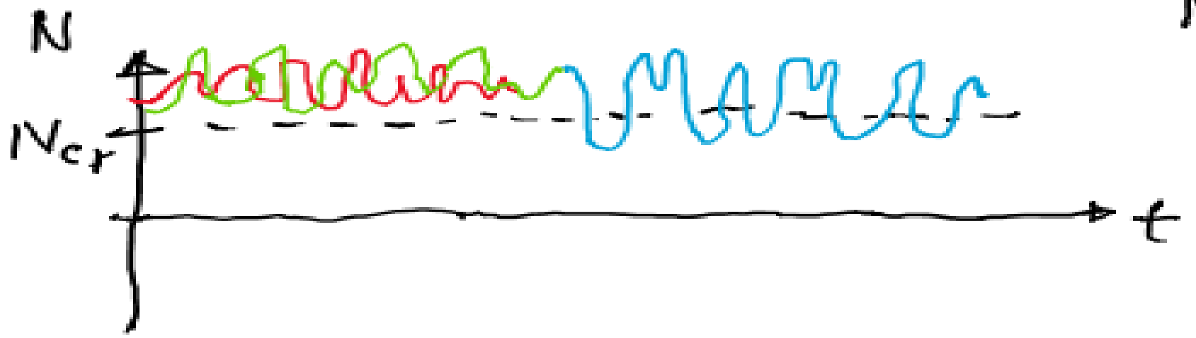
F(t)≠0



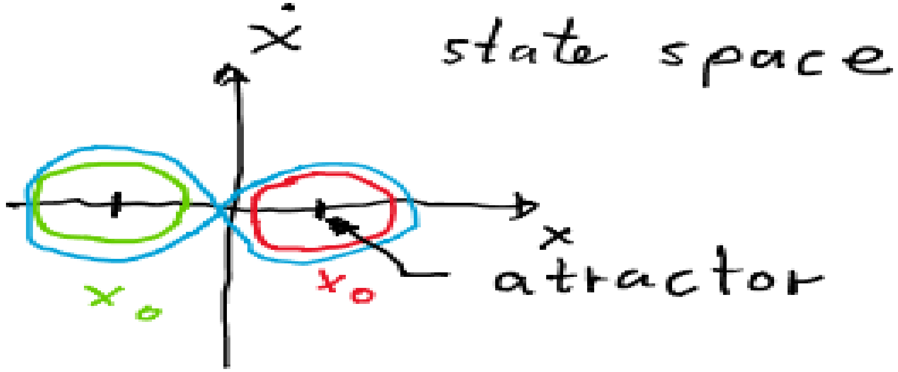








$$N_{cr} = \pi^2 \frac{EJ}{4l^2}$$



Some indicators

- **Phase portrait**: A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane.
- **Poincaré Section:** is a method to transform a continuous dynamical process in time into a set of difference equations, known in modern parlance as a map.
- **Bifurcation**: A **bifurcation** or **branching** is a qualitative state change in non linunder the influence of a parameter of system
- **Lyapunov exponent**: is a quantity that characterizes the rate of separation of infinitesimally close trajectories.
- The Basin of Attraction offers a global perspective to analyze robustness of limit-cycle walking It represents the set of all initial conditions on the Poincaré section from which the system will converge to the fixed point(s).

- Shilnikov Method: Condition to have chaos on a system

- Lelnikov Method : Starting point for successive route to chaos

The transverse intersection between pertubed and non pertubed separateix

In the littérature the only way to prevent that, is to derive in the space parameters of the systems the Melnikov function

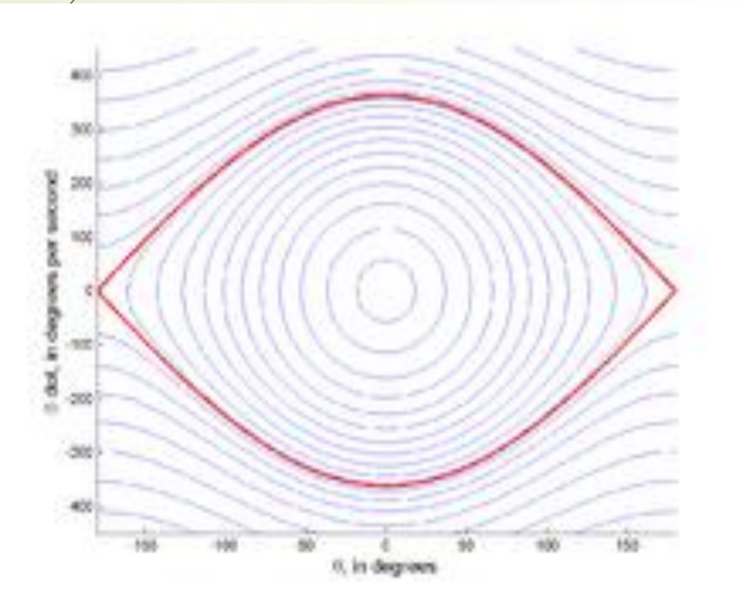
The Melnikov function help to obtain the starting point for a successive route to chaotic dynamics

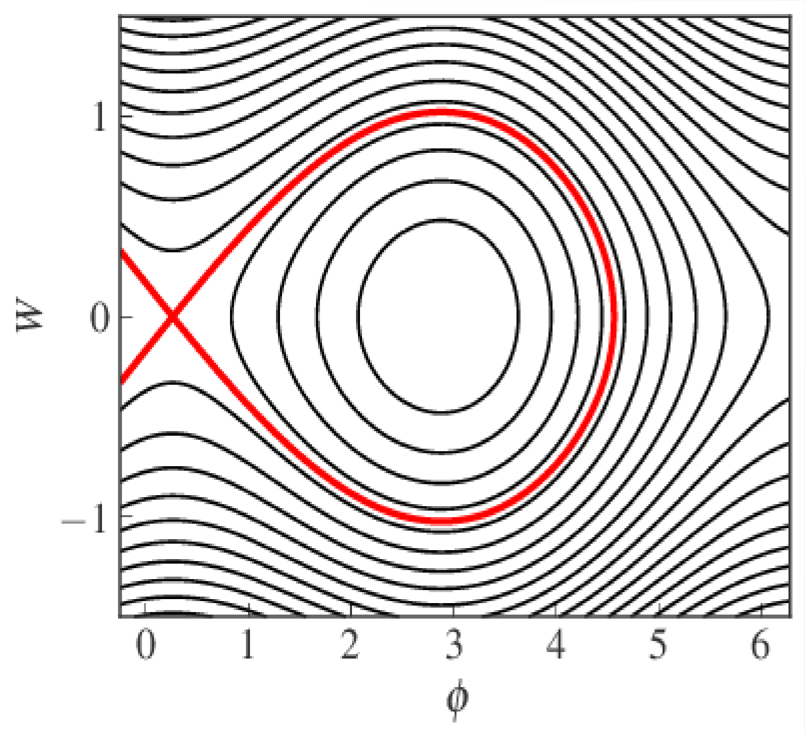
$$M(t_0) = \int_{-\infty}^{+\infty} g_0\left(\bar{u}(t)\right) \times g_p\left(\bar{u}(t), t + t_0\right) dt$$

t_0 is the distance between the transversere intersection between pertubed and unpertubed separatrix

This means that if the potential of the system in non degenerated one can not calculate the Menikov function

If the potential is degenarated that means that we could Homoclinic or Heteroclinic separatrix





- The equation of separatrix can be calculated using the hamiltonian of the system
- This has nothing to do with phase diagram

g_p is a periodic pertubation function and $g_0(g_1; g_2)$

the vector field chosen Hamiltonian with the energy $\boldsymbol{H_0}$ so that:

$$\begin{cases} g_1 = \dfrac{\partial H_0}{\partial \dot{x}} \\ g_2 = -\dfrac{\partial H_0}{\partial x} \end{cases} \qquad (9)$$

AS an example, let us considered the case of Duffing equation

The simplest example is The Duffing equation (or Duffing oscillator), named after Georg Duffing (1861–1944), is a non-linear differential equation of the second order used to model certain forced and damped oscillators. The equation is written:

$$\ddot{x} + \delta\dot{x} + ax + bx^3 = f_0 \cos(\omega t) \qquad (1)$$

Where δ is the damping, a the frequency of system and b the nonlinear term due to the stiffness.

This mathematical model describes the evolution of any physical system in the vicinity of a stable equilibrium position, making it a transversal tool used in many fields: mechanics, electricity and electronics, optics.

Depending on the signs of parameters a and b, his oscillator describes two symmetrical configurations of the potential well:

- **Case 1 :** $a<0$ and $b>0$ we have a two-well potential (see figure 1a)
- **Case 2 :** $a>0$ and $b<0$ we have a catastrophic potential well (see figure 2a). The latter is a typical characteristic encountered in real physical structures

Eq 1 can be rewritten as a system of first order equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ax - bx^3 - \delta y + f_0 \cos \omega t \end{cases} \qquad (2)$$

For the determination of fixed points, we suppose that: $\delta = 0 \;\; and \;\; f_0 = 0$

On obtains:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ax - bx^3 \end{cases} \qquad (3)$$

For initial conditions $\dot{x} = 0 \;\; and \;\; \dot{y} = 0$

One obtains three fixe points depending of signs of a and b:

$$\begin{pmatrix} x = -\sqrt{-\dfrac{a}{b}} \\ y = 0 \end{pmatrix}, \begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix} \; and \; \begin{pmatrix} x = \sqrt{-\dfrac{a}{b}} \\ y = 0 \end{pmatrix} \qquad (4)$$

Depending of values of signs of a and b these fixe points can be stables or unstables.

2- Potential curves and separatrix equations

The equilibrium condition is written by :

$$\vec{F} = -\overrightarrow{gradU} \qquad and \qquad U = -\int F dx \qquad with \quad F = -(\alpha x + \beta x^3)$$

Consider the two forms of Duffing equation:

$$U(x) = \frac{1}{2} ax + \frac{1}{4} bx^4 \qquad (5)$$

$\tau = F = 0$

•**<u>Stable equilibrium point:</u>**

Un point X_0 is a stable equilibrium point if for $x = x_0 \Leftrightarrow U(x_0)$ is minimal

$$\frac{dU}{dx}(x_0) = 0 \qquad and \qquad \frac{d^2U}{dx^2}(x_0) \succ 0 \qquad (6)$$

•**<u>Unstable equilibrium point:</u>**

Un point X_0 is an ustable equilibrium point if for $x = x_0 \Leftrightarrow U(x_0)$ is maximal

$$\frac{dU}{dx}(x_0) = 0 \qquad and \qquad \frac{d^2U}{dx^2}(x_0) \prec 0 \qquad (7)$$

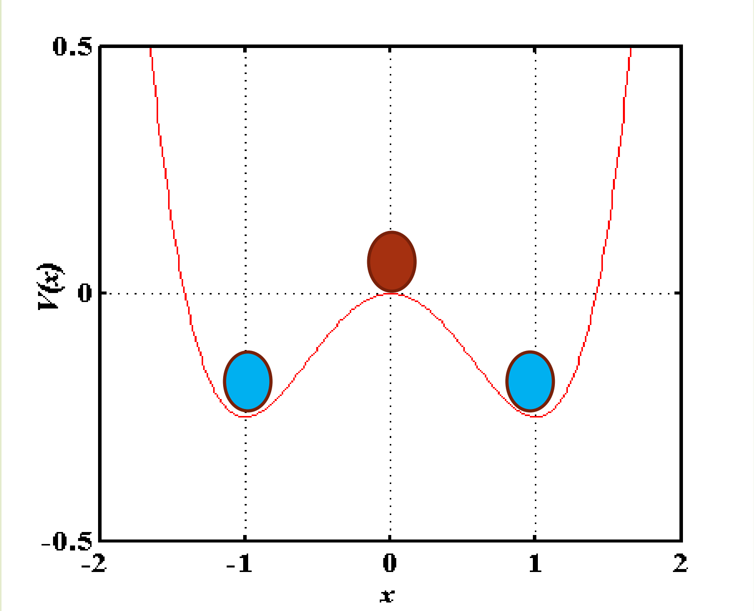


Figure 1a: Potential curve a<0 and b>0

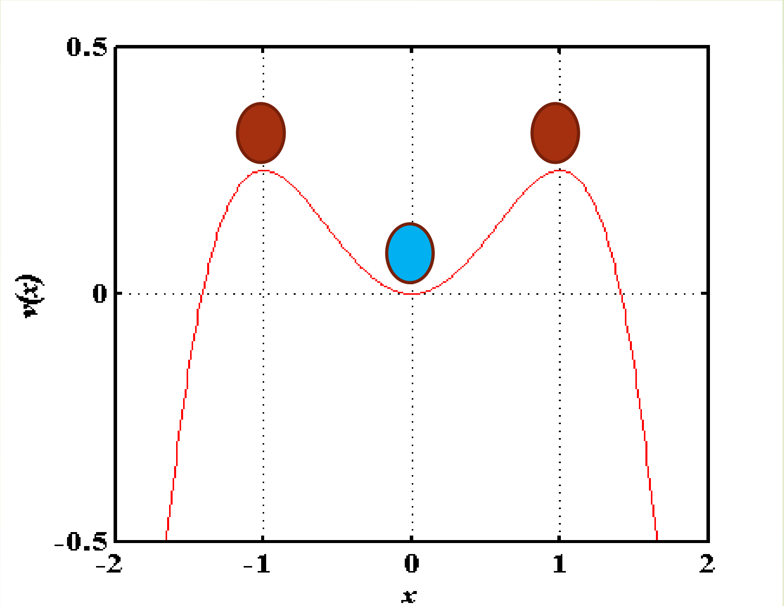


Figure2a : Potential curve a>0 and b<

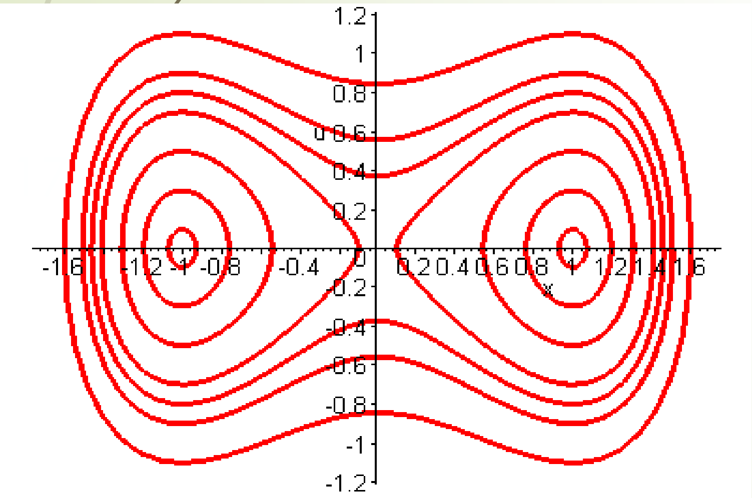


Figure1b: Phase diagram a<0 and b>0

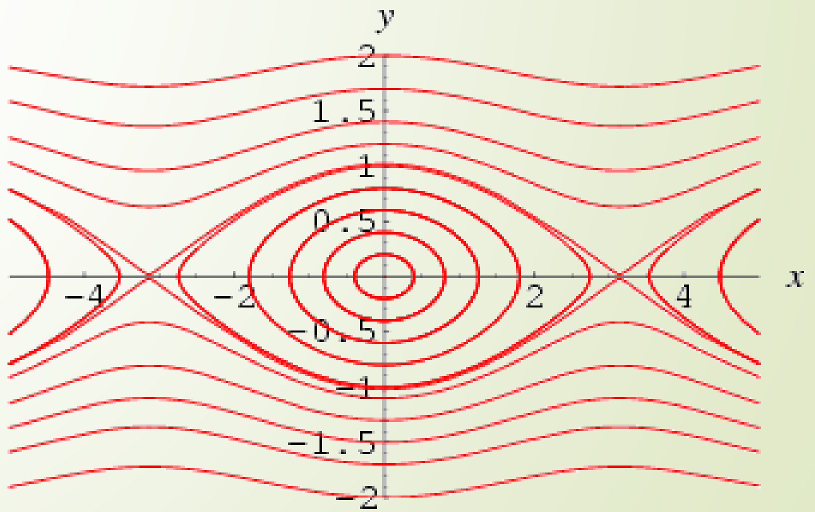
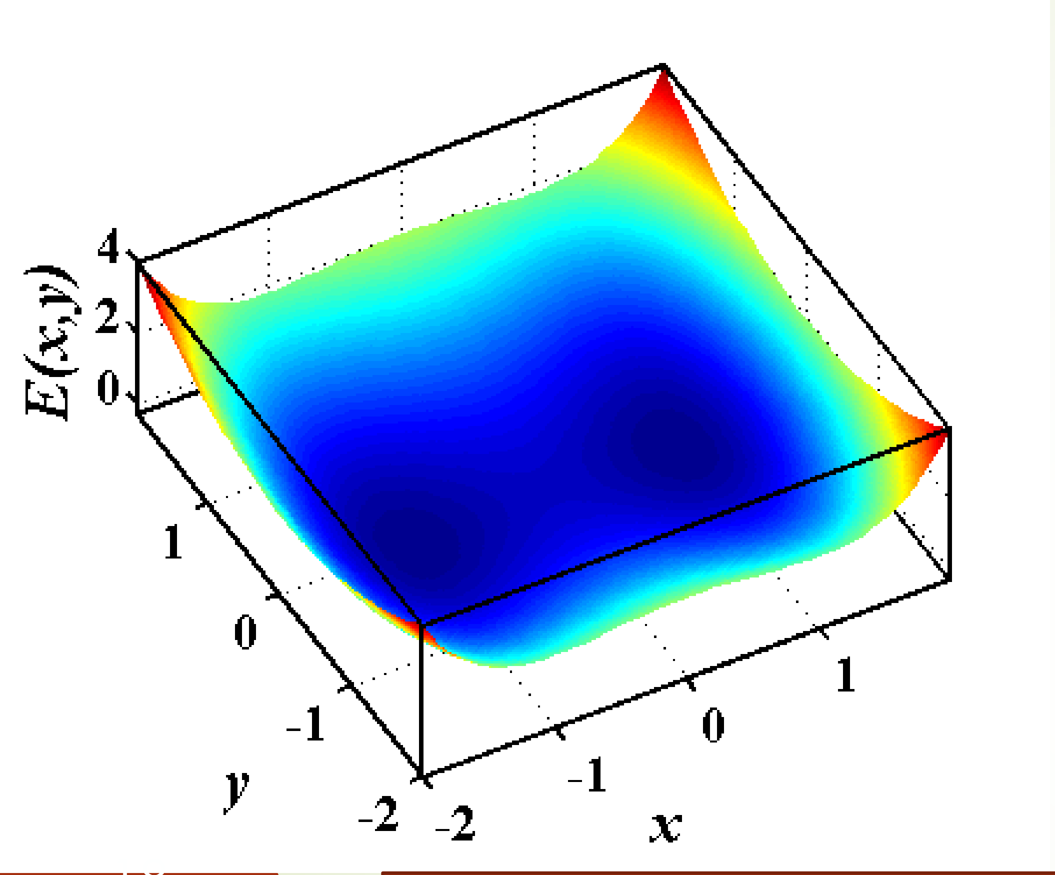


Figure2b: Phase diagram a>0 and b<0



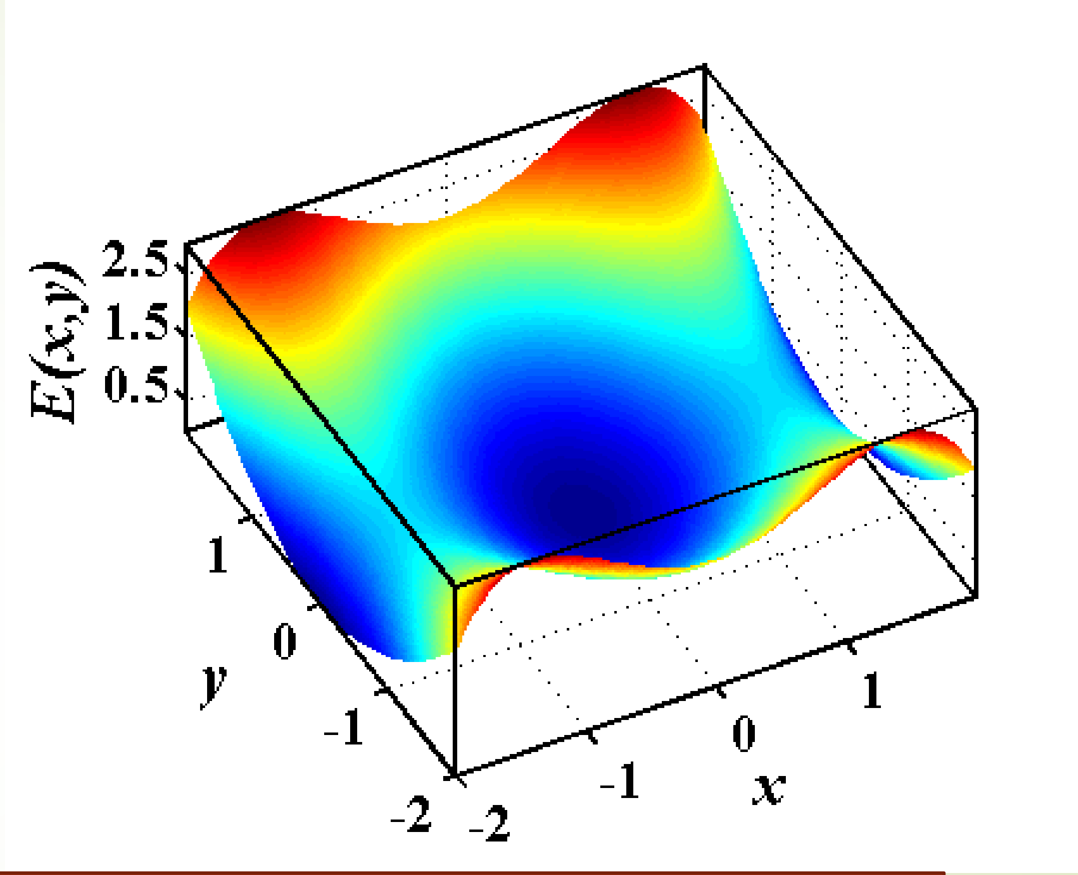


Fig 3: Sphace Diagram in 3D describing the motion of the particle in the the differents wells

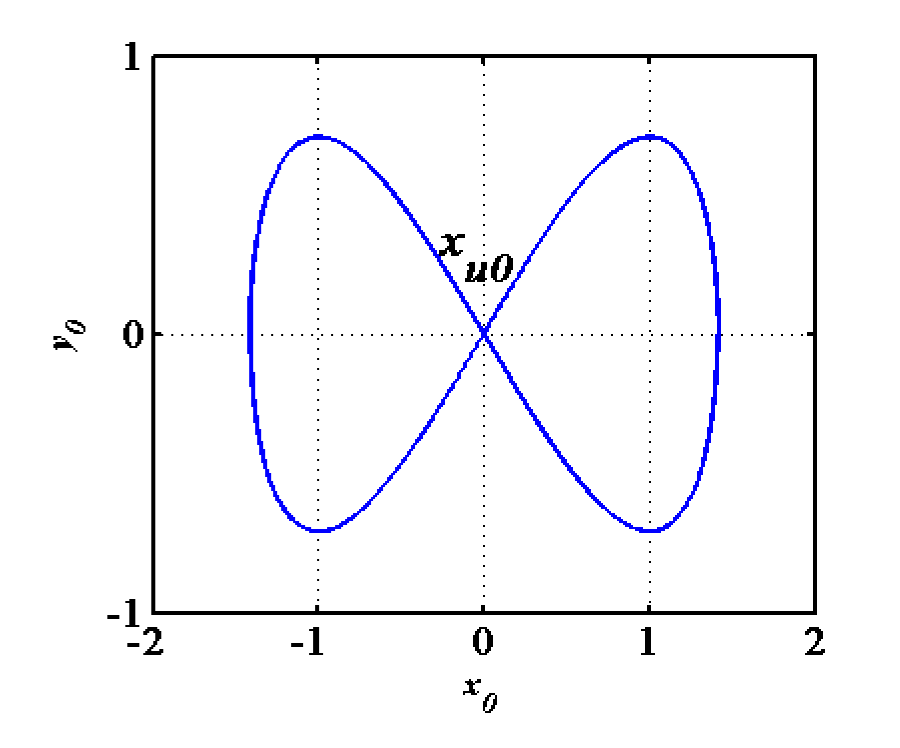


Figure 4a: Homoclinic orbit

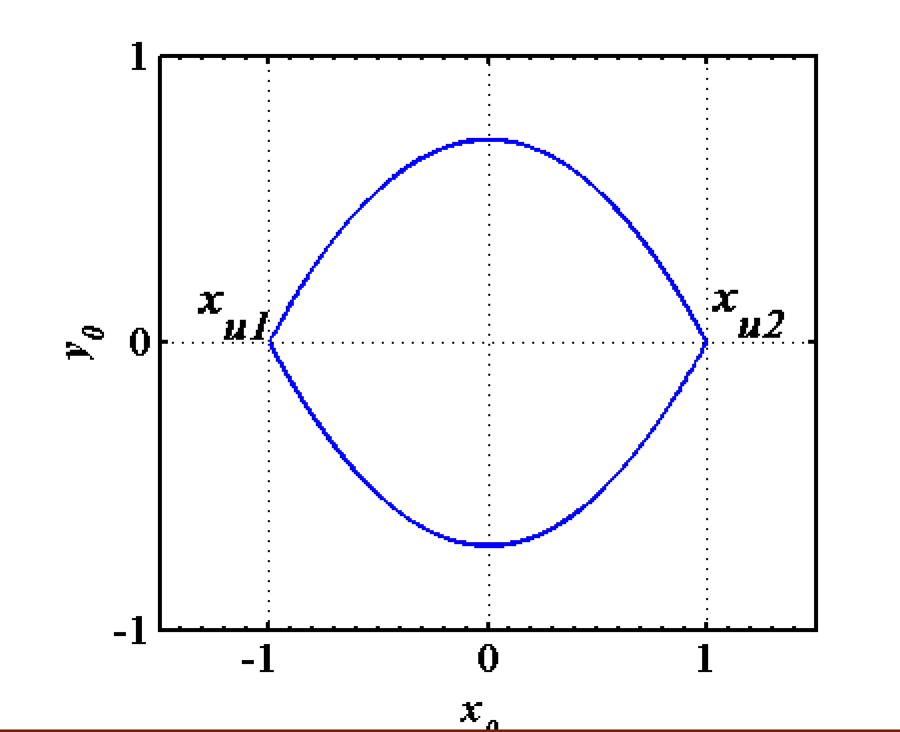


Figure 4b: Heteroclinic orbit

The homoclinic orbit connectes the unstable point X_{u0} to itself (see Fig. (4a)) and the heteroclinic orbit (see Fig. 4(b)) connectes the unstable points X_{u2} and X_{u1}.

3-Melnikov theory

The Melnikov's theory consists to determine analytically the threshold value of external force, where the Horseshoe chaos appears in the system.

Now, we consider the case of double Well.

Consider the generalized dynamical equation of a given system written in vector form:

$$\dot{u} = g_0(u) + \varepsilon g_p(u,t) \qquad (8)$$

Where $u = (x, y)$ is the state vector, g_p is a periodic pertubation function and

$g_0(g_1; g_2)$ the vector field chosen Hamiltonian with the energy $\boldsymbol{H_0}$ so that:

$$\begin{cases} g_1 = \dfrac{\partial H_0}{\partial \dot{x}} \\ g_2 = -\dfrac{\partial H_0}{\partial x} \end{cases} \qquad (9)$$

For this model, we have:

$$\begin{cases} g_0 = \left(y, -ax - bx^3\right) \\ g_p = \left(0, -\delta y + f_0 \cos \omega t\right) \end{cases} \quad (10)$$

Let us assume that the unperturbed Hamiltonian system possesses saddle points connected by a separatrix or heteroclinic orbit $\bar{u}$(t) or only one hyperbolic saddle point with a homoclinic orbit $\bar{u}$(t). In the presence of the perturbation $g_p(u, t)$, the orbits are perturbed. When the perturbed and the unperturbed manifolds intersect transverssaly, the geometry of the basin of attraction may become fractal, indicating the high sensitivity to initial conditions, thus chaos. The Melnikov's theory which gives the condition for the fractal basin boundary can be given as follows:

$$M\left(t_0\right) = \int_{-\infty}^{+\infty} g_0\left(\bar{u}\left(t\right)\right) \times g_p\left(\bar{u}\left(t\right), t + t_0\right) dt \quad (11)$$

With $-\infty \prec t_0 \prec +\infty$. If $M(t_0)$ has simple zeros so that for a given t_0^lone has $M(t_0) = 0$ with $dH(t_0)/dt$ at $t = t_0^l$ (condition for transversal intersection), then Eq (1) can present fractal boundaries for motions around different stable equilibrium points. To apply the Melnikov theorem to our model, we derive the equations for the homoclinic and heteroclinic orbits. Let us first consider the case of the potential with two wells (see Fig. 1(a)). For this case, we have to find the homoclinic orbits connecting the unstable point (see Fig. 2(a)) $x = 0$ to itself. Making use of integrals tables or the method of residues, we obtain the homoclinic orbits defined by:

$$M(t_0) = \int_{-\infty}^{\infty} y_{ho}(t)\left[f_0 \cos\omega(t+t_0) - \delta y_{ho}(t)\right] dt \qquad (12)$$

$$\frac{f_0}{\delta} \geq \frac{\int_{-\infty}^{+\infty} y_{ho}^2 \, dt}{\int_{-\infty}^{+\infty} y_{ho} \cos(\omega t) \, dt}$$

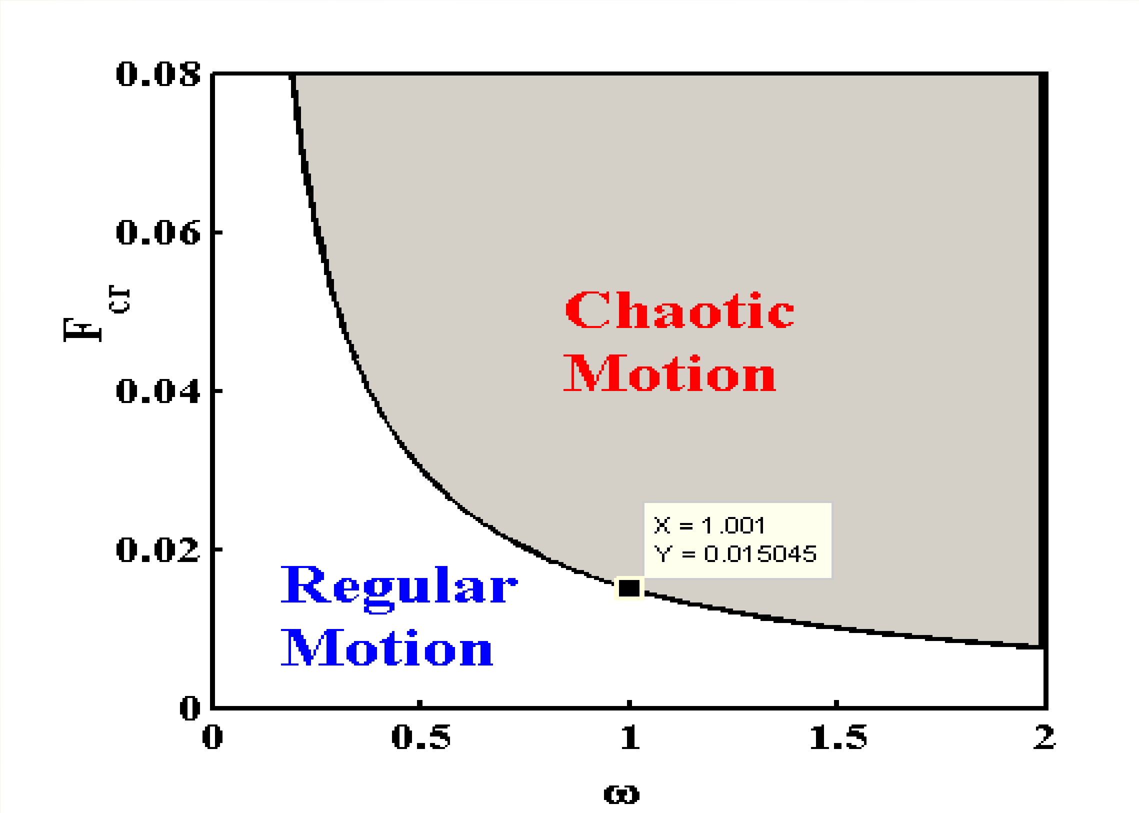


Fig 5: Melnikov Curve for appearance or disappearance of Chaos

4- Fractal basin boundary

In order to confirm the analytical predictions from Melnikov's theory, we analyse in this section the regular or irregular geometries of the attraction basins by numerical resolution of Eq (1) by means of Runge-Kutta algorithm of the fourth order. This irregular geometry of the basin of attraction is characterized by the appearance of fractality [1] on the boundary of basin of attraction which reflects the chaos, resulting indisputably from the greater sensitivity due to initial conditions.

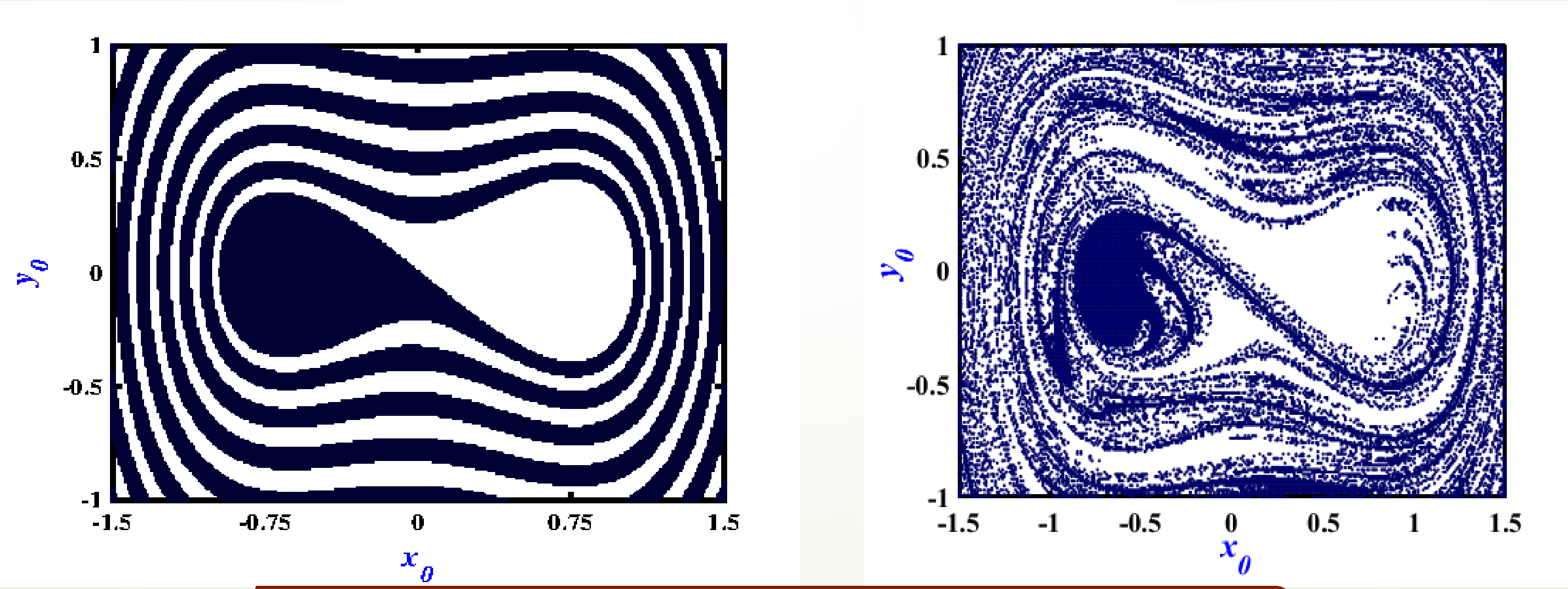


Figure 6: Basins of attraction

Exercice

- Consider a structure model by a Duffing oscillator with catastrophic single well potential
- Derive the equilibrium points
- Calculate equation governing the Heteroclinic separatrix
- Calculate the Melnikov distance for transition to chaos
- Derive the condition for suppression of chaos on the structures

$$\begin{pmatrix} x=-\sqrt{-\frac{a}{b}} \\ y=0 \end{pmatrix},\begin{pmatrix} x=0 \\ y=0 \end{pmatrix} \text{ and } \begin{pmatrix} x=\sqrt{-\frac{a}{b}} \\ y=0 \end{pmatrix}$$

Let the Hamiltonian ***H*** of the system

$$H(x,y)=\frac{1}{2}y^2-a\frac{1}{2}x^2-\frac{1}{4}bx^4$$

For a<0 and b>0 the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ belongs at the separatrix

The equation of separatrix is obtained by posing: **H=cst**

One Obtains: $y^2=ax^2+\frac{1}{2}bx^4 \quad \Rightarrow \int \frac{dx}{x\sqrt{-a+\frac{b}{2}x^2}}=\int dt$

$$\Rightarrow \begin{cases} x_{ho}=\pm\dfrac{\sqrt{2}\sqrt{-a}Cot[\sqrt{-a}t]\sqrt{1+Tan[\sqrt{-a}t]^2}}{\sqrt{b}} \\ y_{ho}=\pm\dfrac{\sqrt{2}aSec[\sqrt{-a}t]^2}{\sqrt{b}\sqrt{1+Tan[\sqrt{-a}t]^2}}\pm\dfrac{\sqrt{2}aCsc[\sqrt{-a}t]^2\sqrt{1+Tan[\sqrt{-a}t]^2}}{\sqrt{b}} \end{cases}$$

For a>0 and b<0 the point $\begin{pmatrix} x = \sqrt{-\frac{a}{b}} \\ y = 0 \end{pmatrix}$ belongs at the separatrix

- The equation of separatrix is obtained by posing: **H=cst =**$\boldsymbol{\frac{-a^2}{4b}}$

$$y^2 = -\frac{a^2}{4b} + \frac{a}{2}x^2 - \frac{1}{4}bx^4 \qquad \Rightarrow \int \frac{dx}{\left(\frac{a}{2b} - \frac{1}{2}x^2\right)} = \int -bdt$$

$$\Rightarrow \begin{cases} x_{he} = \pm \dfrac{\sqrt{-\frac{a}{b}}\, Tanh\left(\frac{1}{2}\sqrt{-\frac{a}{b}}t\right)}{b} \\ y_{he} = \pm \dfrac{\sqrt{-\frac{a}{b}}\, Sech^2\left(\frac{1}{2}\sqrt{-\frac{a}{b}}t\right)}{\sqrt{2b}} \end{cases}$$

$$M\left(t_0\right)=\int_{-\infty}^{+\infty} g_0\left(\bar{u}\left(t\right)\right)\times g_p\left(\bar{u}\left(t\right),t+t_0\right)dt$$

$$\begin{cases} g_0=\left(y,-ax-bx^3\right) \\ g_p=\left(0,-\delta y+f_0\cos\omega t\right) \end{cases}$$

$$M(t_0)=\int_{-\infty}^{+\infty}\begin{pmatrix} y_h \\ -ax_h-bx_h^3 \end{pmatrix}\times\begin{pmatrix} 0 \\ -\delta y_h+f_0\cos\omega t \end{pmatrix}dt$$

$$M\left(t_0\right)=\int_{-\infty}^{\infty} y_{ho}\left(t\right)\left[f_0\cos\omega\left(t+t_0\right)-\delta y_{ho}\left(t\right)\right]dt$$