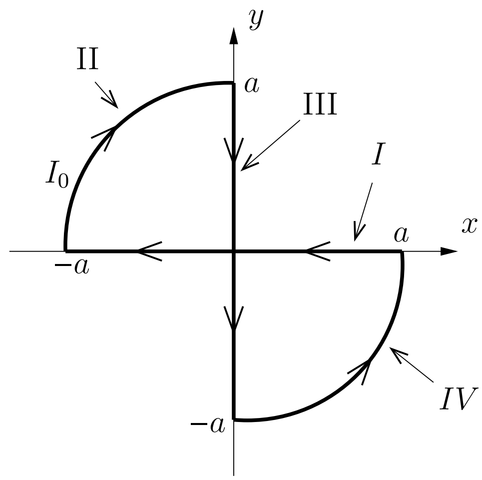
Solution for Problem 11

Problem:

Given is a wire loop (see figure) with $N = 5$ windings consisting of four parts: I, II, III and IV. The wire loop is placed in the xy plane and a constant current I_0 is flowing through the loop. The crossing wires in the origin are not connected.
The terminal wires are to be neglected and are not shown in the figure.



Determine the magnetic field strength on the z axis using the law of Biot-Savart.
Hint:

$$\int \frac{1}{(\alpha^2 + x^2)^{\frac{3}{2}}} dx = \frac{x}{\alpha^2 \sqrt{\alpha^2 + x^2}}$$

Solution: Law of Biot-Savart:

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_C \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

In this equation C is the contour of the wire loop with 5 windings: $C = 5 * (C_I + C_{II} + C_{III} + C_{IV})$.
First, we will evaluate the law of Biot-Savart for a loop of the given geometry with only one winding.

For **Part I** we calculate:

$$\begin{aligned} \underline{\mathbf{R}}' &= x'\underline{\mathbf{e}}_x \qquad\qquad -a \le x' \le a \\ \underline{\mathbf{R}} - \underline{\mathbf{R}}' &= z\underline{\mathbf{e}}_z - x'\underline{\mathbf{e}}_x \\ |\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3 &= (x'^2 + z^2)^{3/2} \end{aligned}$$

$$
\begin{aligned}
\underline{\mathbf{dR}}' &= \underline{\mathbf{e}}_x dx' \\
\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') &= -z dx' \underline{\mathbf{e}}_y \\
\underline{\mathbf{H}}_I(\underline{\mathbf{R}}) = \int\limits_{C_I} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3} &= \int\limits_a^{-a} \frac{-z dx' \underline{\mathbf{e}}_y}{(x'^2 + z^2)^{3/2}} \\
&= \frac{2a\underline{\mathbf{e}}_y}{z\sqrt{z^2 + a^2}}
\end{aligned}
$$

For **Part II** we calculate:

$$
\begin{aligned}
\underline{\mathbf{R}}' &= a\cos\varphi' \underline{\mathbf{e}}_x + a\sin\varphi' \underline{\mathbf{e}}_y \qquad \frac{\pi}{2} \le \varphi' \le \pi \\
\underline{\mathbf{R}} - \underline{\mathbf{R}}' &= z\underline{\mathbf{e}}_z - a\cos\varphi' \underline{\mathbf{e}}_x - a\sin\varphi' \underline{\mathbf{e}}_y \\
|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3 &= (a^2 + z^2)^{3/2} \\
\underline{\mathbf{dR}}' &= (a(-\sin\varphi')\underline{\mathbf{e}}_x + a\cos\varphi' \underline{\mathbf{e}}_y)\ d\varphi' \\
&= a(-\sin\varphi' \underline{\mathbf{e}}_x + \cos\varphi' \underline{\mathbf{e}}_y)\ d\varphi' \\
\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') &= (az\cos\varphi' \underline{\mathbf{e}}_x + az\sin\varphi' \underline{\mathbf{e}}_y + a^2 \underline{\mathbf{e}}_z) d\varphi' \\
\underline{\mathbf{H}}_{II}(\underline{\mathbf{R}}) = \int\limits_{C_{II}} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3} &= \frac{1}{(a^2 + z^2)^{3/2}} \int\limits_{\pi}^{\pi/2} (az\cos\varphi' \underline{\mathbf{e}}_x + az\sin\varphi' \underline{\mathbf{e}}_y + a^2 \underline{\mathbf{e}}_z) d\varphi' \\
&= \frac{az}{(a^2 + z^2)^{3/2}} (\underline{\mathbf{e}}_x - \underline{\mathbf{e}}_y) - \frac{\pi a^2 \underline{\mathbf{e}}_z}{2(a^2 + z^2)^{3/2}}
\end{aligned}
$$

For **Part III** we calculate:

$$
\begin{aligned}
\underline{\mathbf{R}}' &= y' \underline{\mathbf{e}}_y \qquad -a \le x' \le a \\
\underline{\mathbf{R}} - \underline{\mathbf{R}}' &= z\underline{\mathbf{e}}_z - y' \underline{\mathbf{e}}_y \\
|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3 &= (y'^2 + z^2)^{3/2} \\
\underline{\mathbf{dR}}' &= \underline{\mathbf{e}}_y dy' \\
\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') &= z dy' \underline{\mathbf{e}}_x \\
\underline{\mathbf{H}}_{III}(\underline{\mathbf{R}}) = \int\limits_{C_{III}} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3} &= \int\limits_a^{-a} \frac{z dy' \underline{\mathbf{e}}_x}{(x'^2 + z^2)^{3/2}}
\end{aligned}
$$

$$= \frac{-2a\underline{\mathbf{e}}_x}{z\sqrt{z^2+a^2}}$$

For **Part IV** we calculate:

$$\begin{aligned}
\underline{\mathbf{R}}' &= a\cos\varphi'\underline{\mathbf{e}}_x + a\sin\varphi'\underline{\mathbf{e}}_y \qquad \frac{3\pi}{2} \leq \varphi' \leq 2\pi \\
\underline{\mathbf{R}} - \underline{\mathbf{R}}' &= z\underline{\mathbf{e}}_z - a\cos\varphi'\underline{\mathbf{e}}_x - a\sin\varphi'\underline{\mathbf{e}}_y \\
|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3 &= (a^2+z^2)^{3/2} \\
\underline{\mathbf{dR}}' &= (a(-\sin\varphi')\underline{\mathbf{e}}_x + a\cos\varphi'\underline{\mathbf{e}}_y)\ d\varphi' \\
&= a(-\sin\varphi'\underline{\mathbf{e}}_x + \cos\varphi'\underline{\mathbf{e}}_y)\ d\varphi' \\
\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') &= (az\cos\varphi'\underline{\mathbf{e}}_x + az\sin\varphi'\underline{\mathbf{e}}_y + a^2\underline{\mathbf{e}}_z)d\varphi' \\
\underline{\mathbf{H}}_{IV}(\underline{\mathbf{R}}) = \int_{C_{IV}} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3} &= \frac{1}{(a^2+z^2)^{3/2}} \int_{3\pi/2}^{2\pi} (az\cos\varphi'\underline{\mathbf{e}}_x + az\sin\varphi'\underline{\mathbf{e}}_y + a^2\underline{\mathbf{e}}_z)d\varphi' \\
&= \frac{az}{(a^2+z^2)^{3/2}}(\underline{\mathbf{e}}_x - \underline{\mathbf{e}}_y) + \frac{\pi a^2\underline{\mathbf{e}}_z}{2(a^2+z^2)^{3/2}}
\end{aligned}$$

The whole loop (with one winding) then yields:

$$\begin{aligned}
\underline{\mathbf{H}}(\underline{\mathbf{R}}) &= \underline{\mathbf{H}}_I(\underline{\mathbf{R}}) + \underline{\mathbf{H}}_{II}(\underline{\mathbf{R}}) + \underline{\mathbf{H}}_{III}(\underline{\mathbf{R}}) + \underline{\mathbf{H}}_{IV}(\underline{\mathbf{R}}) \\
&= \frac{I_0}{4\pi}\left[\left(\frac{2az}{(a^2+z^2)^{\frac{3}{2}}} - \frac{2a}{z\sqrt{z^2+a^2}}\right)(\underline{\mathbf{e}}_x - \underline{\mathbf{e}}_y)\right]
\end{aligned}$$

Considering there are 5 windings, we finally find:

$$\underline{\mathbf{H}}(\underline{\mathbf{R}}) = 5\frac{I_0}{4\pi}\left[\left(\frac{2az}{(a^2+z^2)^{\frac{3}{2}}} - \frac{2a}{z\sqrt{z^2+a^2}}\right)(\underline{\mathbf{e}}_x - \underline{\mathbf{e}}_y)\right]$$