$\frac{\partial}{\partial t}\underline{\boldsymbol{B}}=-\nabla\times\underline{\boldsymbol{E}}$

$\frac{\partial}{\partial t}\underline{\boldsymbol{D}}=\nabla\times\underline{\boldsymbol{H}}$

GhK TET

$\frac{\partial}{\partial t}\underline{\boldsymbol{p}}=\nabla\cdot\underline{\underline{\boldsymbol{T}}}$

$\frac{\partial}{\partial t}\underline{\underline{\boldsymbol{S}}}=\mathrm{sym}\{\nabla\underline{\boldsymbol{v}}\}$

EFT I – Exam

19.09.01

Solution

Electromagnetic Field Theory I

Problem 1

1a) The magnetic flux density $\underline{\mathbf{B}}(z)$ on the z–axis of a circular wire loop centered around the z–axis in the xy–plane at $z - 0$ (from an old lecture, exercise, or exam):

$$\underline{\mathbf{B}}\,(x=y=0,z) \;=\; \frac{\mu_0\, I_0}{2}\frac{a^2}{(z^2+a^2)^{\frac{3}{2}}}\underline{\mathbf{e}}_z$$

with I_0 constant current in the loop and a ($a>0$) the radius of the loop.
In the origin the magnetic flux density is

$$\underline{\mathbf{B}}\,(x=y=z=0) \;=\; \frac{\mu_0\, I_0}{2}\frac{1}{a}\,\underline{\mathbf{e}}_z\,.$$

This yields for the given problem:
Inner loop:

$$\underline{\mathbf{B}}_1\,(x=y=z=0) \;=\; \frac{I_1\mu_0}{2}\frac{1}{a}\,\underline{\mathbf{e}}_z\,.$$

Outer loop:

$$\underline{\mathbf{B}}_2\,(x=y=z=0) \;=\; \frac{I_2\mu_0}{2}\frac{1}{3a}\,\underline{\mathbf{e}}_z\,.$$

Superposition of both flux densities:

$$\begin{aligned}
\underline{\mathbf{B}}\,(\underline{\mathbf{0}}) &= \frac{I_1\mu_0}{2}\frac{1}{a}\underline{\mathbf{e}}_z + \frac{I_2\mu_0}{2}\frac{1}{3a}\underline{\mathbf{e}}_z \\
\underline{\mathbf{B}}\,(\underline{\mathbf{0}}) &= \frac{\mu_0}{2}\frac{1}{a}\left(I_1+\frac{I_2}{3}\right)\underline{\mathbf{e}}_z \\
&\overset{!}{=} \underline{\mathbf{0}} \\
\Rightarrow \qquad I_1+\frac{I_2}{3} &\overset{!}{=} 0 \\
\Rightarrow \qquad I_2 &= -3I_1\,.
\end{aligned}$$

1b)

$$\begin{aligned}\underline{\mathbf{J}}(\underline{\mathbf{R}}) &= I_1\,\delta(z)\;\delta(r-a)\;\underline{\mathbf{e}}_\varphi(\varphi) + I_2\,\delta(z)\;\delta(r-3a)\;\underline{\mathbf{e}}_\varphi(\varphi)\\ &= \delta(z)\;[I_1\,\delta(r-a) + I_2\,\delta(r-3a)]\;\underline{\mathbf{e}}_\varphi(\varphi)\end{aligned}$$

1c) Solution in Cartesian coordinates:

$$\int_{z=-\infty}^{\infty}\int_{x=0}^{\infty}\underline{\mathbf{J}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{e}}_y dxdz$$

$$= \int_{z=-\infty}^{\infty}\int_{x=0}^{\infty}\delta(z)\;\left[I_1\,\delta(\sqrt{x^2+y^2}-a) + I_2\,\delta\left(\sqrt{x^2+y^2}-3a\right)\right]\;\underline{\mathbf{e}}_\varphi(\varphi)\;\cdot\underline{\mathbf{e}}_y dxdz\Big|_{\varphi=0;y=0} \tag{1.1}$$

$$= \int_{z=-\infty}^{\infty}\int_{x=0}^{\infty}\delta(z)\;[I_1\,\delta(x-a) + I_2\,\delta(x-3a)]\underbrace{\underline{\mathbf{e}}_y\cdot\underline{\mathbf{e}}_y}_{=1}dxdz \tag{1.2}$$

$$= \int_{x=0}^{\infty}[I_1\,\delta(x-a) + I_2\,\delta(x-3a)]\underbrace{\underline{\mathbf{e}}_y\cdot\underline{\mathbf{e}}_y}_{=1}dx \tag{1.3}$$

$$= I_1 + I_2 \tag{1.4}$$

Solution in cylindrical coordinates:

$$\int_{z=-\infty}^{\infty}\int_{r=0}^{\infty}\underline{\mathbf{J}}(\underline{\mathbf{R}})\Big|_{\varphi=0;y=0}\cdot\underline{\mathbf{e}}_\varphi(\varphi)\Big|_{\varphi=0;y=0}dr\,dz$$

$$= \left\{\int_{z=-\infty}^{\infty}\int_{r=0}^{\infty}\delta(z)\;[I_1\,\delta(r-a) + I_2\,\delta(r-3a)]\underbrace{\underline{\mathbf{e}}_\varphi(\varphi)\;\cdot\underline{\mathbf{e}}_\varphi(\varphi)}_{=1}dr\,dz\right\}_{\varphi=0;y=0} \tag{1.5}$$

$$= \int_{z=-\infty}^{\infty}\int_{r=0}^{\infty}\delta(z)\;[I_1\,\delta(r-a) + I_2\,\delta(r-3a)]\;dr\,dz \tag{1.6}$$

$$= \int_{r=0}^{\infty}[I_1\,\delta(r-a) + I_2\,\delta(r-3a)]\;dr \tag{1.7}$$

$$= I_1\int_{r=0}^{\infty}\delta(r-a)\,dr + I_2\int_{r=0}^{\infty}\delta(r-3a)\;dr \tag{1.8}$$

$$= I_1 + I_2 \tag{1.9}$$

Problem 2

a)

$$\begin{aligned}
\underline{\mathbf{D}}(\underline{\mathbf{R}}) &= D_R(R)\,\underline{\mathbf{e}}_R \\
\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{dS}} &= \iiint_V \varrho(\underline{\mathbf{R}})\,dV \\
\underline{\mathbf{dS}} &= \underline{\mathbf{e}}_R\,R^2\sin\vartheta\,d\vartheta\,d\varphi \\
dV &= R^2\,\sin\vartheta\,dR\,d\vartheta\,d\varphi \\
\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{dS}} &= \int_0^{2\pi}\int_0^{\pi} D_R(R)\underline{\mathbf{e}}_R\cdot\underline{\mathbf{e}}_R\,R^2\sin\vartheta\,d\vartheta\,d\varphi \\
&= 4\pi R^2\,D_R(R)
\end{aligned}$$

Inside sphere:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \varrho(\underline{\mathbf{R}}')\,dV' = \varrho_0\pi\frac{R^4}{R_0}$$

Outside sphere:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R_0} \varrho(\underline{\mathbf{R}}')\,\mathrm{d}V' = \varrho_0\pi R_0^3$$

$$\mathrm{D}_R(R) = \begin{cases} \dfrac{\varrho_0}{4}\dfrac{R^2}{R_0} & : \quad 0\le R\le R_0 \\[2ex] \dfrac{\varrho_0}{4}\dfrac{R_0^3}{R^2} & : \quad R_0 < R \le \infty \end{cases}$$

b)

$$\underline{\mathbf{E}} = \frac{\underline{\mathbf{D}}}{\varepsilon_0\varepsilon_r}$$

$$\varepsilon_\mathrm{r}(R) = \begin{cases} 5\dfrac{R}{R_0} & 0\le R\le R_0 \\[2ex] 1 & R_0 < R\le\infty \end{cases}$$

(Note that this is a fictitious relative permittivity, because the value start at $R=0$ with zero, but we all known that ε_r must be always greater than zero.)

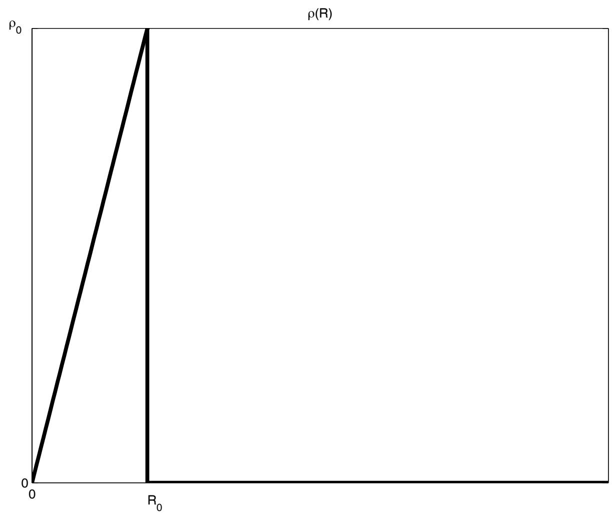
$$\mathrm{E}_R(R) = \begin{cases} \dfrac{\varrho_0}{20\varepsilon_0}R & 0\le R\le R_0 \\[2ex] \dfrac{\varrho_0}{4\varepsilon_0}\dfrac{R_0^3}{R^2} & R_0 < R\le\infty \end{cases}$$

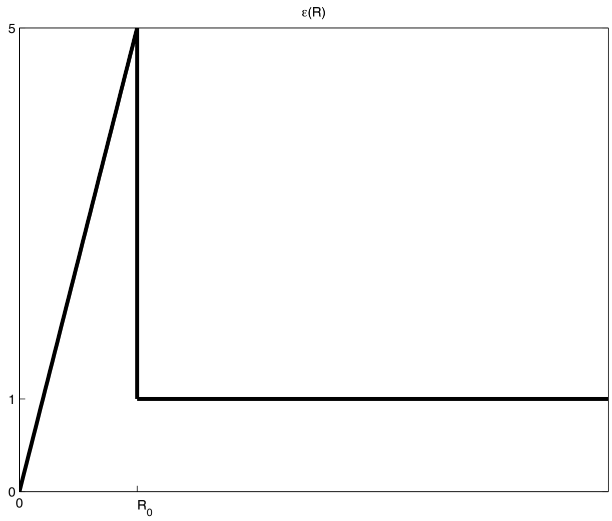
c)

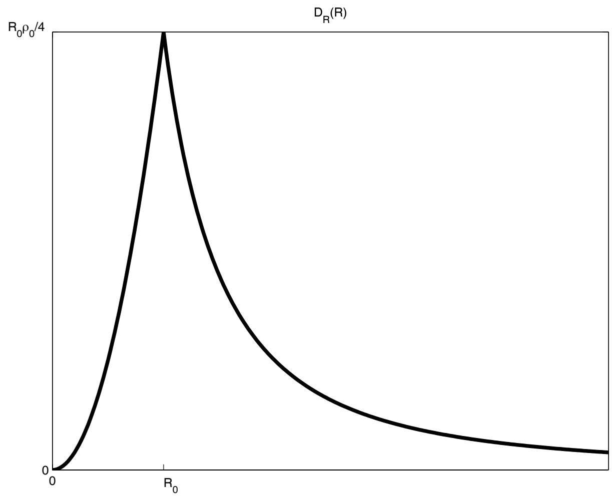
$$\underline{\mathbf{P}} = \underline{\mathbf{D}} - \varepsilon_0 \underline{\mathbf{E}}$$

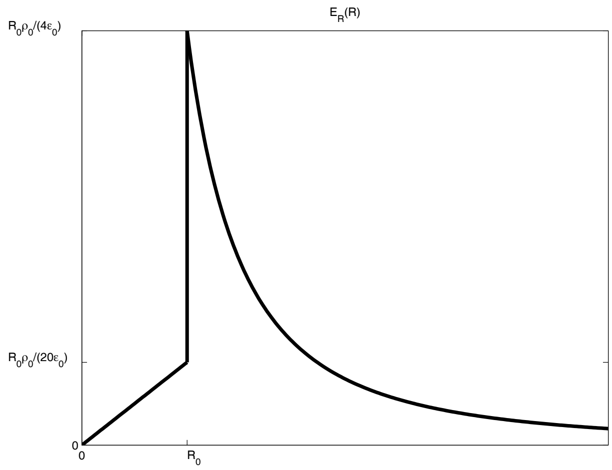
$$\mathrm{P}_R(R) = \begin{cases} \frac{\varrho_0}{4}\left[\frac{R^2}{R_0} - \frac{R}{5}\right] & 0 \le R \le R_0 \\ 0 & R_0 < R \le \infty \end{cases}$$

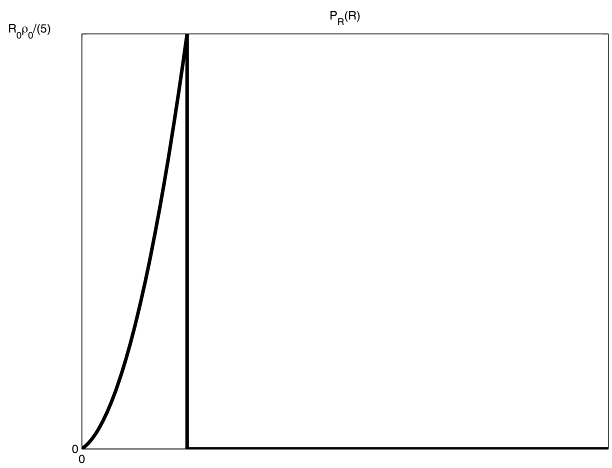
d)






D_R(R)
R_0ρ_0/4
0
0
R_0


E_R(R)
R_0ρ_0/(4ε_0)
R_0ρ_0/(20ε_0)
0
0
R_0


P_R(R)
R_0ρ_0/(5)
0
0

Problem 3

a)

$$
\begin{aligned}
Q_1 = +2Q &\ : \ \underline{\mathbf{R}}_1 = \underline{\mathbf{0}} \\
Q_2 = -Q &\ : \ \underline{\mathbf{R}}_2 = D\,\underline{\mathbf{e}}_y + D\,\underline{\mathbf{e}}_z \\
Q_3 = -Q &\ : \ \underline{\mathbf{R}}_3 = -D\,\underline{\mathbf{e}}_y + D\,\underline{\mathbf{e}}_z \\
Q_4 = -Q &\ : \ \underline{\mathbf{R}}_4 = -D\,\underline{\mathbf{e}}_y - D\,\underline{\mathbf{e}}_z \\
Q_5 = -Q &\ : \ \underline{\mathbf{R}}_5 = D\,\underline{\mathbf{e}}_y - D\,\underline{\mathbf{e}}_z
\end{aligned}
$$

$$
\begin{aligned}
\varrho(\underline{\mathbf{R}}) &= Q_1\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_1) + Q_2\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_2) + Q_3\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_3) + Q_4\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_4) \\
&= \sum_{n=1}^{5} Q_n\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)
\end{aligned}
$$

b)

$$
\begin{aligned}
\underline{\mathbf{p}}_{\mathrm{e}} &= \iiint_{-\infty}^{\infty} \varrho(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\,\underline{\mathbf{R}}\,d^3\underline{\mathbf{R}} \\
&= \iiint_{-\infty}^{\infty} \sum_{n=1}^{5} Q_n\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\underline{\mathbf{R}}\,d^3\underline{\mathbf{R}} \\
&= \sum_{n=1}^{5} Q_n \underbrace{\iiint_{-\infty}^{\infty} \delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\underline{\mathbf{R}}\,d^3\underline{\mathbf{R}}}_{\underline{\mathbf{R}}_n} \\
&= \sum_{n=1}^{5} Q_n\underline{\mathbf{R}}_n \\
&= \underline{\mathbf{0}}
\end{aligned}
$$

c)

$$
\begin{aligned}
\underline{\underline{\mathbf{q}}}_{\mathrm{e}} &= \iiint_{-\infty}^{\infty} \varrho(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\,\underline{\mathbf{R}}\,\underline{\mathbf{R}}\,d^3\underline{\mathbf{R}} \\
&= \iiint_{-\infty}^{\infty} \sum_{n=1}^{5} Q_n\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\,\underline{\mathbf{R}}\,\underline{\mathbf{R}}\,d^3\underline{\mathbf{R}} \\
&= \sum_{n=1}^{5} Q_n \underbrace{\iiint_{-\infty}^{\infty} \delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\,\underline{\mathbf{R}}\,\underline{\mathbf{R}}\,d^3\underline{\mathbf{R}}}_{\underline{\mathbf{R}}_n\,\underline{\mathbf{R}}_n} \\
&= \sum_{n=1}^{5} Q_n\,\underline{\mathbf{R}}_n\,\underline{\mathbf{R}}_n \\
&= -4QD^2\,\left(\underline{\mathbf{e}}_y\,\underline{\mathbf{e}}_y + \underline{\mathbf{e}}_z\,\underline{\mathbf{e}}_z\right)
\end{aligned}
$$

Problem 4

4a) Charges:

$$Q_1 = -Q \quad : \quad \underline{\mathbf{R}}_1 = x_1\,\underline{\mathbf{e}}_x + z_1\,\underline{\mathbf{e}}_z$$
$$Q_2 = +Q \quad : \quad \underline{\mathbf{R}}_2 = y_2\,\underline{\mathbf{e}}_y + z_2\,\underline{\mathbf{e}}_z$$

Images:

$$Q_3 = +Q \quad : \quad \underline{\mathbf{R}}_3 = x_1\,\underline{\mathbf{e}}_x - z_1\,\underline{\mathbf{e}}_z$$
$$Q_4 = -Q \quad : \quad \underline{\mathbf{R}}_4 = y_2\,\underline{\mathbf{e}}_y - z_2\,\underline{\mathbf{e}}_z$$

$$\Phi(\underline{\mathbf{R}}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{n=1}^{4}\frac{Q_n}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_n|} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

4b) We find the electric surface density as the normal component of the electric flux density at the xy–plane:

$$\eta(\underline{\mathbf{R}})\Big|_{z=0} = \underline{\mathbf{n}}\cdot\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\Big|_{z=0} \tag{4.1}$$

where $\underline{\mathbf{n}}$ is the surface normal unit vector.

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}}) \tag{4.2}$$

$$= \begin{cases} \dfrac{1}{4\pi}\displaystyle\sum_{n=1}^{4} Q_n\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_n}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_n|^3} & z \geq 0 \\ \underline{\mathbf{0}} & z < 0 \end{cases} \tag{4.3}$$

$$\eta(\underline{\mathbf{R}})\Big|_{z=0} = \underline{\mathbf{n}}\cdot\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{z=0} \tag{4.4}$$

$$= \underline{\mathbf{e}}_z\cdot\frac{1}{4\pi}\sum_{n=1}^{4} Q_n\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_n}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_n|^3}\Big|_{z=0}. \tag{4.5}$$

At $z = 0$ we find

$$\eta(x,y) = \underline{\mathbf{n}}\cdot\underline{\mathbf{D}}(x,y) \tag{4.6}$$

$$= \underline{\mathbf{e}}_z\cdot\underline{\mathbf{D}}(x,y) \tag{4.7}$$

$$= \frac{1}{4\pi}\sum_{n=1}^{4} Q_n\frac{\underline{\mathbf{e}}_z\cdot(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_n|^3}\Big|_{z=0} \tag{4.8}$$

$$= \frac{Q}{4\pi}\left[\frac{2z_1}{[(x-x_1)^2+y^2+z_1^2]^{3/2}} - \frac{2z_2}{[x^2+(y-y_2)^2+z_2^2]^{3/2}}\right] \tag{4.9}$$

$$= \frac{Q}{2\pi}\left[\frac{z_1}{[(x-x_1)^2+y^2+z_1^2]^{3/2}} - \frac{z_2}{[x^2+(y-y_2)^2+z_2^2]^{3/2}}\right]. \tag{4.10}$$

4c) The electric surface charge $\eta(x,y)$ reaches a extremum, i. e. $|\eta(x,y)| = \max$, for the condition

$$\nabla\eta(x,y) = \underline{\mathbf{0}}\,. \tag{4.11}$$

We compute

$$\nabla\eta(x,y) = \frac{Q}{2\pi}\left[z_1\nabla\left[(x-x_1)^2+y^2+z_1^2\right]^{-3/2} - z_2\nabla\left[x^2+(y-y_2)^2+z_2^2\right]^{-3/2}\right] \tag{4.12}$$

$$= -\frac{3Q}{2\pi}\left[z_1\frac{(x-x_1)\underline{\mathbf{e}}_x + y\underline{\mathbf{e}}_y}{\left[(x-x_1)^2+y^2+z_1^2\right]^{5/2}} - z_2\frac{x\underline{\mathbf{e}}_x + (y-y_2)\underline{\mathbf{e}}_y}{\left[x^2+(y-y_2)^2+z_2^2\right]^{5/2}}\right]. \tag{4.13}$$

$$\nabla\eta_1(x,y) = -\frac{3Q}{2\pi}z_1\frac{(x-x_1)\underline{\mathbf{e}}_x + y\underline{\mathbf{e}}_y}{\left[(x-x_1)^2+y^2+z_1^2\right]^{5/2}}\,. \tag{4.14}$$

$$\nabla\eta_2(x,y) = \frac{3Q}{2\pi}z_2\frac{x\underline{\mathbf{e}}_x + (y-y_2)\underline{\mathbf{e}}_y}{\left[x^2+(y-y_2)^2+z_2^2\right]^{5/2}}\,. \tag{4.15}$$

The vector function $\nabla\eta_1(x,y)$ reaches an extremum at $\underline{\mathbf{R}}_1(x,y,z) = (x_1,0,0)$ and the vector function $\nabla\eta_2(x,y)$ reaches an extremum at $\underline{\mathbf{R}}_2(x,y,z) = (0,y_2,0)$. This means that the extrema of the electric surface charge are measured at the position $\underline{\mathbf{R}}_1$ and $\underline{\mathbf{R}}_2$, because η_1 and η_2 are continues and smooth functions. These are the direct projections of the source points to the perfectly electrically conducting xy–plane.

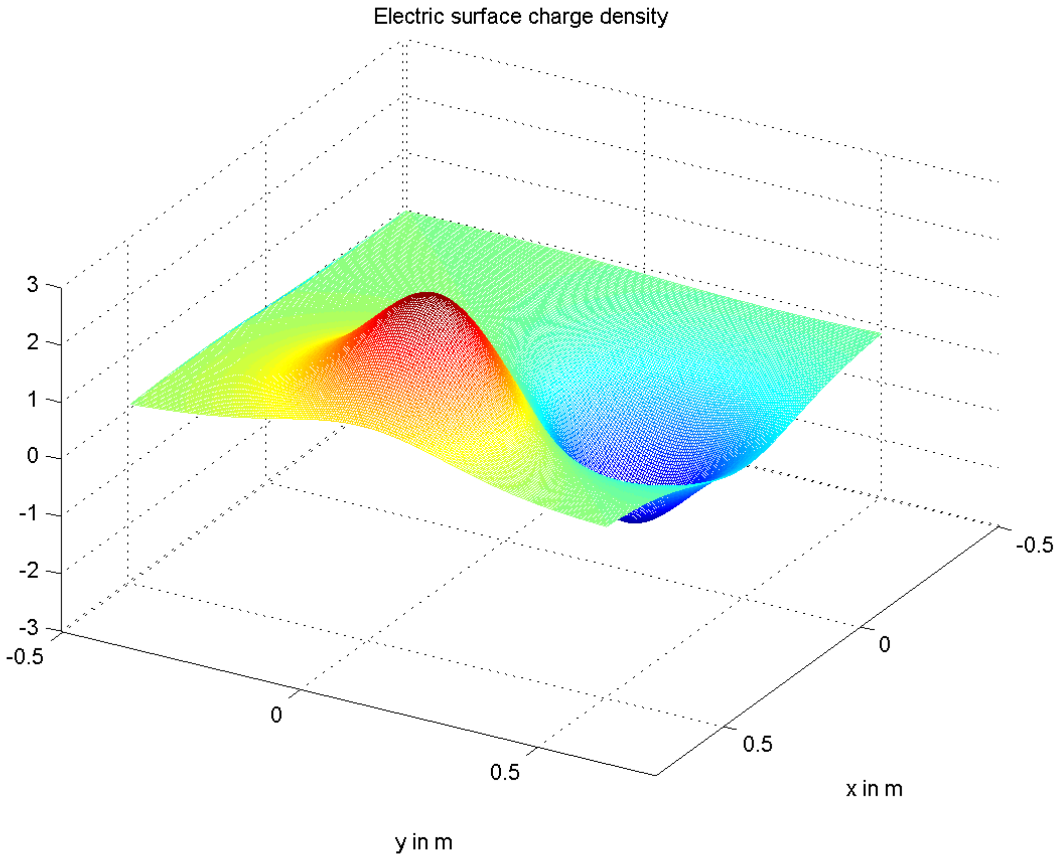


Figure 4.1: Electric surface charge density