Aufgabe 1

a)

$$\begin{aligned}\varrho(\underline{\mathbf{R}}) &= Q_1\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_1) + Q_2\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_2) + Q_3\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_3)\\ &= \sum_{n=1}^{3} Q_n\,\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\end{aligned}$$

$$\begin{aligned}\underline{\mathbf{R}}_1 &= -a_1\,\underline{\mathbf{e}}_x - b_1\,\underline{\mathbf{e}}_y\\ \underline{\mathbf{R}}_2 &= a_2\,\underline{\mathbf{e}}_x - b_2\,\underline{\mathbf{e}}_y\\ \underline{\mathbf{R}}_3 &= a_3\,\underline{\mathbf{e}}_x + b_3\,\underline{\mathbf{e}}_y\end{aligned}$$

b)

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{3} Q_n \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_n}{|\ \underline{\mathbf{R}}-\underline{\mathbf{R}}_n\ |^3}$$

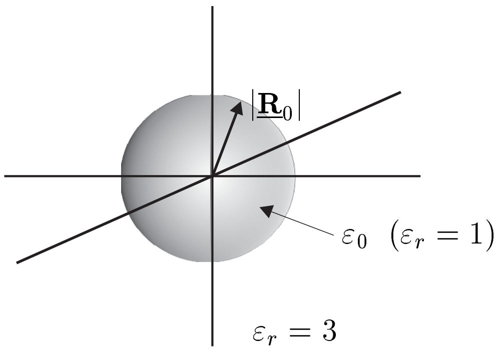
c)

$$\begin{aligned}\underline{\mathbf{p}}_e &= \iiint_{-\infty}^{\infty} \varrho(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\underline{\mathbf{R}}\,\mathrm{d}^3\underline{\mathbf{R}}\\ &= \iiint_{-\infty}^{\infty} \sum_{n=1}^{3} Q_n\delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\underline{\mathbf{R}}\,\mathrm{d}^3\underline{\mathbf{R}}\\ &= \sum_{n=1}^{3} Q_n \underbrace{\iiint_{-\infty}^{\infty} \delta(\underline{\mathbf{R}}-\underline{\mathbf{R}}_n)\underline{\mathbf{R}}\,\mathrm{d}^3\underline{\mathbf{R}}}_{\underline{\mathbf{R}}_n}\\ &= \sum_{n=1}^{3} Q_n\underline{\mathbf{R}}_n\end{aligned}$$

d) $Q_1 = Q_2 = Q_3 = Q, \qquad a_2 = a_3 = b_2 = b_3 = a, \qquad a_1^2$ und $b_1^2 \qquad$ für $\quad p_e = \underline{\mathbf{0}}$

$$\begin{aligned}p_e &= Q\sum_{n=1}^{3}\underline{\mathbf{R}}_n\\ &= Q\left[-a_1\,\underline{\mathbf{e}}_x - b_1\,\underline{\mathbf{e}}_y + 2a\,\underline{\mathbf{e}}_x\right]\\ &= \underline{\mathbf{0}} \quad \text{für} \quad \underline{a_1 = 2a} \quad \text{und} \quad \underline{b_1 = 0}\end{aligned}$$

Aufgabe 2



$\varrho(\underline{\mathbf{R}}) = \varrho_0 \left(\frac{R}{R_0}\right)^2 \qquad (R < R_0)$

a) $Q(R)$ in der Kugel $R < R_0$

$$\begin{aligned}
\mathrm{d}V' &= R' \sin\vartheta' \mathrm{d}\varphi' \mathrm{d}\vartheta' \mathrm{d}R' \\
Q(R) &= \iiint\limits_{\underline{\mathbf{R}}' \in K} \varrho(R')\,\mathrm{d}V' \\
&= 4\pi \int\limits_{R'=0}^{R} \varrho_0 \frac{R'^2}{R_0^2} R'^2\,\mathrm{d}R'^2 \\
&= \frac{4\pi}{R_0^2}\varrho_0 \underbrace{\int\limits_{R'=0}^{R} R'^4\,\mathrm{d}R'}_{= \underbrace{\frac{1}{5}R'^5\Big|_{R'}^{R}}_{\frac{1}{5}R^5}}
\end{aligned}$$

$$Q(R) = \frac{4\pi\varrho_0}{5R_0^2}R^5 \qquad R < R_0$$

b) $Q(R)$ in der Kugel $R > R_0$

$$\begin{aligned}
Q(R) &= \frac{4\pi\varrho_0}{R_0^2} \int\limits_{R'=0}^{R_0} R'^4\,\mathrm{d}R' \\
&= \frac{4\pi\varrho_0}{R_0^2}\frac{1}{5}R_0^5 \\
&= \frac{4\pi\varrho_0}{5}R_0^3 \qquad R \geq R_0
\end{aligned}$$

c)

$$\iint_{S=\partial V} \underline{\mathbf{n}} \cdot \underline{\mathbf{D}}(\underline{\mathbf{R}}) \underbrace{\mathrm{d}S}_{R^2 \sin\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\varphi} = Q(\underline{\mathbf{R}})$$

$$\begin{aligned} \underline{\mathbf{n}} &= \underline{\mathbf{e}}_R \\ \underline{\mathbf{n}} &\cdot \underline{\mathbf{D}}(\underline{\mathbf{R}}) = D_R(R) \end{aligned}$$

$$D_R(R) = \frac{1}{4\pi R^2} Q(R)$$

$$\text{mit} \quad Q(R) = \frac{4\pi \varrho_0}{5R_0^2} \begin{cases} R^5 & \text{für} \quad R < R_0 \\ \\ R_0^5 & \text{für} \quad R \geq R_0 \end{cases}$$

$$D_R(R) = \frac{\varrho_0}{5R_0^2} \frac{1}{R^2} \begin{cases} R^5 & \text{für} \quad R < R_0 \\ \\ R_0^5 & \text{für} \quad R \geq R_0 \end{cases}$$

d)

$$\begin{aligned} D_R(R) &= \varepsilon(R) E_R(R) \\ E_R(R) &= \frac{1}{\varepsilon(R)} D_R(R) \end{aligned}$$

$$\varepsilon(R) = \begin{cases} \varepsilon_0 & R < R_0 \\ \\ \varepsilon_0 \varepsilon_r & R \geq R_0 \end{cases}$$

$$E_R(R) = \frac{\varrho_0}{5\varepsilon_0 R_0^2} \frac{1}{R^2} \begin{cases} R^5 & R < R_0 \\ \\ \dfrac{R_0^5}{\varepsilon_r} & R \geq R_0 \end{cases}$$

e)

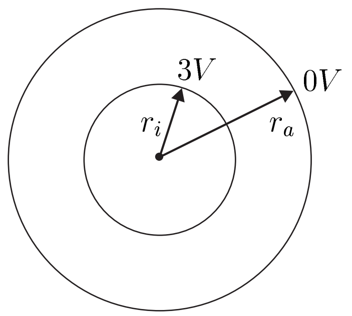
$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}(\underline{\mathbf{R}})$$

$$D_R(R) = \varepsilon_0 E_R(R) + P_R(R)$$

$$
\begin{aligned}
P_R(R) &= -\varepsilon_0 E_R(R) + D_R(R) \\
&= -\varepsilon_0 \frac{1}{\varepsilon(R)} D_R(R) + D_R(R) \\
&= D_R(R)\left[-\frac{\varepsilon_0}{\varepsilon(R)} + 1\right] \\
&= D_R(R)\left[-\frac{1}{\varepsilon_r(R)} + 1\right]
\end{aligned}
$$

$$
P_R(R) = \begin{cases} 0 & R < R_0 \\ \frac{\varrho_0}{5}\frac{1}{R^2}R_0^3 \underbrace{\left[-\frac{1}{3} + 1\right]}_{+\frac{2}{3}} & R \geq R_0 \end{cases}
$$

Aufgabe 3



$$\Phi(r_i = 0,25m) = 3V$$
$$\Phi(r_a = 0,5m) = 0V$$

a)

$$\begin{aligned} \Delta\Phi(r) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Phi(r)\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} r\frac{\partial}{\partial r}\Phi(r) &= C_1 \\ \Phi(r) &= C_1 \ln r + C_2 \end{aligned}$$

$r = r_i$:
$$\begin{aligned} \Phi(r) &= C_1 \ln r_i + C_2 \\ &= 3V \end{aligned}$$

$r = r_a$:
$$\begin{aligned} \Phi(r) &= C_1 \ln r_a + C_2 \\ &= 0V \end{aligned}$$

$$C_2 = -C_1 \ln r_a$$

$$\begin{aligned} \Phi(r) &= C_1 \ln r_i - C_1 \ln r_a \\ &= C_1 \ln \frac{r_i}{r_a} \\ &= 3V \end{aligned}$$

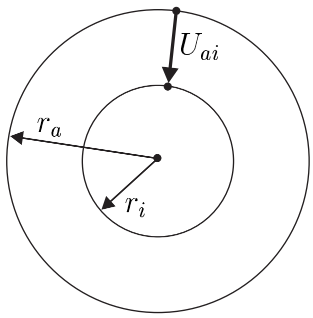
$$C_1 = \frac{3V}{\ln \frac{r_i}{r_a}}$$

$$
\begin{aligned}
\Phi(r) &= C_1 \ln r + C_2 \\
&= C_1 \ln r - C_1 \ln r_a \\
&= C_1 \ln \frac{r}{r_a}
\end{aligned}
$$

$$
\Phi(r) = \frac{3V}{\ln \frac{r_i}{r_a}} \ln \frac{r}{r_a} \qquad\qquad (r_i < r < r_a)
$$

b)

$$
\begin{aligned}
\underline{\mathbf{E}}(\underline{\mathbf{R}}) &= -\nabla \Phi(\underline{\mathbf{R}}) \\
&= -\frac{\partial}{\partial r} \Phi(r)\, \underline{\mathbf{e}}_r \\
&= -\frac{3V}{\ln\left(\frac{r_i}{r_a}\right)} \underbrace{\frac{\partial}{\partial r} \ln\left(\frac{r}{r_a}\right)}_{\frac{1}{r}} \underline{\mathbf{e}}_r \\
&= -\frac{3V}{\ln\left(\frac{r_i}{r_a}\right)} \frac{1}{r}\, \underline{\mathbf{e}}_r \qquad\qquad (r_i < r < r_a)
\end{aligned}
$$



c)

$$
\begin{aligned}
U_{ai} &= \int\limits_{r=r_a}^{r_i} E_r(r)\, \mathrm{d}r \\
&= \Phi(r_a) - \Phi(r_i)
\end{aligned}
$$

$$
U_{ai} = -3V
$$

Aufgabe 4

a)

Ladungen:

$$\begin{aligned} Q_1 = -Q \quad &: \quad \underline{\mathbf{R}}_- = a\,\underline{\mathbf{e}}_x - b\,\underline{\mathbf{e}}_y + c\,\underline{\mathbf{e}}_z = \underline{\mathbf{R}}_1 \\ Q_2 = +Q \quad &: \quad \underline{\mathbf{R}}_+ = d\,\underline{\mathbf{e}}_x + e\,\underline{\mathbf{e}}_y - f\,\underline{\mathbf{e}}_z = \underline{\mathbf{R}}_2 \end{aligned}$$

Spiegelladungen:

$$\begin{aligned} Q_3 = +Q^{Sp} = Q \quad &: \quad \underline{\mathbf{R}}_+^{Sp} = -a\,\underline{\mathbf{e}}_x - b\,\underline{\mathbf{e}}_y + c\,\underline{\mathbf{e}}_z = \underline{\mathbf{R}}_3 \\ Q_4 = -Q^{Sp} = -Q \quad &: \quad \underline{\mathbf{R}}_-^{Sp} = -d\,\underline{\mathbf{e}}_x + e\,\underline{\mathbf{e}}_y - f\,\underline{\mathbf{e}}_z = \underline{\mathbf{R}}_4 \end{aligned}$$

$$\Phi(\underline{\mathbf{R}}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{n=1}^{4} \dfrac{Q_n}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_n|} & x > 0 \\[2ex] 0 & x \le 0 \end{cases}$$

b)

$$\begin{aligned} \underline{\mathbf{D}} &= \varepsilon_0\,\underline{\mathbf{E}} \\ &= \frac{1}{4\pi} \sum_{n=1}^{4} Q_n \frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_n}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_n|^3} \qquad (x > 0) \end{aligned}$$

c)

$$\begin{aligned} \eta(x=0,y,z) &= \underline{\mathbf{e}}_x \cdot \underline{\mathbf{D}}\big|_{x=0} \\ &= \frac{1}{4\pi} \sum_{n=1}^{4} Q_n \left. \frac{\underline{\mathbf{e}}_x \cdot (\underline{\mathbf{R}} - \underline{\mathbf{R}}_n)}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_n|^3} \right|_{x=0} \end{aligned}$$

$$\begin{aligned} = \frac{1}{4\pi} \Bigg[& -Q \frac{-a}{[a^2 + (y+b)^2 + (z-c)^2]^{3/2}} \\ & +Q \frac{-d}{[d^2 + (y-e)^2 + (z+f)^2]^{3/2}} \\ & +Q \frac{+a}{[a^2 + (y+b)^2 + (z-c)^2]^{3/2}} \\ & -Q \frac{+d}{[d^2 + (y-e)^2 + (z+f)^2]^{3/2}} \Bigg] \end{aligned}$$

$$= \frac{1}{4\pi} \left[\frac{2Qa}{[a^2 + (y+b)^2 + (z-c)^{3/2}]} - \frac{2Qd}{[d^2 + (y-e)^2 + (z+f)^{3/2}]} \right]$$