Elektromagnetische Feldtheorie I (EFT I) / Electromagnetic Field Theory I (EFT I)

11th Lecture / 11. Vorlesung

Dr.-Ing. René Marklein

marklein@uni-kassel.de

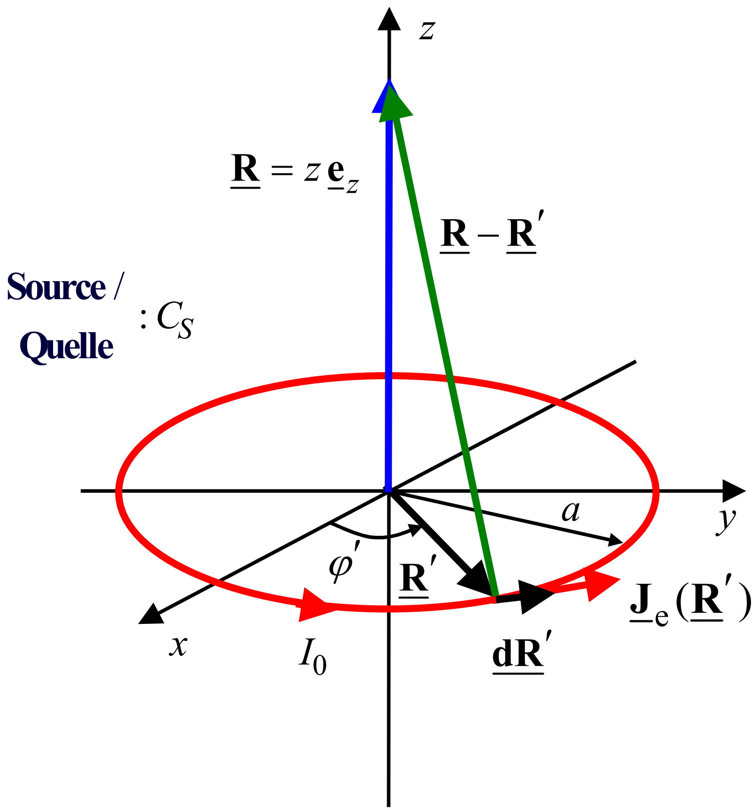
http://www.tet.e-technik.uni-kassel.de

http://www.uni-kassel.de/fb16/tet/marklein/index.html

Universität Kassel
Fachbereich Elektrotechnik / Informatik
(FB 16)
Fachgebiet Theoretische Elektrotechnik
(FG TET)
Wilhelmshöher Allee 71
Büro: Raum 2113 / 2115
D-34121 Kassel

University of Kassel
Dept. Electrical Engineering / Computer Science (FB 16)
Electromagnetic Field Theory
(FG TET)
Wilhelmshöher Allee 71
Office: Room 2113 / 2115
D-34121 Kassel

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)



$$\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}) = I_0\,\delta(r - a)\,\delta(z)\,\underline{\mathbf{e}}_\varphi(\varphi), \qquad a > 0$$

Biot-Savart's Law /
Biot-Savartsches Gesetz

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\,\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

Magnetic Field Strength on the z Axis/
Magnetische Feldstärke auf der z-Achse

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\,\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

$$\underline{\mathbf{R}} = z\,\underline{\mathbf{e}}_z \qquad -\infty < z < \infty$$

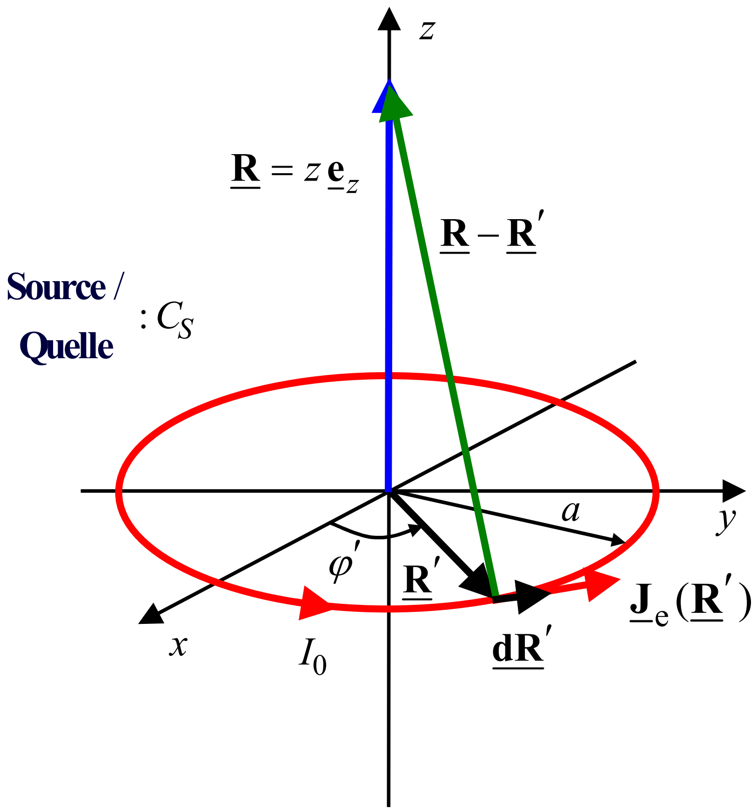
$$\underline{\mathbf{R}}' = a\,\underline{\mathbf{e}}_{r'}(\varphi') \qquad C_s : 0 \le \varphi \le 2\pi$$

$$= a\cos\varphi'\,\underline{\mathbf{e}}_x + a\sin\varphi'\,\underline{\mathbf{e}}_y$$

$$\underline{\mathbf{R}} - \underline{\mathbf{R}}' = z\,\underline{\mathbf{e}}_z - a\cos\varphi'\,\underline{\mathbf{e}}_x - a\sin\varphi'\,\underline{\mathbf{e}}_y$$

$$\mathbf{d}\underline{\mathbf{R}}' = \left[\frac{\mathrm{d}}{\mathrm{d}\varphi'}\underline{\mathbf{R}}'(\varphi')\right]\mathrm{d}\varphi'$$

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)



Magnetic Field Strength on the z Axis/
Magnetische Feldstärke auf der z-Achse

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

$$\underline{\mathbf{R}}' = a\cos\varphi'\,\underline{\mathbf{e}}_x + a\sin\varphi'\,\underline{\mathbf{e}}_y$$

$$\begin{aligned}\mathbf{d}\underline{\mathbf{R}}' &= \left[\frac{\mathrm{d}}{\mathrm{d}\varphi'}\underline{\mathbf{R}}'(\varphi')\right]\mathrm{d}\varphi' \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\varphi'}\left(a\cos\varphi'\underline{\mathbf{e}}_x + a\sin\varphi'\underline{\mathbf{e}}_y\right)\right]\mathrm{d}\varphi' \\ &= a\left(-\sin\varphi'\underline{\mathbf{e}}_x + \cos\varphi'\underline{\mathbf{e}}_y\right)\mathrm{d}\varphi'\end{aligned}$$

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\,\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

$$\underline{\mathbf{R}} - \underline{\mathbf{R}}' = z\underline{\mathbf{e}}_z - a\cos\varphi'\underline{\mathbf{e}}_x - a\sin\varphi'\underline{\mathbf{e}}_y, \qquad 0 \le \varphi' \le 2\pi$$

$$\underline{\mathbf{dR}}' = a\left(-\sin\varphi'\underline{\mathbf{e}}_x + \cos\varphi'\underline{\mathbf{e}}_y\right)\mathrm{d}\varphi'$$

$$\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') = \begin{vmatrix} \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_z \\ -a\sin\varphi'\mathrm{d}\varphi' & a\cos\varphi'\mathrm{d}\varphi' & 0 \\ -a\cos\varphi' & -a\sin\varphi' & z \end{vmatrix}$$

$$= az\cos\varphi'\,\underline{\mathbf{e}}_x\mathrm{d}\varphi' + a^2\sin^2\varphi'\,\underline{\mathbf{e}}_z\mathrm{d}\varphi' + a^2\cos^2\varphi'\,\underline{\mathbf{e}}_z\mathrm{d}\varphi' + az\sin\varphi'\,\underline{\mathbf{e}}_y\mathrm{d}\varphi'$$

$$= az\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right)\mathrm{d}\varphi' + a^2\underbrace{\left(\sin^2\varphi' + \cos^2\varphi'\right)}_{=1}\underline{\mathbf{e}}_z\mathrm{d}\varphi'$$

$$= az\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right)\mathrm{d}\varphi' + a^2\,\underline{\mathbf{e}}_z\mathrm{d}\varphi'$$

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int\limits_{C_S} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\,\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

$$\underline{\mathbf{R}} - \underline{\mathbf{R}}' = z\underline{\mathbf{e}}_z - a\cos\varphi'\underline{\mathbf{e}}_x - a\sin\varphi'\underline{\mathbf{e}}_y, \qquad 0 \le \varphi' \le 2\pi$$

$$\left|\underline{\mathbf{R}} - \underline{\mathbf{R}}'\right|^3 = \Bigg[\underbrace{a^2\cos^2\varphi' + a^2\sin^2\varphi'}_{=a^2\underbrace{\left(\cos^2\varphi' + \sin^2\varphi'\right)}_{=1}} + z^2\Bigg]^{3/2}$$

$$= \left[a^2 + z^2\right]^{3/2}$$

$$\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') = az\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right)\mathrm{d}\varphi' + a^2\,\underline{\mathbf{e}}_z\,\mathrm{d}\varphi'$$

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\,\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

$$\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') = za\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right) - a^2\,\underline{\mathbf{e}}_z$$

$$\left|\underline{\mathbf{R}} - \underline{\mathbf{R}}'\right|^3 = \left[a^2 + z^2\right]^{3/2}$$

$$\begin{aligned}
\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) &= \frac{I_0}{4\pi} \int_{C_S} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\,\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3} \\
&= \frac{I_0}{4\pi} \int_{\varphi'=0}^{2\pi} \frac{za\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right) + a^2\,\underline{\mathbf{e}}_z}{\left[a^2 + z^2\right]^{3/2}} \mathrm{d}\varphi' \\
&= \frac{I_0}{4\pi} \int_{\varphi'=0}^{2\pi} \frac{za\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right)}{\left[a^2 + z^2\right]^{3/2}} \mathrm{d}\varphi' - \frac{I_0}{4\pi} \int_{\varphi'=0}^{2\pi} \frac{a^2\,\underline{\mathbf{e}}_z}{\left[a^2 + z^2\right]^{3/2}} \mathrm{d}\varphi'
\end{aligned}$$

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

$$\mathbf{d}\underline{\mathbf{R}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}') = az\left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right) - a^2\,\underline{\mathbf{e}}_z$$

$$\left|\underline{\mathbf{R}} - \underline{\mathbf{R}}'\right|^3 = \left[a^2 + z^2\right]^{3/2}$$

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \frac{za}{\left[a^2 + z^2\right]^{3/2}} \underbrace{\int_{\varphi'=0}^{2\pi} \left(\cos\varphi'\,\underline{\mathbf{e}}_x + \sin\varphi'\,\underline{\mathbf{e}}_y\right)\mathrm{d}\varphi'}_{=0} - \frac{I_0}{4\pi} \frac{a\,\underline{\mathbf{e}}_z}{\left[a^2 + z^2\right]^{3/2}} \int_{\varphi'=0}^{2\pi} \mathrm{d}\varphi'$$

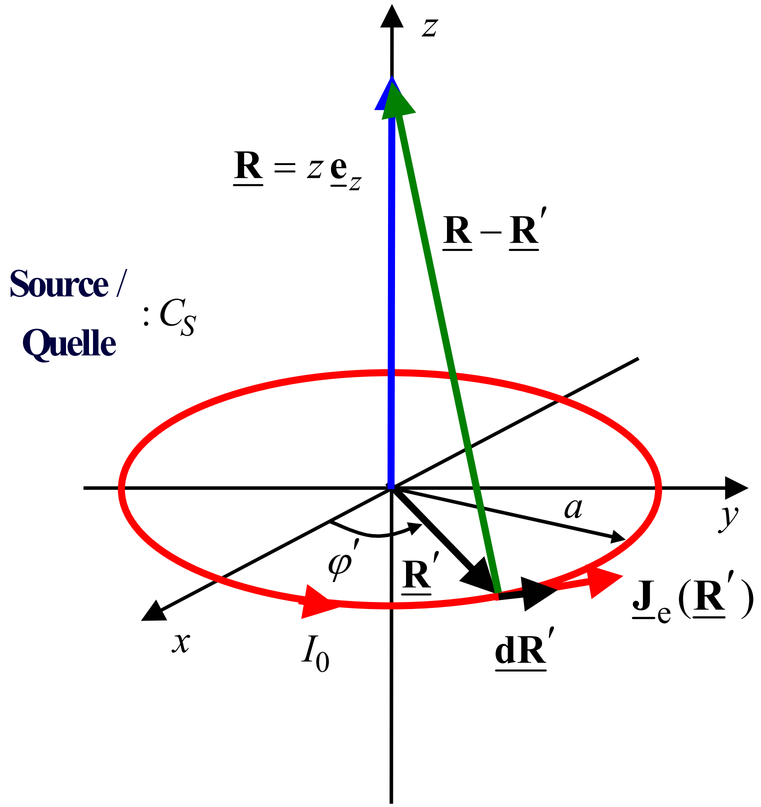
$$= \frac{I_0}{4\pi} \frac{a^2\,\underline{\mathbf{e}}_z}{\left[a^2 + z^2\right]^{3/2}} \underbrace{\int_{\varphi'=0}^{2\pi} \mathrm{d}\varphi'}_{=2\pi}$$

$$= \frac{I_0}{2} \frac{a^2}{\left[a^2 + z^2\right]^{3/2}} \underline{\mathbf{e}}_z$$

$$\int_{\varphi'=0}^{2\pi} \cos\varphi'\mathrm{d}\varphi' = 0$$

$$\int_{\varphi'=0}^{2\pi} \sin\varphi'\mathrm{d}\varphi' = 0$$

MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)



Biot-Savart's Law /
Biot-Savartsches Gesetz

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{4\pi} \int_{C_S} \frac{\underline{\mathbf{dR}}' \times (\underline{\mathbf{R}} - \underline{\mathbf{R}}')}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|^3}$$

Magnetic Field Strength on the z Axis/
Magnetische Feldstärke auf der z-Achse

$$\underline{\mathbf{H}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{I_0}{2} \frac{a^2}{\left[a^2 + z^2\right]^{3/2}} \underline{\mathbf{e}}_z$$

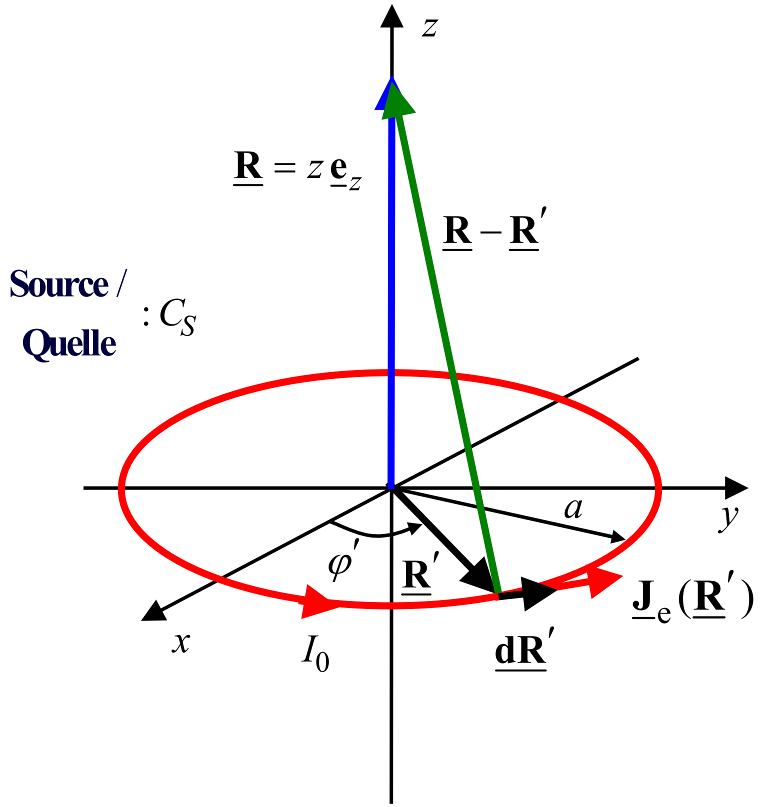
Constitutive Equation for Vacuum /
Materialgleichung für Vakuum

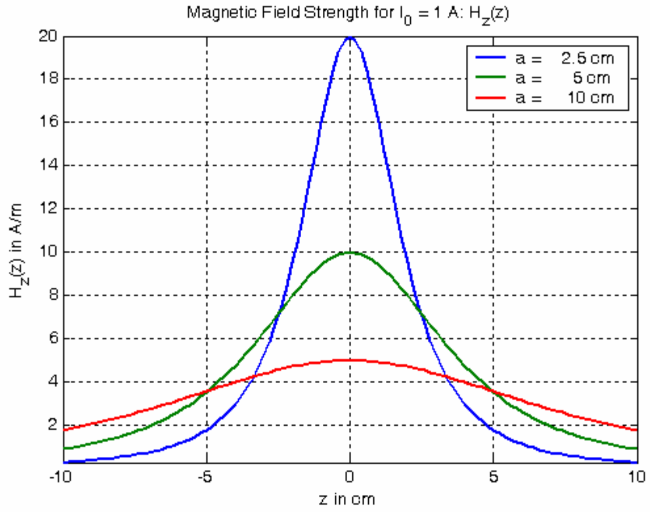
$$\underline{\mathbf{B}}(\underline{\mathbf{R}}) = \mu_0 \underline{\mathbf{H}}(\underline{\mathbf{R}})$$

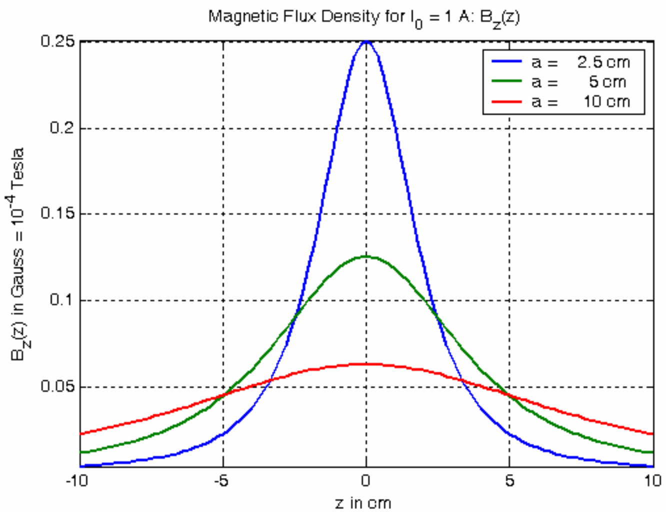
Magnetic Flux Density on the z Axis/
Magnetische Flussdichte auf der z-Achse

$$\underline{\mathbf{B}}(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0}{2} \frac{a^2}{\left[a^2 + z^2\right]^{3/2}} \underline{\mathbf{e}}_z$$

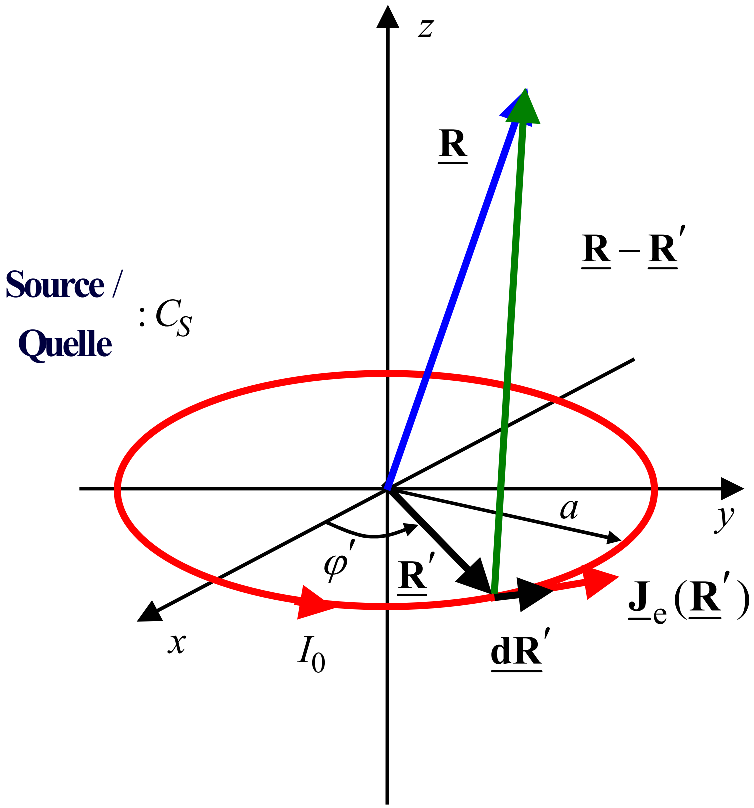
MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)







MS Fields – Biot-Savart's Law – Wire Loop Carrying a Constant Electric Current / MS-Felder – Biot-Savartsches Gesetz – Drahtschleife, die einen konstanten elektrischen Strom führt (...)



Magnetic Flux Density – Arbitrary Observation Point / Magnetische Flussdichte – Beliebiger Beobachtungspunkt

$$\underline{\mathbf{B}}(\underline{\mathbf{R}}) = B_r(\underline{\mathbf{R}})\underline{\mathbf{e}}_r(\varphi) + B_z(\underline{\mathbf{R}})\underline{\mathbf{e}}_z$$

$$B_r(\underline{\mathbf{R}}) = \frac{\mu_0 I_0 k}{4\pi\sqrt{ar}}\left[-K(k) + \frac{a^2 + r^2 + z^2}{(a-r)^2 + z^2}E(k)\right]$$

$$B_z(\underline{\mathbf{R}}) = \frac{\mu_0 I_0 k}{4\pi\sqrt{ar}}\left[K(k) + \frac{a^2 - r^2 - z^2}{(a-r)^2 + z^2}E(k)\right]$$

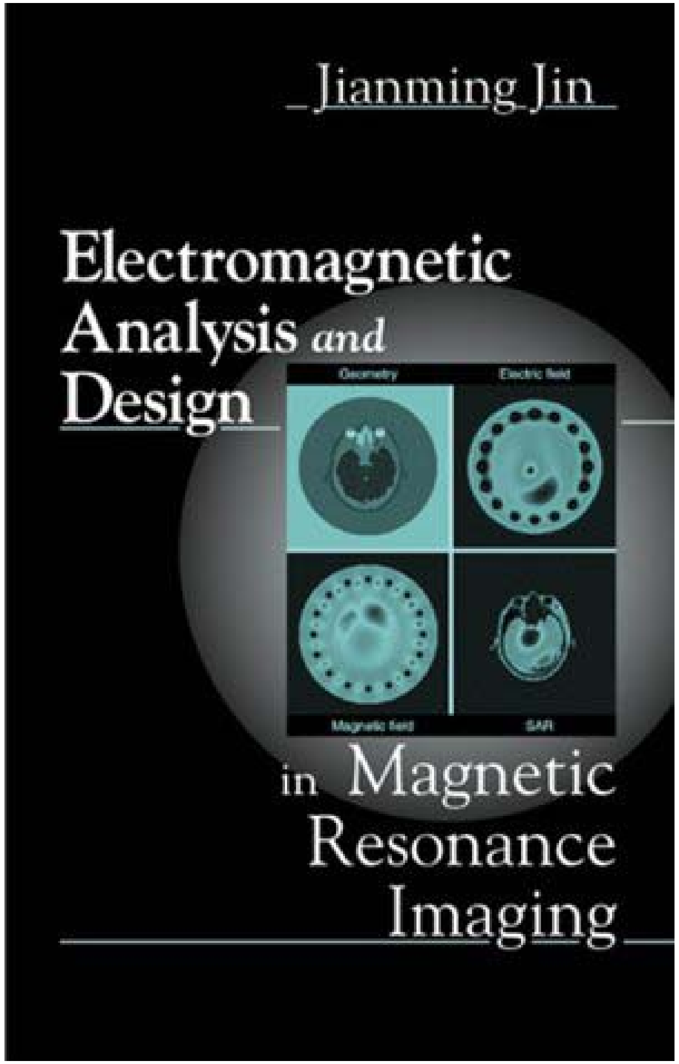
with / mit

$$k = \sqrt{\frac{4ar}{(a+r)^2 + z^2}}$$

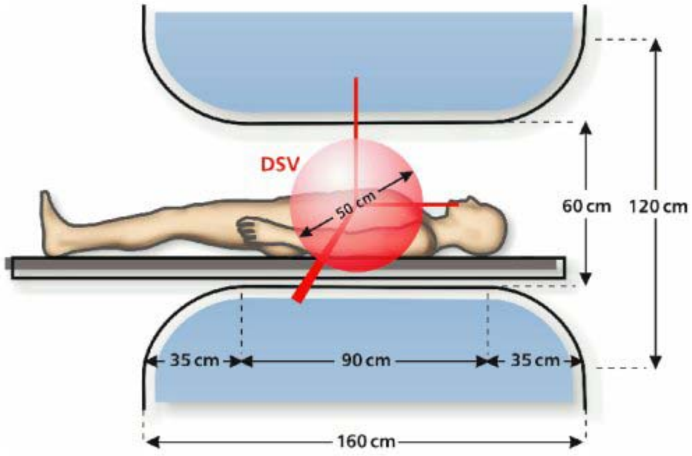
Complete Elliptic Integrals of 1st and 2nd Kind / Kompletten elliptischen Integrale 1. und 2. Art

$$K(k) = \int_{\theta=0}^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\theta}}\,\mathrm{d}\theta \qquad E(k) = \int_{\theta=0}^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\,\mathrm{d}\theta$$

MS Fields – Coils for MRT Systems – MRT: Magnetic Resonance Tomography / MS-Felder – Spulen für MRT-Systeme – MRT: Magnetresonanz Tomographie



MS Fields – Coils for MRT Systems – MRT: Magnetic Resonance Tomography / MS-Felder – Spulen für MRT-Systeme – MRT: Magnetresonanz Tomographie



Magent (Homogeneous Magnetic Field)

- Ultracompact 1.5 Tesla Magnet, Length: 160 cm
- Wide, Patient-friendly Inner Bore Diameter 60 cm
- Magnet Weight Only 4,050 kg
- Large DSV (Diameter Spherical Volume) with Excellent Homogeneity Over 50 cm

Gradient Coil (Gradient Magnetic Field)

- Gradient Field Strength up to 30 mT/m
- Slew Rate up to 125 T/m/s
- Large Field of View up to 50 cm, Optimized for whole Body Examinations
- Ultrafast, Highly Compact, Water-cooled Gradient Amplifier in Solid-State Technology for best min.TR 1.8 ms and min.TE 0.8 ms (Matrix 256²)

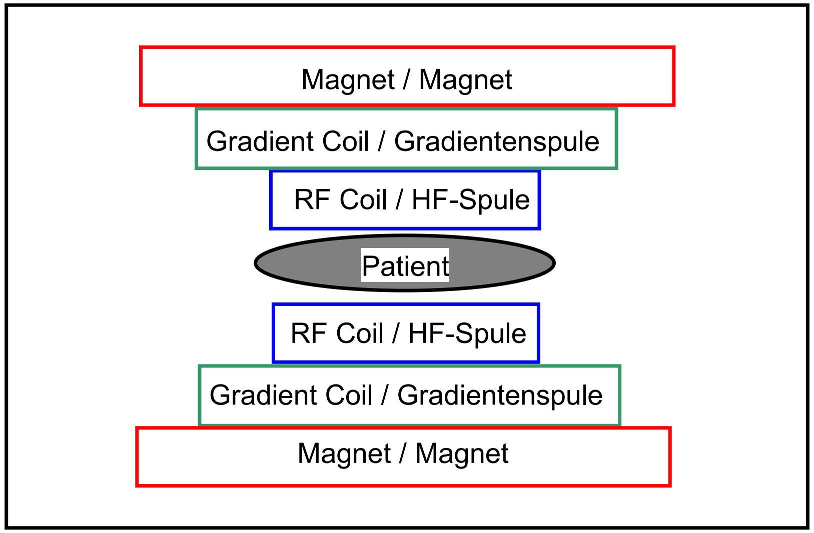
Magnetic Flux Density B_0 of an MRI System /
Magnetische Flussdichte B_0 eines MR-Systems:
1.5 T = 15,000 Gauss = Vs/m² (T = Tesla = 10^4 Gauss)

Magnetic Flux Density of the Earth Magnetic Field /
Magnetische Flussdichte des Erdmagnetfeldes:
0.25 ... 0.5 Gauss = 2.5 mT .... 5 mT

15,000 Gauss / 0.5 Gauss = 30,000

MS Fields – Coils for MRT Systems – MRT: Magnetic Resonance Tomography / MS-Felder – Spulen für MRT-Systeme – MRT: Magnetresonanz Tomographie

Block Diagram of an MRI System /
Blockdiagramm eines MR-Systems

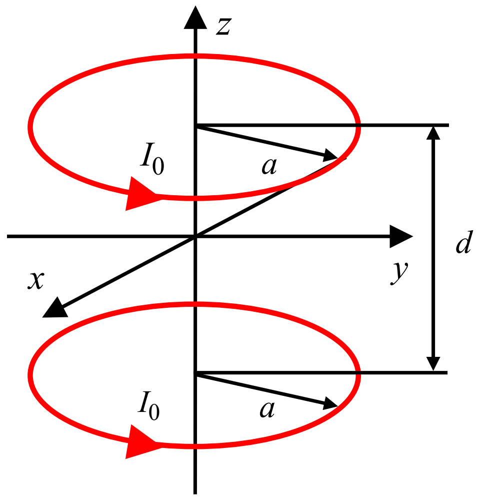


RF Coil / HF-Spule ⇒ Helmholtz Coil / Helmholtz-Spule

Gradient Coil / Gradientenspule ⇒ Maxwell Coil / Maxwell-Spule

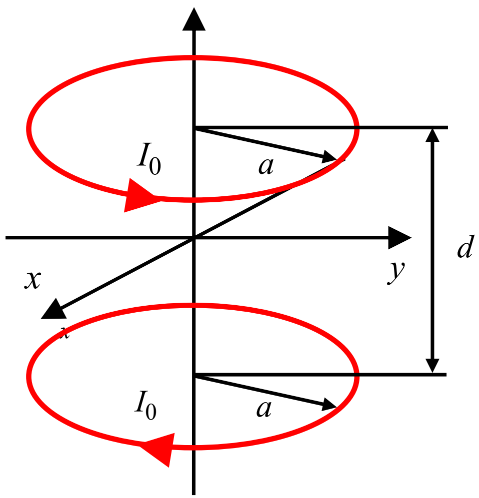
MS Fields – Biot-Savart's Law – Helmholtz and Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Helmholtz- und Maxwell-Spule

Helmholtz Coil / Helmholtz-Spule



$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[a^2 + (d/2 - z)^2\right]^{3/2}} + \frac{\mu_0 I_0 a^2}{2\left[a^2 + (d/2 + z)^2\right]^{3/2}}$$

Maxwell Coil / Maxwell-Spule



$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[a^2 + (d/2 - z)^2\right]^{3/2}} - \frac{\mu_0 I_0 a^2}{2\left[a^2 + (d/2 + z)^2\right]^{3/2}}$$

MS Fields – Biot-Savart's Law – Optimization of a Helmholtz Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Helmholtz-Spule

Goal: Homogeneous Magnetic Field Between the Loops /
Ziel: Homogenes Magnetfeld zwischen den Schleifen

$$\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = 0 \qquad -\frac{d}{2} \le z \le \frac{d}{2}$$

$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2} \left[\frac{1}{\left[a^2 + (d/2 - z)^2\right]^{3/2}} + \frac{1}{\left[a^2 + (d/2 + z)^2\right]^{3/2}} \right]$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}z} B_z(z)\right|_{r=0} = \frac{3\mu_0 I_0 a^2}{2} \left\{ \frac{d/2 - z}{\left[a^2 + (d/2 - z)^2\right]^{5/2}} - \frac{d/2 + z}{\left[a^2 + (d/2 + z)^2\right]^{5/2}} \right\}$$

$$\left.\left[\frac{\mathrm{d}}{\mathrm{d}z} B_z(z)\right]\right|_{r=0, z=0} = \left.\frac{3\mu_0 I_0 a^2}{2} \left\{ \frac{d/2 - z}{\left[a^2 + (d/2 - z)^2\right]^{5/2}} - \frac{d/2 + z}{\left[a^2 + (d/2 + z)^2\right]^{5/2}} \right\}\right|_{z=0}$$

$$= \frac{3\mu_0 I_0 a^2}{2} \left\{ \frac{d/2}{\left[a^2 + (d/2)^2\right]^{5/2}} - \frac{d/2}{\left[a^2 + (d/2)^2\right]^{5/2}} \right\}$$

$$= 0 \quad \text{for / für} \quad \forall d$$

MS Fields – Biot-Savart's Law – Optimization of a Helmholtz Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Helmholtz-Spule

Goal: Homogeneous Magnetic Field Between the Loops /
Ziel: Homogenes Magnetfeld zwischen den Schleifen

$$\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = 0 \qquad -\frac{d}{2} \le z \le \frac{d}{2}$$

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2} B_z(z)\right|_{r=0} = -\frac{3\mu_0 I_0 a^2}{2} \left\{ \frac{a^2 - 4(d/2 - z)^2}{\left[a^2 + (d/2 - z)^2\right]^{7/2}} + \frac{a^2 - 4(d/2 - z)^2}{\left[a^2 + (d/2 + z)^2\right]^{5/2}} \right\}$$

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2} B_z(z)\right|_{r=0, z=0} = -\frac{3\mu_0 I_0 a^2}{2} \left.\left\{ \frac{a^2 - 4(d/2 - z)^2}{\left[a^2 + (d/2 - z)^2\right]^{7/2}} + \frac{a^2 - 4(d/2 - z)^2}{\left[a^2 + (d/2 + z)^2\right]^{5/2}} \right\}\right|_{z=0}$$

$$= -\frac{3\mu_0 I_0 a^2}{2} \left\{ \frac{a^2 - 4(d/2)^2}{\left[a^2 + (d/2)^2\right]^{7/2}} + \frac{a^2 - 4(d/2)^2}{\left[a^2 + (d/2)^2\right]^{5/2}} \right\}$$

$$= -12\mu_0 I_0 a^2 \frac{\left[a^2 - d^2\right]}{\left[a^2 + (d/2)^2\right]^{7/2}}$$

$$= 0 \quad \text{for / für} \quad d = a$$

MS Fields – Biot-Savart's Law – Optimization of a Helmholtz Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Helmholtz-Spule

Goal: Homogeneous Magnetic Field Between the Loops /
Ziel: Homogenes Magnetfeld zwischen den Schleifen

$$\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = 0 \qquad -\frac{d}{2} \leq z \leq \frac{d}{2}$$

$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2} \left[\frac{1}{\left[a^2 + \left(d/2 - z \right)^2 \right]^{3/2}} + \frac{1}{\left[a^2 + \left(d/2 + z \right)^2 \right]^{3/2}} \right]$$

$$\left[\frac{\mathrm{d}^3}{\mathrm{d}z^3} B_z(z) \right] \Bigg|_{r=0, z=0} = 0 \quad \text{for / für} \quad \forall d$$

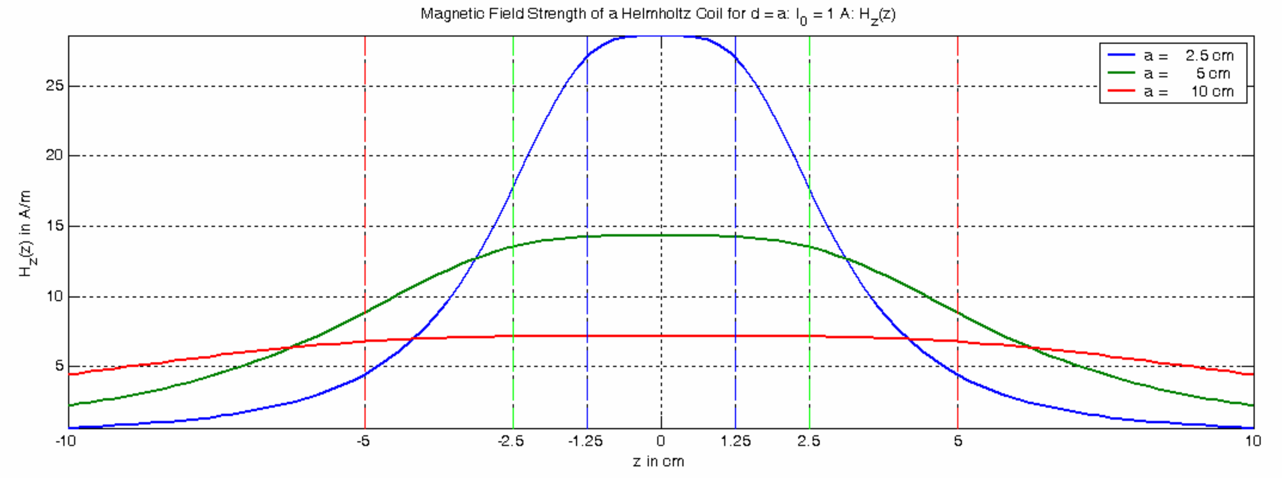
$$d = a$$ Optimized Value / Optimierter Wert

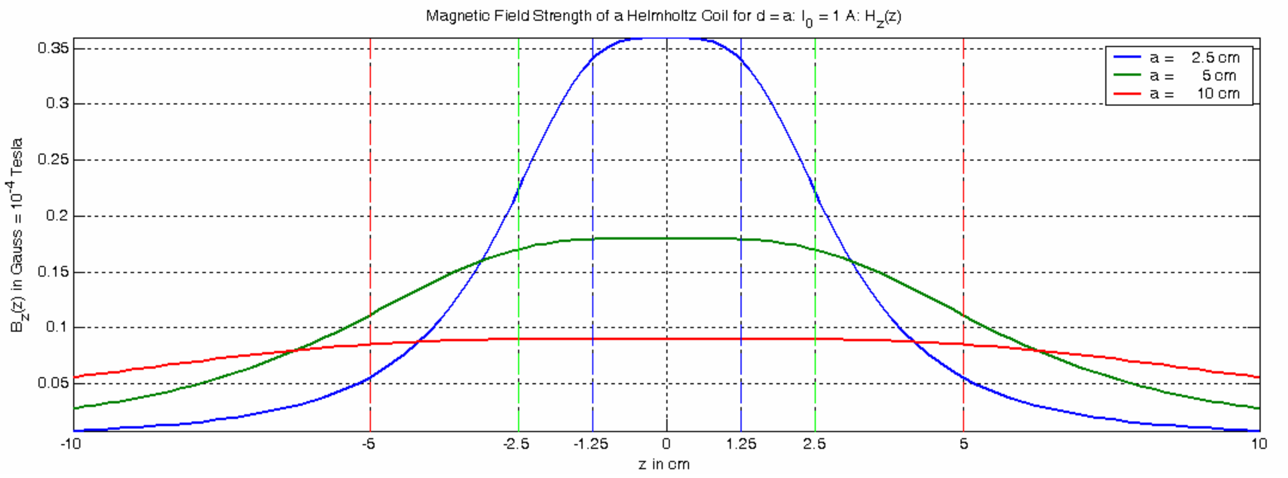
$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2} \left[\frac{1}{\left[a^2 + \left(a/2 - z \right)^2 \right]^{3/2}} + \frac{1}{\left[a^2 + \left(a/2 + z \right)^2 \right]^{3/2}} \right]$$

$$B_z(z) = B_z(0) + \mathcal{O}\left[(z/d)^4 \right]$$

MS Fields – Biot-Savart's Law – Optimized Helmholtz Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierte Helmholtz Spule (...)

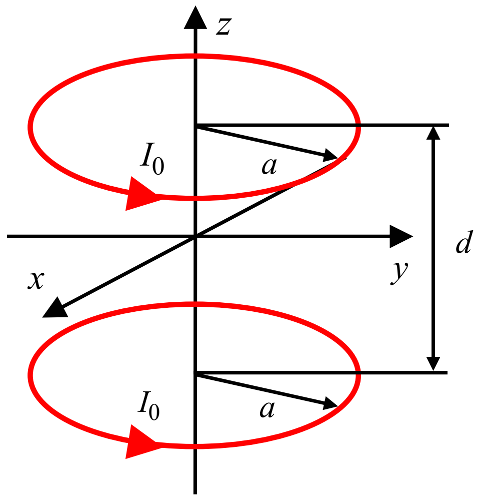
Helmholtz Coil with Optimized Distance $d = a$ /
Helmholtz-Spule mit optimierten Abstand $d = a$





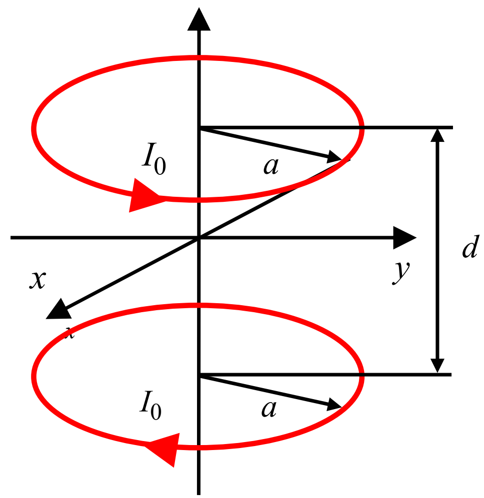
MS Fields – Biot-Savart's Law – Helmholtz and Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Helmholtz- und Maxwell-Spule

Helmholtz Coil / Helmholtz-Spule



$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[a^2+(d/2-z)^2\right]^{3/2}} + \frac{\mu_0 I_0 a^2}{2\left[a^2+(d/2+z)^2\right]^{3/2}}$$

Maxwell Coil / Maxwell-Spule



$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[a^2+(d/2-z)^2\right]^{3/2}} - \frac{\mu_0 I_0 a^2}{2\left[a^2+(d/2+z)^2\right]^{3/2}}$$

MS Fields – Biot-Savart's Law – Optimization of a Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Maxwell-Spule

Goal: Linear Varying Magnetic Field Along the z Axis Between the Loops /
Ziel: Linear variierendes Magnetfeld entlang der z Achse zwischen den beiden Schleifen $\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = C \qquad -\frac{d}{2} \leq z \leq \frac{d}{2}$

$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[(d/2 - z)^2 + a^2\right]^{3/2}} - \frac{\mu_0 I_0 a^2}{2\left[(d/2 + z)^2 + a^2\right]^{3/2}}$$

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}z} B_z(z)\right|_{r=0} &= \frac{3\mu_0 I_0 a^2}{2}\left\{\frac{d/2 - z}{\left[a^2 + (d/2 - z)^2\right]^{5/2}} + \frac{d/2 - z}{\left[a^2 + (d/2 + z)^2\right]^{5/2}}\right\} \\ &= B_z'(z) \end{aligned}$$

$$\begin{aligned} \left.\left[\frac{\mathrm{d}}{\mathrm{d}z} B_z(z)\right]\right|_{r=0,z=0} &= \left.\frac{3\mu_0 I_0 a^2}{2}\left\{\frac{d/2 - z}{\left[a^2 + (d/2 - z)^2\right]^{5/2}} + \frac{d/2 - z}{\left[a^2 + (d/2 + z)^2\right]^{5/2}}\right\}\right|_{z=0} \\ &= 3\mu_0 I_0 a^2 \frac{d/2}{\left[a^2 + (d/2)^2\right]^{5/2}} \\ &= \left.B_z'(0)\right|_{r=0,z=0} \end{aligned}$$

MS Fields – Biot-Savart's Law – Optimization of a Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Maxwell-Spule

Goal: Linear Varying Magnetic Field Along the z Axis Between the Loops /
Ziel: Linear variierendes Magnetfeld entlang der z Achse zwischen den beiden Schleifen

$$\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = C \qquad -\frac{d}{2} \le z \le \frac{d}{2}$$

$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[\left(d/2 - z\right)^2 + a^2\right]^{3/2}} - \frac{\mu_0 I_0 a^2}{2\left[\left(d/2 + z\right)^2 + a^2\right]^{3/2}}$$

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2} B_z(z)\right|_{r=0} = -\frac{3\mu_0 I_0 a^2}{2}\left\{\frac{a^2 + 4\left(d/2 - z\right)^2}{\left[a^2 + \left(d/2 - z\right)^2\right]^{7/2}} + \frac{a^2 + 4\left(d/2 - z\right)^2}{\left[a^2 + \left(d/2 + z\right)^2\right]^{5/2}}\right\}$$

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2} B_z(z)\right|_{r=0,z=0} = -\frac{3\mu_0 I_0 a^2}{2}\left.\left\{\frac{a^2 - 4\left(d/2 - z\right)^2}{\left[a^2 + \left(d/2 - z\right)^2\right]^{7/2}} + \frac{a^2 - 4\left(d/2 - z\right)^2}{\left[a^2 + \left(d/2 + z\right)^2\right]^{5/2}}\right\}\right|_{z=0}$$

$$= -\frac{3\mu_0 I_0 a^2}{2}\left\{\frac{a^2 - 4\left(d/2\right)^2}{\left[a^2 + \left(d/2\right)^2\right]^{7/2}} - \frac{a^2 - 4\left(d/2\right)^2}{\left[a^2 + \left(d/2\right)^2\right]^{7/2}}\right\}$$

$$= 0 \quad \text{for / für} \quad \forall d$$

MS Fields – Biot-Savart's Law – Optimization of a Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Maxwell-Spule

Goal: Linear Varying Magnetic Field Along the z Axis Between the Loops /
Ziel: Linear variierendes Magnetfeld entlang der z Achse zwischen den beiden Schleifen

$$\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = C \qquad -\frac{d}{2} \leq z \leq \frac{d}{2}$$

$$B_z(\underline{\mathbf{R}} = z\underline{\mathbf{e}}_z) = \frac{\mu_0 I_0 a^2}{2\left[\left(d/2 - z\right)^2 + a^2\right]^{3/2}} - \frac{\mu_0 I_0 a^2}{2\left[\left(d/2 + z\right)^2 + a^2\right]^{3/2}}$$

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}z^3} B_z(z)\right|_{r=0} = \frac{15\mu_0 I_0 a^2}{2}\left\{\frac{4\left(d/2 - z\right)^3 - 3\left(d/2 - z\right)a^2}{\left[a^2 + \left(d/2 - z\right)^2\right]^{9/2}} + \frac{4\left(d/2 + z\right)^3 - 3\left(d/2 + z\right)a^2}{\left[a^2 + \left(d/2 + z\right)^2\right]^{9/2}}\right\}$$

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}z^3} B_z(z)\right|_{r=0,z=0} = \left.\frac{15\mu_0 I_0 a^2}{2}\left\{\frac{4\left(d/2 - z\right)^3 - 3\left(d/2 - z\right)a^2}{\left[a^2 + \left(d/2 - z\right)^2\right]^{9/2}} + \frac{4\left(d/2 + z\right)^3 - 3\left(d/2 + z\right)a^2}{\left[a^2 + \left(d/2 + z\right)^2\right]^{9/2}}\right\}\right|_{z=0}$$

$$= \frac{15\mu_0 I_0 a^2}{2}\left\{\frac{4\left(d/2\right)^3 - 3\left(d/2\right)a^2}{\left[a^2 + \left(d/2\right)^2\right]^{9/2}} + \frac{4\left(d/2\right)^3 - 3\left(d/2\right)a^2}{\left[a^2 + \left(d/2\right)^2\right]^{9/2}}\right\}$$

$$= 15\mu_0 I_0 a^2\left\{\frac{4\left(d/2\right)^3 - 3\left(d/2\right)a^2}{\left[a^2 + \left(d/2\right)^2\right]^{9/2}}\right\}$$

MS Fields – Biot-Savart's Law – Optimization of a Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierung einer Maxwell-Spule

Goal: Linear Varying Magnetic Field Along the z Axis Between the Loops /
Ziel: Linear variierendes Magnetfeld entlang der z Achse zwischen den beiden Schleifen

$$\frac{\mathrm{d}}{\mathrm{d}z} B_z(z) = C \qquad -\frac{d}{2} \le z \le \frac{d}{2}$$

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}z^3} B_z(z)\right|_{r=0,z=0} = 15\mu_0 I_0 a^2 \left\{ \frac{4\left(d/2\right)^3 - 3\left(d/2\right)a^2}{\left[a^2 + \left(d/2\right)^2\right]^{9/2}} \right\}$$

$$4\left(d/2\right)^3 = 3\left(d/2\right)a^2$$

$$d^2 = 3a^2$$

$$d = \sqrt{3}a$$

$$\left.\left[\frac{\mathrm{d}^3}{\mathrm{d}z^3} B_z(z)\right]\right|_{r=0,z=0} = 0 \quad \text{for / für} \quad d = \sqrt{3}a$$

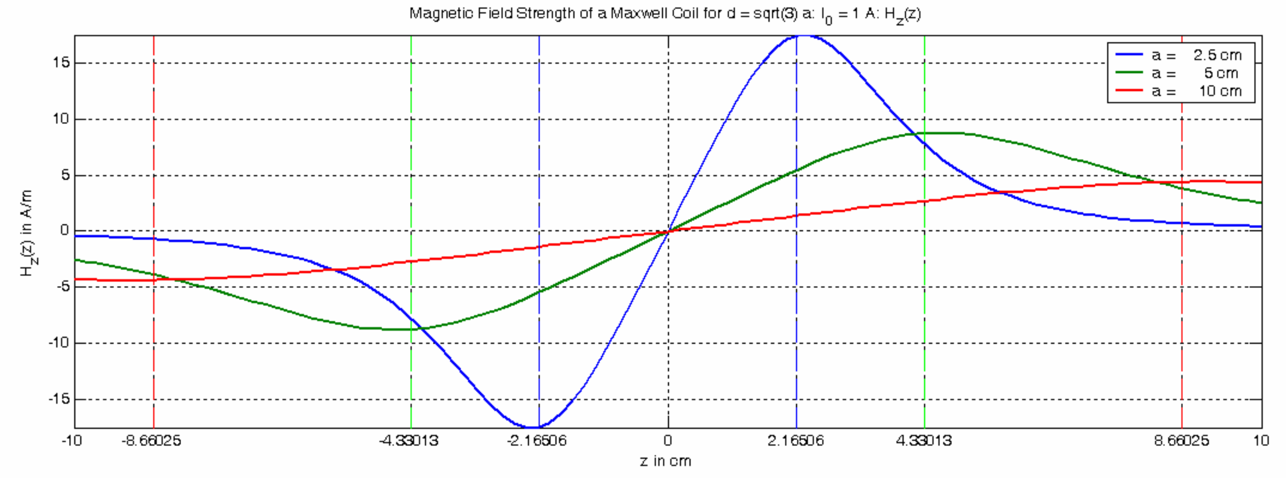
$$d = \sqrt{3}a$$

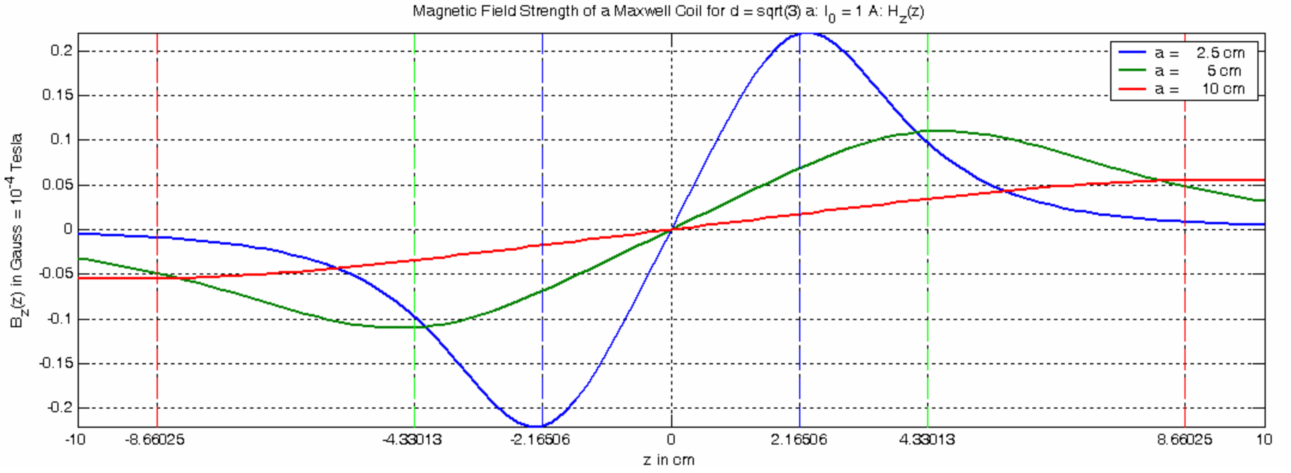
$$\left.B_z(z)\right|_{r=0} = \frac{\mu_0 I_0 a^2}{2} \left(\frac{1}{\left[\left(\sqrt{3}a/2 - z\right)^2 + a^2\right]^{3/2}} - \frac{1}{\left[\left(\sqrt{3}a/2 + z\right)^2 + a^2\right]^{3/2}} \right)$$

$$B_z(z) = \left.B_z'(0)\right|_{r=0,z=0} z + \mathcal{O}\left[\left(\frac{z}{d}\right)^5\right]$$

MS Fields – Biot-Savart's Law – Optimized Maxwell Coil / MS-Felder – Biot-Savartsches Gesetz – Optimierte Maxwell-Spule (...)

Maxwell Coil with Optimized Distance $d = \sqrt{3}\ a$ /
Maxwell-Spule mit optimierten Abstand $d = \sqrt{3}\ a$



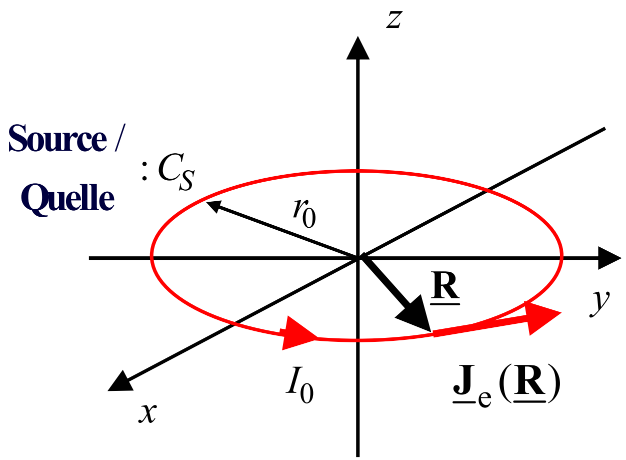


MS Fields – Magnetic Dipole Moment / MS-Felder – Magnetisches Dipolmoment

Magnetic Dipole Moment /
Magnetisches Dipolmoment

$$\underline{\mathbf{p}}_\mathrm{m} = \frac{\mu_0}{2} \iiint_{V_s} \underline{\mathbf{R}} \times \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}})\, \mathrm{d}^3 \underline{\mathbf{R}}$$

Example: Magnetic Dipole Moment of a Current Loop /
Beispiel: Magnetisches Dipolmoment einer elektrischen Stromschleife



$$\underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}) = I_0\, \delta(r - r_0)\, \delta(z)\, \underline{\mathbf{e}}_\varphi(\varphi), \qquad r_0 > 0$$

$$\underline{\mathbf{R}} = r\, \underline{\mathbf{e}}_r(\varphi) + z\, \underline{\mathbf{e}}_z$$

$$\underline{\mathbf{p}}_\mathrm{m} = \frac{\mu_0}{2} \iiint_{V_s} \underline{\mathbf{R}} \times \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}})\, \mathrm{d}^3 \underline{\mathbf{R}}$$

$$\underline{\mathbf{R}} \times \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}}) = \left[r\, \underline{\mathbf{e}}_r(\varphi) + z\, \underline{\mathbf{e}}_z \right] \times \left[I_0\, \delta(r - r_0)\, \delta(z)\, \underline{\mathbf{e}}_\varphi(\varphi) \right]$$

End of the 11th Lecture / Ende der 11. Vorlesung