Elektromagnetische Feldtheorie I (EFT I) / Electromagnetic Field Theory I (EFT I)

8th Lecture / 8. Vorlesung

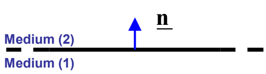
Dr.-Ing. René Marklein
marklein@uni-kassel.de
http://www.tet.e-technik.uni-kassel.de
http://www.uni-kassel.de/fb16/tet/marklein/index.html

Universität Kassel
Fachbereich Elektrotechnik / Informatik
(FB 16)
Fachgebiet Theoretische Elektrotechnik
(FG TET)
Wilhelmshöher Allee 71
Büro: Raum 2113 / 2115
D-34121 Kassel

University of Kassel
Dept. Electrical Engineering / Computer Science
(FB 16)
Electromagnetic Field Theory
(FG TET)
Wilhelmshöher Allee 71
Office: Room 2113 / 2115
D-34121 Kassel

ES Fields – Transition and Boundary Conditions / ES Felder – Übergangs- und Randbedingungen

Transition Conditions / Übergangsbedingungen



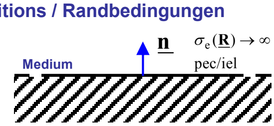
$\underline{\mathbf{R}} \in S_{\mathrm{I(nterface)}}$

$$\underline{\mathbf{n}} \times \left[\underline{\mathbf{E}}^{(2)}(\underline{\mathbf{R}}) - \underline{\mathbf{E}}^{(1)}(\underline{\mathbf{R}}) \right] = \underline{\mathbf{0}}$$

$$\underline{\mathbf{n}} \cdot \left[\underline{\mathbf{D}}^{(2)}(\underline{\mathbf{R}}) - \underline{\mathbf{D}}^{(1)}(\underline{\mathbf{R}}) \right] = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

ws: with sources; sf = source-free /
mq = mit Quellen; qf = quellenfrei

Boundary Conditions / Randbedingungen



$\underline{\mathbf{R}} \in S_{\mathrm{B(oundary)}}$

$$\underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}} \qquad \text{pec / iel}$$

$$\underline{\mathbf{n}} \cdot \underline{\mathbf{D}}(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

pec = perfectly electric conducting /
iel = ideal elektrisch leitend

$$\underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{E}}_{tan}(\underline{\mathbf{R}}) = E_{tan}(\underline{\mathbf{R}})\underline{\mathbf{e}}_{tan}$$

$\underline{\mathbf{E}}_{tan}(\underline{\mathbf{R}})$: Vector Tangential Component of / Vektorielle Tangentialkomponente von $\underline{\mathbf{E}}(\underline{\mathbf{R}})$

$E_{tan}(\underline{\mathbf{R}})$: Scalar Tangential Component of / Skalare Tangentialkomponente von $\underline{\mathbf{E}}(\underline{\mathbf{R}})$

$$E_{tan}^{(2)}(\underline{\mathbf{R}}) - E_{tan}^{(1)}(\underline{\mathbf{R}}) = 0$$

$$D_n^{(2)}(\underline{\mathbf{R}}) - D_n^{(1)}(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{n}} \cdot \underline{\mathbf{D}}(\underline{\mathbf{R}}) = D_n(\underline{\mathbf{R}})$$

$D_n(\underline{\mathbf{R}})$: Scalar Normal Component of / Skalare Normalkomponente von $\underline{\mathbf{D}}(\underline{\mathbf{R}})$

$$E_{tan}(\underline{\mathbf{R}}) = 0 \qquad \text{pec / iel}$$

$$D_n(\underline{\mathbf{R}}) = \eta_{\mathrm{e}}(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

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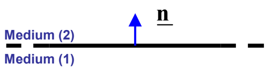
Transition Conditions / Übergangsbedingungen



$\underline{\mathbf{R}} \in S_{\text{I(nterface)}}$

$$E_{tan}^{(2)}(\underline{\mathbf{R}}) - E_{tan}^{(1)}(\underline{\mathbf{R}}) = 0$$
$$D_n^{(2)}(\underline{\mathbf{R}}) - D_n^{(1)}(\underline{\mathbf{R}}) = \eta_e(\underline{\mathbf{R}})$$

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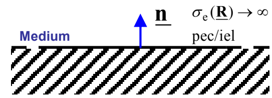
Boundary Conditions / Randbedingungen

$\underline{\mathbf{R}} \in S_{\text{B(oundary)}}$



$$E_{tan}(\underline{\mathbf{R}}) = 0 \qquad \text{pec / iel}$$
$$D_n(\underline{\mathbf{R}}) = \eta_e(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

pec = perfectly electric conducting /
iel = ideal elektrisch leitend

$$\underline{\mathbf{E}}^{(i)}(\underline{\mathbf{R}}) = -\nabla\Phi^{(i)}(\underline{\mathbf{R}}) \qquad \underline{\mathbf{D}}^{(i)}(\underline{\mathbf{R}}) = \varepsilon_0\varepsilon_r^{(i)}\underline{\mathbf{E}}^{(i)}(\underline{\mathbf{R}}) \qquad i = 1,2$$
$$= -\varepsilon_0\varepsilon_r^{(i)}\nabla\Phi_e^{(i)}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{n}} \times \nabla\left[\Phi_e^{(2)}(\underline{\mathbf{R}}) - \Phi_e^{(1)}(\underline{\mathbf{R}})\right] = \underline{\mathbf{0}}$$
$$\Phi_e^{(2)}(\underline{\mathbf{R}}) - \Phi_e^{(1)}(\underline{\mathbf{R}}) = \Phi_{e0} = \text{const.}$$

$$\underbrace{\underline{\mathbf{n}} \cdot \nabla}_{=\frac{\partial}{\partial n}}\left[\varepsilon_r^{(2)}\Phi_e^{(2)}(\underline{\mathbf{R}}) - \varepsilon_r^{(1)}\Phi_e^{(1)}(\underline{\mathbf{R}})\right] = -\frac{\eta_e(\underline{\mathbf{R}})}{\varepsilon_0}$$

$$\frac{\partial}{\partial n}\Phi_e^{(2)}(\underline{\mathbf{R}}) - \frac{\varepsilon_r^{(1)}}{\varepsilon_r^{(2)}}\frac{\partial}{\partial n}\Phi_e^{(1)}(\underline{\mathbf{R}}) = -\frac{\eta_e(\underline{\mathbf{R}})}{\varepsilon_0\varepsilon_r^{(2)}}$$

$$\underline{\mathbf{n}} \times \nabla\Phi_e(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$$
$$\Phi_e(\underline{\mathbf{R}}) = \Phi_{e0} = \text{const.}$$

$$\varepsilon_0\varepsilon_r \underbrace{\underline{\mathbf{n}} \cdot \nabla}_{=\frac{\partial}{\partial n}}\Phi_e(\underline{\mathbf{R}}) = -\eta_e(\underline{\mathbf{R}})$$

$$\frac{\partial}{\partial n}\Phi_e(\underline{\mathbf{R}}) = -\frac{\eta_e(\underline{\mathbf{R}})}{\varepsilon_0\varepsilon_r}$$

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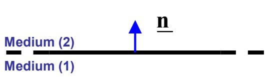
Transition Conditions / Übergangsbedingungen



$\underline{\mathbf{R}} \in S_{\text{I(nterface)}}$

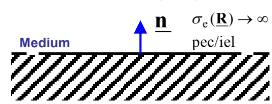
$$E_{tan}^{(2)}(\underline{\mathbf{R}}) - E_{tan}^{(1)}(\underline{\mathbf{R}}) = 0$$
$$D_n^{(2)}(\underline{\mathbf{R}}) - D_n^{(1)}(\underline{\mathbf{R}}) = \eta_e(\underline{\mathbf{R}})$$

$$E_{tan}^{(2)}(\underline{\mathbf{R}}) - E_{tan}^{(1)}(\underline{\mathbf{R}}) = 0$$
$$\Downarrow$$
$$\Phi_e^{(2)}(\underline{\mathbf{R}}) - \Phi_e^{(1)}(\underline{\mathbf{R}}) = \Phi_{e0} = \text{const.}$$

$$D_n^{(2)}(\underline{\mathbf{R}}) - D_n^{(1)}(\underline{\mathbf{R}}) = \eta_e(\underline{\mathbf{R}})$$
$$\Downarrow$$
$$\frac{\partial}{\partial n}\Phi_e^{(2)}(\underline{\mathbf{R}}) - \frac{\varepsilon_r^{(1)}}{\varepsilon_r^{(2)}}\frac{\partial}{\partial n}\Phi_e^{(1)}(\underline{\mathbf{R}}) = -\frac{1}{\varepsilon_0\varepsilon_r^{(2)}}\eta_e(\underline{\mathbf{R}})$$

Boundary Conditions / Randbedingungen

$\underline{\mathbf{R}} \in S_{\text{B(oundary)}}$



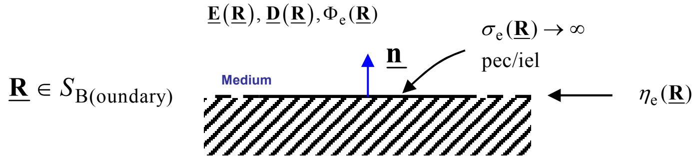
$$E_{tan}(\underline{\mathbf{R}}) = 0 \qquad \text{pec / iel}$$
$$D_n(\underline{\mathbf{R}}) = \eta_e(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

$$E_{tan}(\underline{\mathbf{R}}) = 0$$
$$\Downarrow$$
$$\Phi_e(\underline{\mathbf{R}}) = \Phi_{e0} = \text{const.}$$

$$D_n(\underline{\mathbf{R}}) = \eta_e(\underline{\mathbf{R}})$$
$$\Downarrow$$
$$\frac{\partial}{\partial n}\Phi_e(\underline{\mathbf{R}}) = -\frac{1}{\varepsilon_0\varepsilon_r}\eta_e(\underline{\mathbf{R}})$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Boundary Conditions / Randbedingungen



$$\Phi_e(\underline{\mathbf{R}}) = \Phi_{e0} = \text{const.} \quad \left(\begin{array}{c}\text{Special Case /}\\ \text{Spezialfall}\end{array} \Phi_{e0} = 0\ \text{V}\right)$$

$$\frac{\partial}{\partial n}\Phi_e(\underline{\mathbf{R}}) = -\frac{1}{\varepsilon_0 \varepsilon_r}\eta_e(\underline{\mathbf{R}})$$

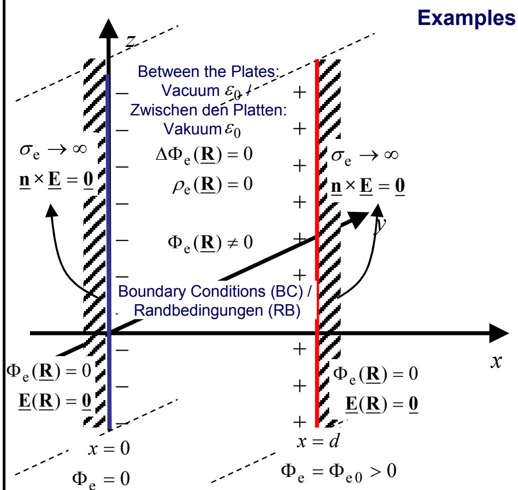
Neumann Boundary Conditions for Φ_e / Neumann-Randbedingung für Φ_e

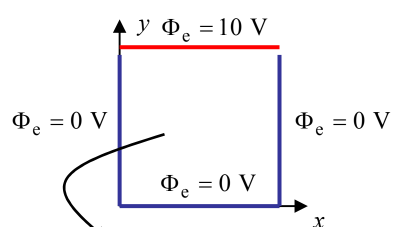
Dirichlet Boundary Conditions for Φ_e / Dirichlet-Randbedingung für Φ_e

Electrostatic (ES) Fields – Boundary Value Problem (BVP) / Elektrostatische (ES) Felder – Randwertproblem (RWP)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi_e(x,y,z) = \begin{cases} -\dfrac{\rho_e(x,y,z)}{\varepsilon_0} & \text{for / für} \quad \rho_e(x,y,z) \neq 0 \quad \text{Poisson Equation / Poisson-Gleichung} \\ 0 & \text{for / für} \quad \rho_e(x,y,z) = 0 \quad \text{Laplace Equation / Laplace-Gleichung} \end{cases}$$

Examples: / Beispiele:



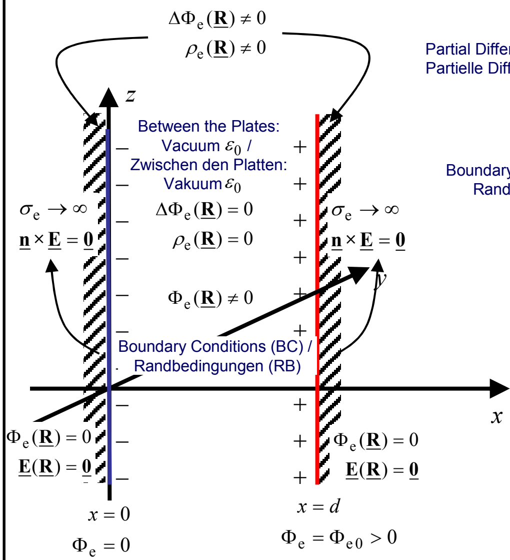


$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y) = 0$$

Separation of Variables / Separation der Variablen !

ES Fields – Electrostatic Field Between Two Parallel PEC Plates / ES Felder – Elektrostatisches Feld zwischen zwei parallelen IEL Platten

Boundary Value Problem (BVP) – Electrostatic Poisson Equation /
Randwertproblem (RWP) – Elektrostatische Poisson-Gleichung



Partial Differential Equation /
Partielle Differentialgleichung

$$\Delta\Phi_e(\underline{\mathbf{R}}) \begin{cases} = 0 & \text{for / für} \quad 0 < x < d \\ \neq \text{const.} & \text{for / für} \quad x = 0,\; x = d \end{cases}$$

Boundary Conditions (BC) /
Randbedingungen (RB)

$$x = 0: \quad \Phi_e = 0$$
$$x = d: \quad \Phi_e = \Phi_{e0} > 0$$

Between the Plates Laplace Equation:
Zwischen den Platten: Laplace-Gleichung

$$\Delta\Phi_e(\underline{\mathbf{R}}) = 0$$

... Cartesian Coordinates /
... Kartesische Koordinaten

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi_e(x, y, z) = 0$$

... Because of the Symmetry /
... wegen der Symmetrie

$$\frac{d^2}{dx^2}\Phi_e(x) = 0$$

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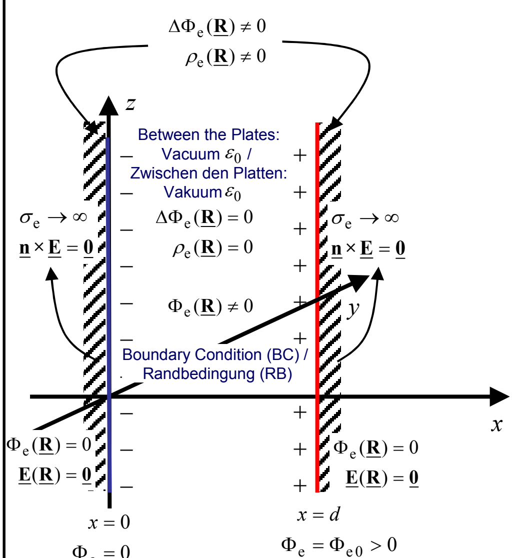

$$\frac{d^2}{dx^2}\Phi_e(x) = 0 \qquad 0 < x < d$$

Integrating once / Integriere einmal

$$\int \frac{d^2}{dx^2}\Phi_e(x)\,dx = \left[\frac{d}{dx}\Phi_e(x)\right] = \text{const} = a$$

$$\left[\frac{d}{dx}\Phi_e(x)\right] = \text{const} = a$$

Integrating twice / Zweifache Integration ergibt

$$\int\left\{\left[\frac{d}{dx}\Phi_e(x)\right] = \text{const} = a\right\}dx = \Phi_e(x) = ax + b$$

$$\Phi_e(x) = ax + b$$

$$\Rightarrow \quad \Phi_e(x) = ax + b \qquad 0 < x < d$$

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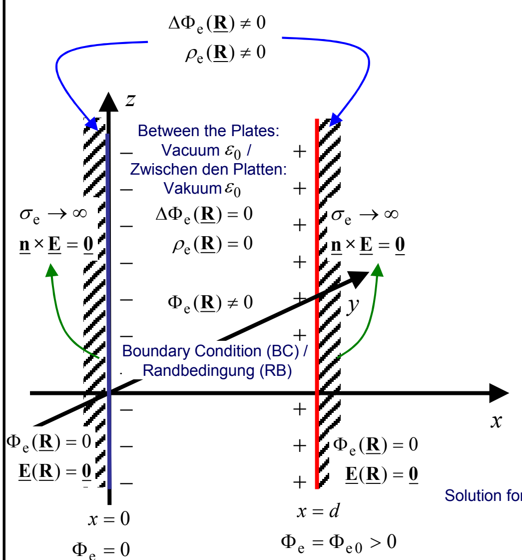

$\Rightarrow \quad \Phi_e(x) = ax + b$

Boundary Conditions (BC) / Randbedingungen (RB)

$x = 0: \quad \Phi_e = 0$

$x = d: \quad \Phi_e = \Phi_{e0} > 0$

$$\Phi_e(x=0) = a(x=0) + b = 0$$

$$b = 0$$

$$\Phi_e(x) = ax$$

$$\Phi_e(x=b) = a(x=b) = \Phi_{e0}$$

$$a = \frac{\Phi_{e0}}{d}$$

Solution for the Electrostatic Potential / Lösung für das elektrostatische Potential

$$\Rightarrow \quad \Phi_e(x) = \frac{\Phi_{e0}}{d}x \qquad 0 \le x \le d$$

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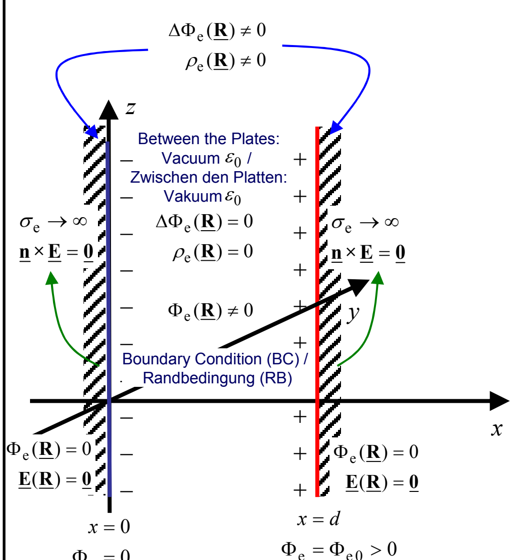
Boundary Value Problem (BVP) – Electrostatic Poisson Equation /
Randwertproblem (RWP) – Elektrostatische Poisson-Gleichung



Partial Differential Equation (PDE) /
Partielle Differentialgleichung (DGL)

$$\frac{d^2}{dx^2}\Phi_e(x) = 0 \qquad 0 < x < d$$

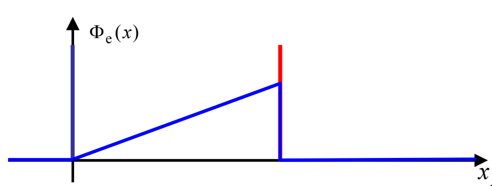
Boundary Conditions (BC) / Randbedingungen (RB)

$x = 0: \quad \Phi_e = 0$

$x = d: \quad \Phi_e = \Phi_{e0} > 0$

Solution for the Electrostatic Potential /
Lösung für das elektrostatische Potential

$$\Phi_e(x) = \begin{cases} \frac{\Phi_{e0}}{d}x & 0 \le x \le d \\ 0 & \text{else / sonst} \end{cases}$$



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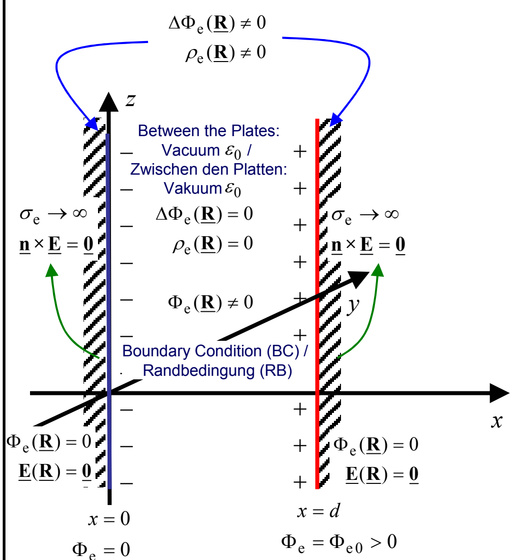

Electrostatic Potential / Elektrostatisches Potential

$$\Phi_e(x) = \begin{cases} \dfrac{\Phi_{e0}}{d} x & 0 \le x \le d \\ 0 & \text{else / sonst} \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_e(\underline{\mathbf{R}})$$

$$\underline{\mathbf{E}}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\Phi_e(x)\underline{\mathbf{e}}_x \qquad -\infty < x < \infty$$

$$= \begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\Phi_{e0}}{d} x\right)\underline{\mathbf{e}}_x & 0 \le x \le d \\ 0 & \text{else / sonst} \end{cases}$$

$$= \begin{cases} -\dfrac{\Phi_{e0}}{d}\underline{\mathbf{e}}_x & 0 \le x \le d \\ 0 & \text{else / sonst} \end{cases}$$

⇨ The Electrostatic Potential and Electrostatic Field Strenth are Discontinuous at the Plates /
Das elektrostatisches Potential und die elektrostatische Feldstärke sind unstetig an den Platten

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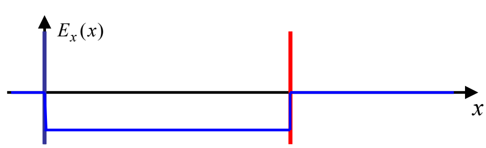
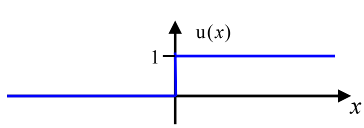
Boundary Value Problem (BVP) – Electrostatic Poisson Equation /
Randwertproblem (RWP) – Elektrostatische Poisson-Gleichung

$$\underline{\mathbf{E}}(x) = \begin{cases} -\dfrac{\Phi_{e0}}{d}\underline{\mathbf{e}}_x & 0 \le x \le d \\ 0 & \text{else / sonst} \end{cases}$$

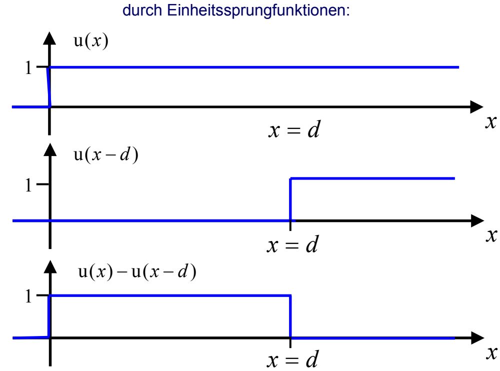


Step Functions / Einheitssprungfunktionen

$$\mathrm{u}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$



Representation of the Electrostatic Field Strenth using the Unit Step Functions: /
Darstellung der elektrostatischen Feldstärke durch Einheitssprungfunktionen:



$$\underline{\mathbf{E}}(x) = -\frac{\Phi_{e0}}{d}\left[\mathrm{u}(x) - \mathrm{u}(x-d)\right]\underline{\mathbf{e}}_x \qquad -\infty < x < \infty$$

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Boundary Value Problem (BVP) – Electrostatic Poisson Equation /
Randwertproblem (RWP) – Elektrostatische Poisson-Gleichung

$$\nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) = \rho_e(\underline{\mathbf{R}})$$

$$\frac{d}{dx} D_x(x) = \rho_e(x)$$

$$\varepsilon_0 \frac{d}{dx} E_x(x) = \rho_e(x)$$

$$\frac{d}{dx} E_x(x) = \frac{\rho_e(x)}{\varepsilon_0}$$

$$\frac{d}{dx} E_x(x) = -\frac{\Phi_{e0}}{d}\frac{d}{dx}\left[u(x) - u(x-d)\right]$$

$$= -\frac{\Phi_{e0}}{d}\left[\underbrace{\frac{d}{dx}u(x)}_{\delta(x)} - \underbrace{\frac{d}{dx}u(x-d)}_{\delta(x-d)}\right]$$

$$= -\frac{\Phi_{e0}}{d}\left[\delta(x) - \delta(x-d)\right]$$

$$= \frac{\rho_e(x)}{\varepsilon_0}$$

$$\underbrace{\frac{d}{dx}u(x)}_{=u'(x)} = \delta(x)$$

$$\int_{-\infty}^{\infty} u'(x) f(x)\,dx = u(x) f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x) f'(x)\,dx$$

$$= \left[\underbrace{\underbrace{u(\infty)}_{=1} f(\infty)}_{=f(\infty)} - \underbrace{\underbrace{u(-\infty)}_{0} f(-\infty)}_{=0}\right] - \int_0^{\infty} f'(x)\,dx$$

$$= f(\infty) - f(x)\Big|_0^{\infty}$$

$$= f(\infty) - \left[f(\infty) - f(0)\right]$$

$$= f(0)$$

$$\int_{-\infty}^{\infty} \underbrace{u'(x)}_{=\delta(x)} f(x)\,dx = f(0)$$

$$u'(x) = \delta(x)$$

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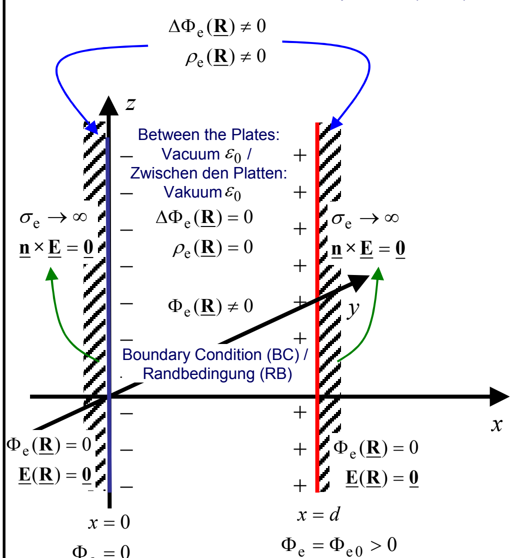
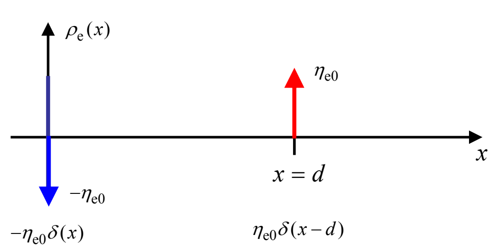


$$\rho_e(x) = \underbrace{\varepsilon_0 \frac{\Phi_{e0}}{d}}_{=\eta_{e0}}\left[-\delta(x) + \delta(x-d)\right]$$

Electric Surface Charge Density /
Elektrische Flächenladungsdichte

$$= -\eta_{e0}\delta(x) + \eta_{e0}\delta(x-d)$$



Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ... (2)

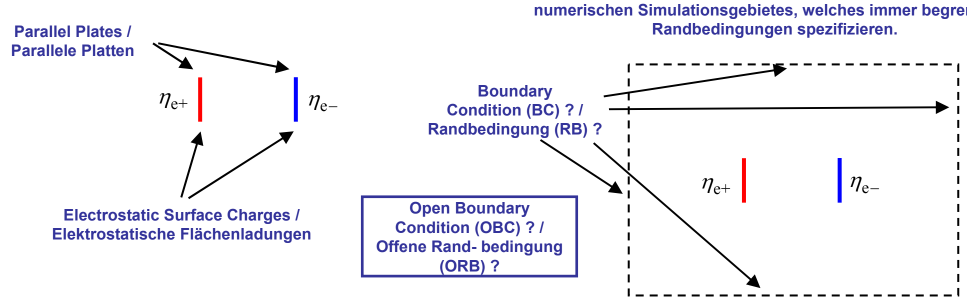
Application: Numerical Solution of Unbounded Static Problems / Anwendung: Numerische Lösung von unbegrenzten statischen Problemen

$$\Delta\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon_0}$$

Problem: Parallel Plate Capacitor in an Unbounded Region / Problem: Paralleler Plattenkondensator in einem unbegrenzten Gebiet

Outline of the Problem / Entwurf des Problems

Numerical Solution: We need to Specify Boundary Conditions at the Boundaries of the Simulation Area which is always bounded. / Numerische Lösung: Wir müssen für die Ränder des numerischen Simulationsgebietes, welches immer begrenzt ist, Randbedingungen spezifizieren.

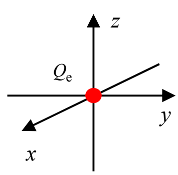


Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono-, Di- und Quadrupol

Monopole / Monopol

One Point Charge / Eine Punktladung



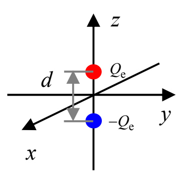
$$\rho_{\mathrm{e}}(\underline{\mathbf{R}}) = Q_{\mathrm{e}}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$$

with / mit

$$\underline{\mathbf{R}}_+ = \underline{\mathbf{0}}$$

Dipole / Dipol

Two Point Charges / Zwei Punktladungen



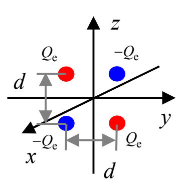
$$\rho_{\mathrm{e}}(\underline{\mathbf{R}}) = Q_{\mathrm{e}}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_{\mathrm{e}}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with / mit

$$\underline{\mathbf{R}}_+ = \frac{d}{2}\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\frac{d}{2}\underline{\mathbf{e}}_z$$

Quadrupole / Quadrupol

Four Point Charges / Vier Punktladungen



$$\rho_{\mathrm{e}}(\underline{\mathbf{R}}) = \sum_{i=1}^{4} Q_{\mathrm{e}}^{(i)}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}^{(i)})$$

with / mit

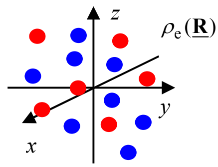
$$Q_{\mathrm{e}}^{(1)} = -Q_{\mathrm{e}},\ \underline{\mathbf{R}}^{(1)} = \frac{d}{2}\underline{\mathbf{e}}_y + \frac{d}{2}\underline{\mathbf{e}}_z \qquad Q_{\mathrm{e}}^{(2)} = Q_{\mathrm{e}},\ \underline{\mathbf{R}}^{(2)} = -\frac{d}{2}\underline{\mathbf{e}}_y + \frac{d}{2}\underline{\mathbf{e}}_z$$

$$Q_{\mathrm{e}}^{(3)} = -Q_{\mathrm{e}},\ \underline{\mathbf{R}}^{(3)} = -\frac{d}{2}\underline{\mathbf{e}}_y - \frac{d}{2}\underline{\mathbf{e}}_z \qquad Q_{\mathrm{e}}^{(4)} = Q_{\mathrm{e}},\ \underline{\mathbf{R}}^{(4)} = \frac{d}{2}\underline{\mathbf{e}}_y - \frac{d}{2}\underline{\mathbf{e}}_z$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

Arbitrary Point Charge /
Beliebige Punktsladungsverteilungen



$$\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_{\mathrm{e}}(\underline{\mathbf{R}}') d^3\underline{\mathbf{R}}'$$

Expansion of $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in a Taylor Series for $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ yields:
Entwicklung von $\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|}$ in eine Taylor-Reihe für $\underline{\mathbf{R}}' = \underline{\mathbf{0}}$ ergibt:

$$\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} = \frac{1}{R} + \frac{1}{R^3}\underline{\mathbf{R}} \bullet \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^5}\underline{\mathbf{R}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right] \bullet \underline{\mathbf{R}} + \mathcal{HOT}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

$$\begin{aligned}
\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) &= \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\, \mathrm{d}^3\underline{\mathbf{R}}' \\
&= \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{\frac{1}{R} + \frac{1}{R^3}\underline{\mathbf{R}} \bullet \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^5}\underline{\mathbf{R}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right] \bullet \underline{\mathbf{R}} + \mathcal{HOT}\right\} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\, \mathrm{d}^3\underline{\mathbf{R}}' \\
&= \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{\frac{1}{R} + \frac{1}{R^2}\underline{\hat{\mathbf{R}}} \bullet \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^3}\underline{\hat{\mathbf{R}}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}'\underline{\underline{\mathbf{I}}}\right] \bullet \underline{\hat{\mathbf{R}}} + \mathcal{HOT}\right\} \rho_{\mathrm{e}}(\underline{\mathbf{R}}')\, \mathrm{d}^3\underline{\mathbf{R}}'
\end{aligned}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \widehat{\underline{\mathbf{R}}} \bullet \underline{\mathbf{R}}' + \frac{1}{2}\frac{1}{R^3} \widehat{\underline{\mathbf{R}}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\underline{\mathbf{I}}} \right] \bullet \widehat{\underline{\mathbf{R}}} + \mathcal{HOT} \right\} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3\underline{\mathbf{R}}'$$

$$\begin{aligned}
\widehat{\underline{\mathbf{R}}} \bullet \left[3\underline{\mathbf{R}}'\underline{\mathbf{R}}' - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underline{\underline{\mathbf{I}}} \right] \bullet \widehat{\underline{\mathbf{R}}} &= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \widehat{\underline{\mathbf{R}}} \bullet \underbrace{\underline{\underline{\mathbf{I}}} \bullet \widehat{\underline{\mathbf{R}}}}_{=\widehat{\underline{\mathbf{R}}}} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}' \underbrace{\widehat{\underline{\mathbf{R}}} \bullet \widehat{\underline{\mathbf{R}}}}_{=1} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underbrace{\underline{\mathbf{R}}' \bullet \underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\underline{\mathbf{I}}}} \\
&= 3\underline{\mathbf{R}}'\underline{\mathbf{R}}' : \widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \underline{\underline{\mathbf{I}}} \\
&= \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left(3\widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right)
\end{aligned}$$

$$\Phi_e(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \underline{\mathbf{R}}' \bullet \widehat{\underline{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3} \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left(3\widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right) + \mathcal{HOT} \right\} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3\underline{\mathbf{R}}'$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

$$\begin{aligned}
\Phi_e(\underline{\mathbf{R}}) &= \frac{1}{4\pi\varepsilon_0} \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}'|} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3\underline{\mathbf{R}}' \\
&= \frac{1}{4\pi\varepsilon_0} \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left\{ \frac{1}{R} + \frac{1}{R^2} \underline{\mathbf{R}}' \bullet \widehat{\underline{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3} \underline{\mathbf{R}}'\underline{\mathbf{R}}' : \left(3\widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right) + \mathcal{HOT} \right\} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3\underline{\mathbf{R}}' \\
&= \frac{1}{4\pi\varepsilon_0} \left\{ \frac{1}{R} \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \, \mathrm{d}^3\underline{\mathbf{R}}'}_{=Q_e} \right. \\
&\quad + \frac{1}{R^2} \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \underline{\mathbf{R}}' \, \mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{p}}_e} \bullet \widehat{\underline{\mathbf{R}}} \\
&\quad \left. + \frac{1}{2}\frac{1}{R^3} \left[\underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}') \underline{\mathbf{R}}'\underline{\mathbf{R}}' \, \mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\underline{\mathbf{q}}}_e} : \left(3\widehat{\underline{\mathbf{R}}}\widehat{\underline{\mathbf{R}}} - \underline{\underline{\mathbf{I}}} \right) \right] \right\} + \mathcal{HOT}
\end{aligned}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{R}\underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=Q_\mathrm{e}}\right.$$

$$+\frac{1}{R^2}\underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{p}}_\mathrm{e}}\bullet\hat{\underline{\mathbf{R}}}$$

$$\left.+\frac{1}{2}\frac{1}{R^3}\left[\underbrace{\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'}_{=\underline{\underline{\mathbf{q}}}_\mathrm{e}} : \left(3\hat{\underline{\mathbf{R}}}\,\hat{\underline{\mathbf{R}}}-\underline{\underline{\mathbf{I}}}\right)\right]\right\} + \mathcal{HOT}$$

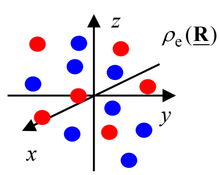
$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{R}Q_\mathrm{e} + \frac{1}{R^2}\underline{\mathbf{p}}_\mathrm{e}\bullet\hat{\underline{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3}\underline{\underline{\mathbf{q}}}_\mathrm{e} : \left[3\hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}}-\underline{\underline{\mathbf{I}}}\right] + \mathcal{HOT}\right\}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ...

Arbitrary Point Charge / Beliebige Punktsladungsverteilungen



$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{\underline{\mathbf{R}}'=-\infty}^{\infty}\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}\rho_\mathrm{e}(\underline{\mathbf{R}}')\mathrm{d}^3\underline{\mathbf{R}}'$$

Expansion of / Entwicklung von $\frac{1}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}'|}$ in a Taylor Series for / in eine Taylor-Reihe für $\underline{\mathbf{R}}'=\underline{\mathbf{0}}$ yields / ergibt :

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{R}Q_\mathrm{e} + \frac{1}{R^2}\underline{\mathbf{p}}_\mathrm{e}\bullet\hat{\underline{\mathbf{R}}} + \frac{1}{2}\frac{1}{R^3}\underline{\underline{\mathbf{q}}}_\mathrm{e} : \left[3\hat{\underline{\mathbf{R}}}\hat{\underline{\mathbf{R}}}-\underline{\underline{\mathbf{I}}}\right] + \mathcal{HOT}\right\}$$

$\mathcal{HOT}$: Higher Order Terms / Terme höherer Ordnung

Monopole Moment / Monopolmoment
$$Q_\mathrm{e} = \iiint\limits_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\mathrm{d}^3\underline{\mathbf{R}}'$$

Dipole Moment / Dipolmoment
$$\underline{\mathbf{p}}_\mathrm{e} = \iiint\limits_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'$$

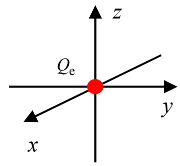
Quadrupole Moment / Quadrupolmoment
$$\underline{\underline{\mathbf{q}}}_\mathrm{e} = \iiint\limits_{\underline{\mathbf{R}}=-\infty}^{\infty} \rho_\mathrm{e}(\underline{\mathbf{R}}')\,\underline{\mathbf{R}}'\,\underline{\mathbf{R}}'\,\mathrm{d}^3\underline{\mathbf{R}}'$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Point Charge(s): Monopole, Dipole, and Quadrupole / Punktladung(en): Mono-, Di- und Quadrupol

Monopole Moment / Monopolmoment

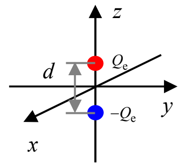
One Point Charge / Eine Punktladung



$$Q_e \neq 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

Dipole Moment / Dipolmoment

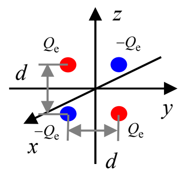
Two Point Charges / Zwei Punktladungen



$$Q_e = 0, \quad \underline{\mathbf{p}}_e \neq \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e = \underline{\underline{\mathbf{0}}}$$

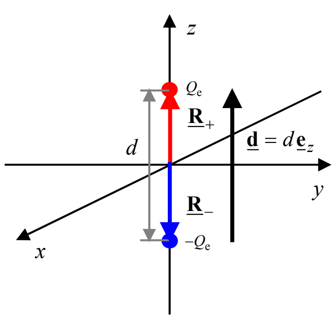
Quadrupole Moment/ Quadrupolmoment

Four Point Charges / Vier Punktladungen



$$Q_e = 0, \quad \underline{\mathbf{p}}_e = \underline{\mathbf{0}}, \quad \underline{\underline{\mathbf{q}}}_e \neq \underline{\underline{\mathbf{0}}}$$

Electrostatic Dipole / Elektrostatischer Dipol



Electrostatic Volume Charge Density / Elektrostatische Raumladungsdichte

$$\rho_e(\underline{\mathbf{R}}) = Q_e\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-) = Q_e\delta(\underline{\mathbf{R}} - \frac{d}{2}\underline{\mathbf{e}}_z) - Q_e\delta(\underline{\mathbf{R}} + \frac{d}{2}\underline{\mathbf{e}}_z)$$

with $\underline{\mathbf{R}}_+ = \frac{d}{2}\underline{\mathbf{e}}_z$, $\underline{\mathbf{R}}_- = -\frac{d}{2}\underline{\mathbf{e}}_z$

Electrostatic Potential / Elektrostatisches Potential

$$\Phi_e(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$

Electrostatic Field Strength / Elektrostatische Feldstärke

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi\varepsilon_0}\left(\frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_+}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_-}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|^3}\right)$$

Electrostatic Dipole Moment / Elektrische Dipolmoment

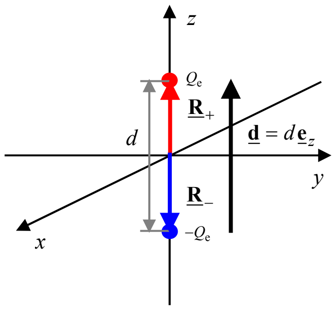
$$\underline{\mathbf{p}}_e = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}' = \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e\delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\right]\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'$$

$$= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}' - \underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\,d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-} = Q_e\underline{\mathbf{R}}_+ - Q_e\underline{\mathbf{R}}_- = Q_e(\underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_-) = Q_e\underline{\mathbf{d}}$$

Distance Vector / Abstandsvektor $\underline{\mathbf{d}} = \underline{\mathbf{R}}_+ - \underline{\mathbf{R}}_- = \frac{d}{2}\underline{\mathbf{e}}_z + \frac{d}{2}\underline{\mathbf{e}}_z = d\,\underline{\mathbf{e}}_z$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Electrostatic Dipole / Elektrostatischer Dipol



Electrostatic Dipole Moment / Elektrostatisches Dipolmoment

$$\underline{\mathbf{p}}_e = Q_e \underline{\mathbf{d}} = p_e \hat{\underline{\mathbf{p}}}_e \qquad \text{with / mit} \qquad p_e = Q_e |\underline{\mathbf{d}}| = Q_e |\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|, \quad \hat{\underline{\mathbf{p}}}_e = \hat{\underline{\mathbf{d}}} = \widehat{|\underline{\mathbf{R}}_+ + \underline{\mathbf{R}}_-|}$$

Electrostatic Quadrupole Moment / Elektrostatisches Quadrupolmoment

$$\begin{aligned}
\underline{\underline{\mathbf{q}}}_e &= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \rho_e(\underline{\mathbf{R}}')\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}' \\
&= \iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \left[Q_e\delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_+) - Q_e\delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_-)\right]\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}' \\
&= Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_+)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_+\underline{\mathbf{R}}_+} - Q_e \underbrace{\iiint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \delta(\underline{\mathbf{R}}'-\underline{\mathbf{R}}_-)\underline{\mathbf{R}}'\underline{\mathbf{R}}' d^3\underline{\mathbf{R}}'}_{=\underline{\mathbf{R}}_-\underline{\mathbf{R}}_-} \\
&= Q_e \underline{\mathbf{R}}_+\underline{\mathbf{R}}_+ - Q_e \underline{\mathbf{R}}_-\underline{\mathbf{R}}_- \\
&= Q_e\left[\left(\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(\frac{d}{2}\underline{\mathbf{e}}_z\right) - \left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\left(-\frac{d}{2}\underline{\mathbf{e}}_z\right)\right] \\
&= Q_e \underbrace{\left[d\,\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z - d\,\underline{\mathbf{e}}_z\underline{\mathbf{e}}_z\right]}_{=\underline{\underline{\mathbf{0}}}} \\
&= \underline{\underline{\mathbf{0}}}
\end{aligned}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

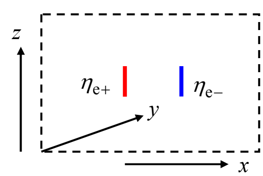
Point Charge(s): Monopole, Dipole, and Quadrupole … / Punktladung(en): Mono-, Di- und Quadrupol ... (2)

Application: Numerical Solution of Unbounded Static Problems / Anwendung: Numerische Lösung von unbegrenzten statischen Problemen

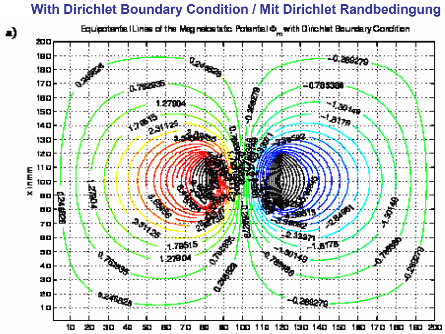
$$\Delta\Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon_0}$$

$$\rho_e(\underline{\mathbf{R}}) = \eta_{e+}(y,z)\delta(x-x_+) + \eta_{e-}(y,z)\delta(x-x_-)$$

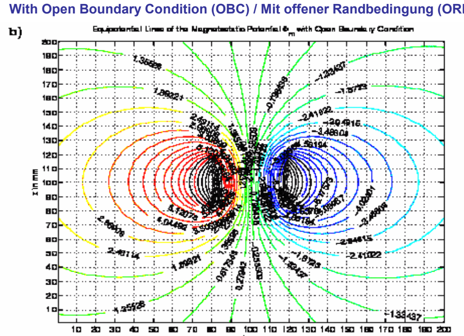
$$\eta_{e+}(y,z) = \begin{cases} \eta_{e0} & y_- \le y \le y_+ \\ & z_- \le z \le z_+ \\ 0 & \text{else / sonst} \end{cases} = -\eta_{e-}(y,z)$$



With Dirichlet Boundary Condition / Mit Dirichlet Randbedingung



With Open Boundary Condition (OBC) / Mit offener Randbedingung (ORB)

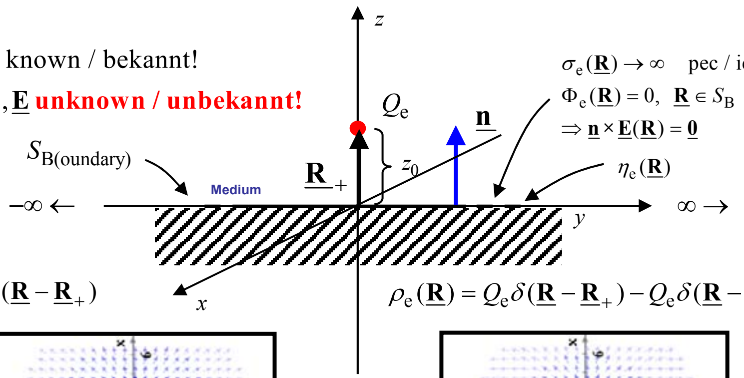


ES Fields – Method of Images / ES-Felder – Spiegelungsmethode

Boundary Value Problem (BVP) – Randwertproblem (RWP)

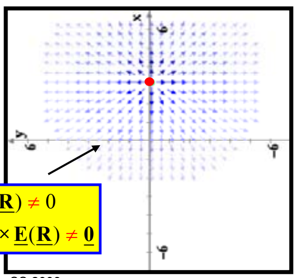
Q_e known / bekannt!

$\Phi_\mathrm{e}, \underline{\mathbf{E}}$ **unknown / unbekannt!**



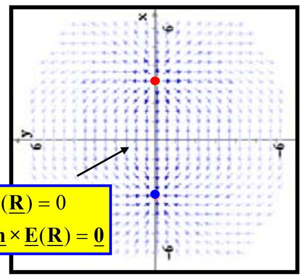
$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+)$

$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$



$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) \neq 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) \neq \underline{\mathbf{0}}$

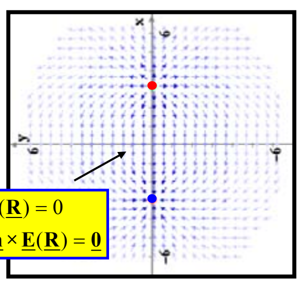


$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

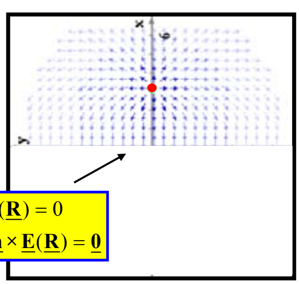
Method of Images / Spiegelungsmethode



$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \frac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right)$$



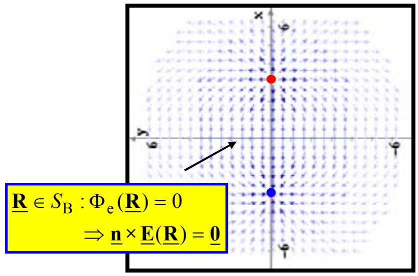
$\underline{\mathbf{R}} \in S_\mathrm{B} : \Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0$

$\Rightarrow \underline{\mathbf{n}} \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$

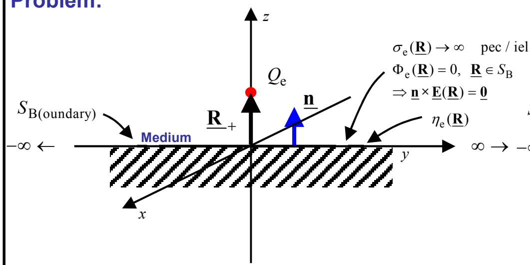
$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

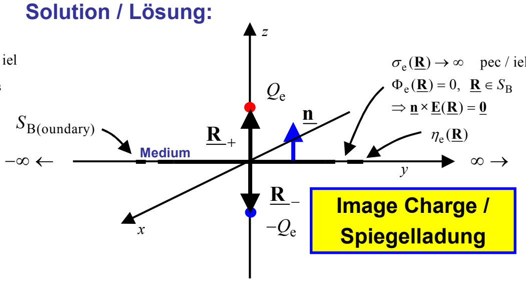
Method of Images / Spiegelungsmethode



Problem:

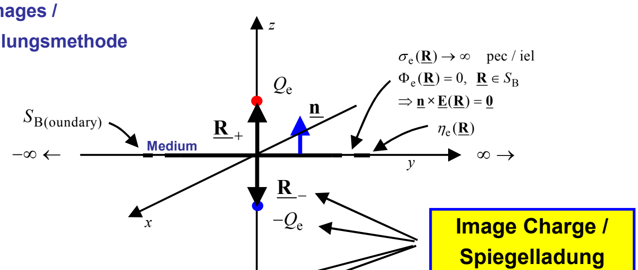


Solution / Lösung:



ES Fields – Method of Images / ES Felder – Spiegelungsmethode

Solution by Applying the Method of Images / Lösung durch Anwendung der Spiegelungsmethode



$$\rho_e(\underline{\mathbf{R}}) = Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_e \delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with / mit $\quad \underline{\mathbf{R}}_+ = z_0 \underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0 \underline{\mathbf{e}}_z$

$$\Phi_e(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\rho_\mathrm{e}(\underline{\mathbf{R}}) = Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_+) - Q_\mathrm{e}\delta(\underline{\mathbf{R}} - \underline{\mathbf{R}}_-)$$

with / mit $\underline{\mathbf{R}}_+ = z_0\underline{\mathbf{e}}_z \qquad \underline{\mathbf{R}}_- = -\underline{\mathbf{R}}_+ = -z_0\underline{\mathbf{e}}_z$

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} - \dfrac{1}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|}\right) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

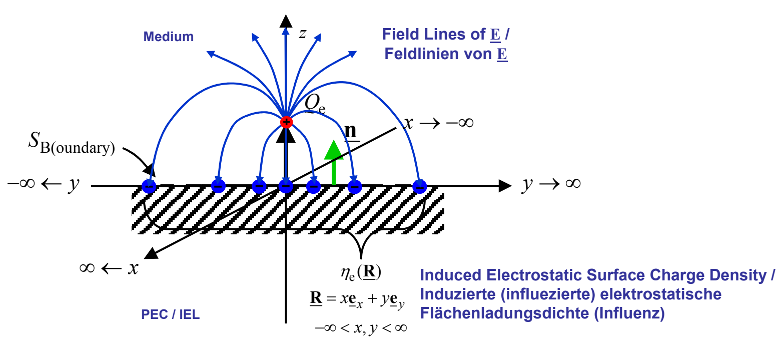
$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\left(\dfrac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_+}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_-}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|^3}\right) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$= \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi}\left(\dfrac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_+}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|^3} - \dfrac{\underline{\mathbf{R}} - \underline{\mathbf{R}}_-}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_-|^3}\right) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



Without the Method of Images we have to Solve the Following Integral Equation for the Unknown Induced Electrostatic Surface Charge / Ohne die Spiegelungsmethode muss man die folgende Integralgleichung für die induzierte (influezierte) elektrostatische Flächenladungsdichte lösen

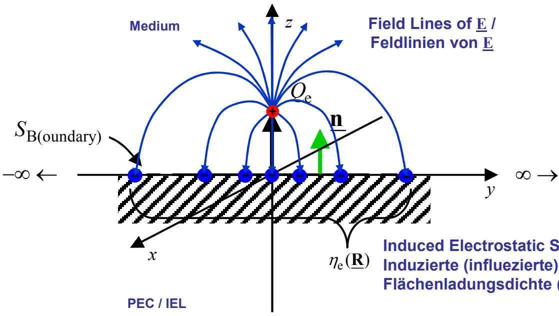
Unknown / Unbekannt

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_\mathrm{e}}{|\underline{\mathbf{R}} - \underline{\mathbf{R}}_+|} + \iint_{\underline{\mathbf{R}}'=-\infty}^{\infty} \frac{\eta_\mathrm{e}(\underline{\mathbf{R}}')}{|\underline{\mathbf{R}}' - \underline{\mathbf{R}}_+|} d^2\underline{\mathbf{R}}' \Bigg|_{z=0}\right] = 0$$

for / für $\Phi_\mathrm{e}(\underline{\mathbf{R}})\big|_{z=0} = 0$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode



If $\underline{\mathbf{D}}$ is known from the Method of Images /
Falls $\underline{\mathbf{D}}$ über die Spiegelungsmethode
bekannt ist

$$\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B} = \begin{matrix}\text{known}\\ \text{bekannt}\end{matrix} \;!$$

$\eta_e(\underline{\mathbf{R}})$ is Defined by the Normal Component of $\underline{\mathbf{D}}$ / $\eta_e(\underline{\mathbf{R}})$ ist definiert über die Normalkomponente von $\underline{\mathbf{D}}$

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}}\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B}$$

$$= \underline{\mathbf{n}}\bullet\frac{Q_e}{4\pi}\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{\underline{\mathbf{R}}\in S_B} \quad \begin{matrix}\text{for}\\ \text{für}\end{matrix} \quad z=0$$

$$= \frac{Q_e}{4\pi}\underline{\mathbf{e}}_z\bullet\left[\frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_+}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{R}}-\underline{\mathbf{R}}_-}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{z=0}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}}\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}}\in S_B}$$

$$= \frac{Q_e}{4\pi}\left[\frac{\underline{\mathbf{e}}_z\bullet(\underline{\mathbf{R}}-\underline{\mathbf{R}}_+)}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_+|^3} - \frac{\underline{\mathbf{e}}_z\bullet(\underline{\mathbf{R}}-\underline{\mathbf{R}}_-)}{|\underline{\mathbf{R}}-\underline{\mathbf{R}}_-|^3}\right]\Bigg|_{z=0}$$

$$= \frac{Q_e}{4\pi}\left[\frac{z-z_0}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}} - \frac{z+z_0}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}}\right]\Bigg|_{z=0}$$

$$= \frac{Q_e}{4\pi}\left[\frac{-z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}} - \frac{z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}}\right]$$

$$= -\frac{Q_e}{2\pi}\frac{z_0}{\left[x^2+y^2+z_0^2\right]^{3/2}}$$

$$= -\frac{Q_e}{2\pi}\frac{z_0}{\left[r^2+z_0^2\right]^{3/2}}$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Method of Images / Spiegelungsmethode

$$\eta_e(\underline{\mathbf{R}}) = \underline{\mathbf{n}} \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\Big|_{\underline{\mathbf{R}} \in S_B} = -\frac{Q_e}{2\pi} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}}$$

⇨ **Total Electric Charge at the xy Plane at $z=0$ / Gesamtladung auf der xy Ebene bei $z=0$**

$$Q_e^{\text{tot}} = -\frac{Q_e}{2\pi} \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} \, r \, dr \, d\varphi$$

$$= -\frac{Q_e}{2\pi} \int_{r=0}^{\infty} \frac{z_0}{\left[r^2 + z_0^2\right]^{3/2}} \, r \, dr \underbrace{\int_{\varphi=0}^{2\pi} d\varphi}_{=2\pi}$$

$$= -Q_e z_0 \int_{r=0}^{\infty} \frac{r}{\left[r^2 + z_0^2\right]^{3/2}} \, dr$$

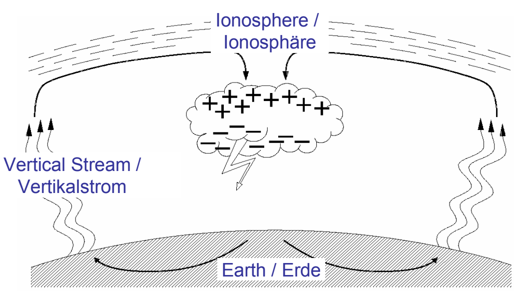
$$\int \frac{x}{\left[x^2 + a^2\right]^{3/2}} \, dx = -\frac{1}{\sqrt{x^2 + a^2}}$$

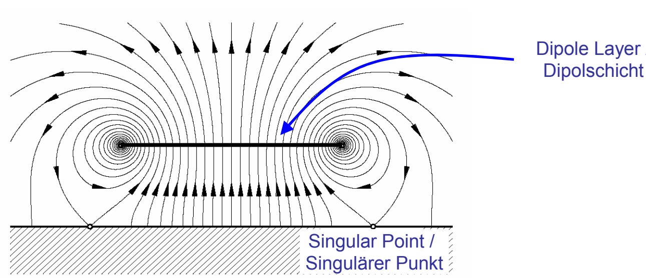
$$= Q_e z_0 \left[\underbrace{\frac{1}{\sqrt{(r \to \infty)^2 + z_0^2}}}_{\to 0} - \frac{1}{z_0} \right]$$

$$= -Q_e$$

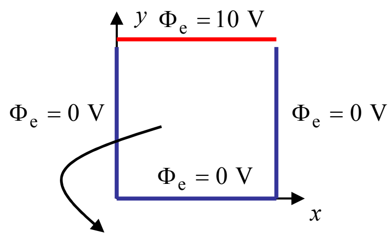
$$\boxed{Q_e^{\text{tot}} = -Q_e}$$

ES Fields – Method of Images – Applications / ES Felder – Spiegelungsmethode – Anwendungen





Electrostatic (ES) Fields – Separation of Variables – Example / Elektrostatische (ES) Felder – Separation der Variablen – Beispiel



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi_\mathrm{e}(x,y) = 0$$

Separation of Variables / Separation der Variablen !

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\Delta \Phi_\mathrm{e}(x,y,z) = 0$$

Elliptic Partial Differential Equation / Elliptische partielle Differentialgleichung

Laplace Equation in Cartesian Coordinates / Laplace-Gleichung in Kartesischen Koordinaten

3-D / 3D

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi_\mathrm{e}(x,y,z) = 0$$

Function of Three Variables / Funktion von drei Variablen

2-D / 2D

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi_\mathrm{e}(x,y) = 0$$

Function of Two Variables / Funktion von zwei Variablen

$$\frac{\partial^2}{\partial x^2} \Phi_\mathrm{e}(x,y) + \frac{\partial^2}{\partial y^2} \Phi_\mathrm{e}(x,y) = 0$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=0$$

Solution Strategy: Reduce the Partial Differential Equation (PDE) to an Ordinary Differential Equation (ODE) and Find a Solution of the PDE by Solving the ODE

Lösungsstrategie: Reduziere die partielle Differentialgleichung (PDG) auf eine gewöhnliche (ordinäre) Differentialgleichung (GDG) und finde eine Lösung der PDG durch Lösung der GDG

Ansatz of Separation / Separationsansatz

$$\Phi_e(x,y)=\overbrace{X(x)\;Y(y)}$$

Product of 2 Functions / Produkt aus 2 Funktionen

Function of 2 Variables: x and y / Funktion von 2 Variablen: x und y

Function of x only / Nur eine Funktion von x

Function of y only / Nur eine Funktion von y

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=0$$

Ansatz of Separation / Separationsansatz

$$\Phi_e(x,y)=X(x)Y(y)$$

⇨ Inserted in the Above Laplace Equation Yields / Eingesetzt in die obere Laplace-Gleichung ergibt

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=\frac{\partial^2}{\partial x^2}\left[X(x)Y(y)\right]+\frac{\partial^2}{\partial y^2}\left[X(x)Y(y)\right]$$

$$=Y(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+X(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=\underbrace{Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)}$$

$$\frac{1}{X(x)Y(y)}$$

$$\frac{1}{X(x)Y(y)}\overbrace{\left[Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)\right]}=\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=0$$

$$\underbrace{\underbrace{\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)}_{=\text{Function of } x\,/\,\text{Funktion von } x}}_{=-\alpha^2}+\underbrace{\underbrace{\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)}_{=\text{Function of } y\,/\,\text{Funktion von } y}}_{=-\beta^2}=0$$

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=\underbrace{\left(-\alpha^2\right)+\left(-\beta^2\right)}_{=0}$$

Separation Condition / Separationsbedingung $\quad \alpha^2+\beta^2=0$ ⇦

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{d^2}{dx^2}X(x)=-\alpha^2X(x)$$

$$\frac{d^2}{dy^2}Y(y)=-\beta^2Y(y)$$

Separation Condition / Separationsbedingung

$$\alpha^2+\beta^2=0$$

With / Mit $\quad \alpha^2=-\beta^2=k^2$

We Obtain Two ODE / Wir erhalten zwei GDG

$$\frac{d^2}{dx^2}X(x)=-k^2X(x)$$

$$\frac{d^2}{dy^2}Y(y)=k^2Y(y)$$

Solutions of these Equations are / Lösungen dieser Gleichungen sind

$X(x)\sim\cos(kx)$ or / oder $\sim\sin(kx)$

$Y(y)\sim\cosh(ky)$ or / oder $\sim\sinh(ky)$

For $k=0$ these Solutions Degenerate to / Für $k=0$ diese Lösungen degenerieren zu

$X(x)\sim\text{const.}$ or / oder $\sim x$

$Y(y)\sim\text{const.}$ or / oder $\sim y$

End of 8th Lecture /
Ende der 8. Vorlesung