Elektromagnetische Feldtheorie I (EFT I) / Electromagnetic Field Theory I (EFT I)

9th Lecture / 9. Vorlesung

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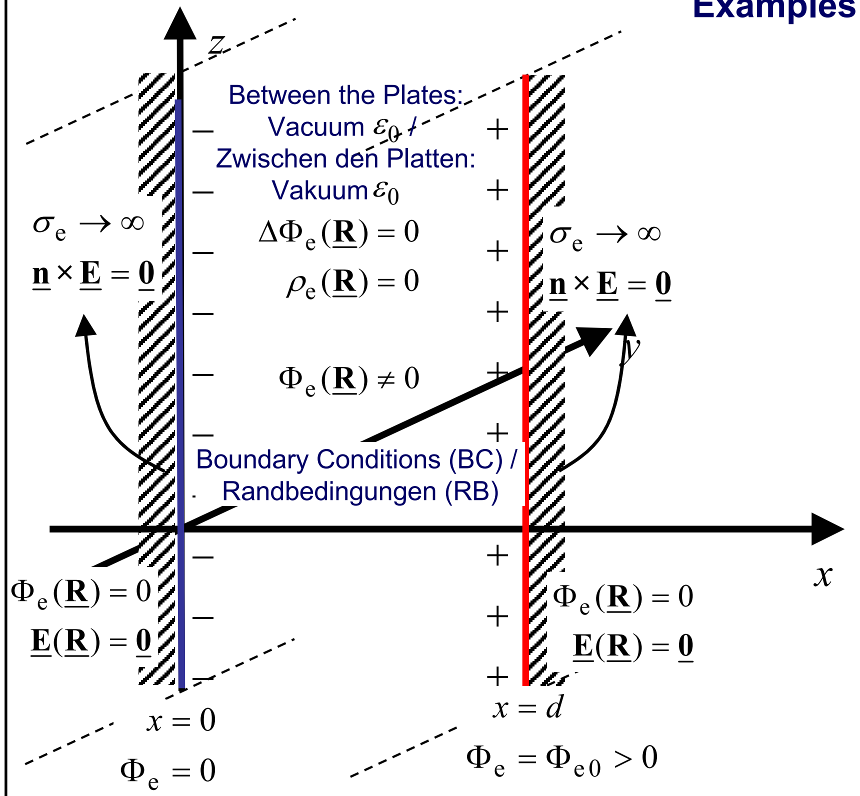
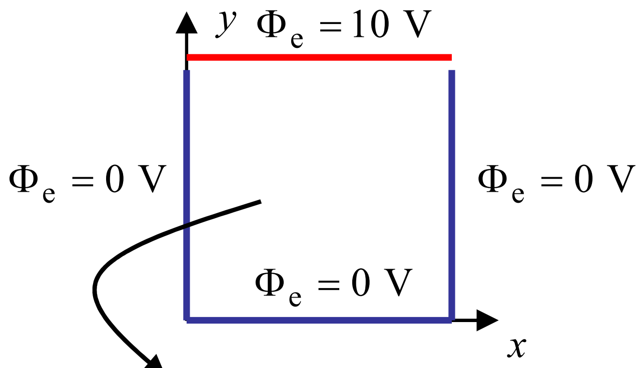
Electrostatic (ES) Fields – Boundary Value Problem (BVP) / Elektrostatische (ES) Felder – Randwertproblem (RWP)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi_\mathrm{e}(x,y,z)=\begin{cases}-\dfrac{\rho_\mathrm{e}(x,y,z)}{\varepsilon_0} & \text{for / für } \rho_\mathrm{e}(x,y,z)\neq 0 \quad \text{Poisson Equation / Poisson-Gleichung}\\ 0 & \text{for / für } \rho_\mathrm{e}(x,y,z)=0 \quad \text{Laplace Equation / Laplace-Gleichung}\end{cases}$$

Examples: / Beispiele:

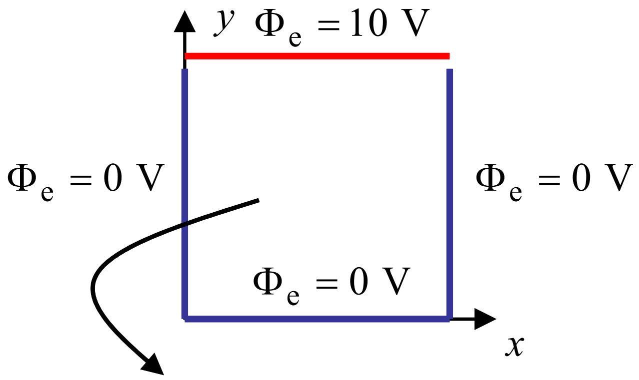




$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_\mathrm{e}(x,y)=0$$

➡ **Separation of Variables / Separation der Variablen** **!**

Electrostatic (ES) Fields – Separation of Variables – Example / Elektrostatische (ES) Felder – Separation der Variablen – Beispiel



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y) = 0$$

➡ **Separation of Variables / Separation der Variablen !**

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\Delta\Phi_{\mathrm{e}}(x, y, z) = 0$$

Elliptic Partial Differential Equation / Elliptische partielle Differentialgleichung

Laplace Equation in Cartesian Coordinates / Laplace-Gleichung in Kartesischen Koordinaten

3-D / 3D

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi_{\mathrm{e}}(x, y, z) = 0$$

Function of Three Variables / Funktion von drei Variablen

2-D / 2D

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_{\mathrm{e}}(x, y) = 0$$

Function of Two Variables / Funktion von zwei Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_{\mathrm{e}}(x, y) + \frac{\partial^2}{\partial y^2}\Phi_{\mathrm{e}}(x, y) = 0$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation /
Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=0$$

Solution Strategy: Reduce the Partial Differential Equation (PDE) to an Ordinary Differential Equation (ODE) and Find a Solution of the PDE by Solving the ODE

Lösungsstrategie: Reduziere die partielle Differentialgleichung (PDG) auf eine gewöhnliche (ordinäre) Differentialgleichung (GDG) und finde eine Lösung der PDG durch Lösung der GDG

Ansatz of Separation /
Separationsansatz

$$\Phi_\mathrm{e}(x,y)=\overbrace{X(x)\;Y(y)}$$

Product of 2 Functions /
Produkt aus 2 Funktionen

Function of 2 Variables: x and y /
Funktion von 2 Variablen: x und y

Function of x only /
Nur eine Funktion von x

Function of y only /
Nur eine Funktion von y

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

Laplace Equation / Laplace-Gleichung

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=0$$

Ansatz of Separation / Separationsansatz

$$\Phi_\mathrm{e}(x,y)=X(x)\,Y(y)$$

⇨ Inserted in the Above Laplace Equation Yields / Eingesetzt in die obere Laplace-Gleichung ergibt

$$\frac{\partial^2}{\partial x^2}\Phi_\mathrm{e}(x,y)+\frac{\partial^2}{\partial y^2}\Phi_\mathrm{e}(x,y)=\frac{\partial^2}{\partial x^2}\big[X(x)\,Y(y)\big]+\frac{\partial^2}{\partial y^2}\big[X(x)\,Y(y)\big]$$

$$=Y(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x)+X(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y)$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\partial^2}{\partial x^2}\Phi_e(x,y)+\frac{\partial^2}{\partial y^2}\Phi_e(x,y)=\underbrace{Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)}$$

$$\frac{1}{X(x)Y(y)}$$

$$\frac{1}{X(x)Y(y)}\overbrace{\left[Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)\right]}=\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=0$$

$$\underbrace{\underbrace{\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)}_{=\text{Function of } x\,/\,\text{Funktion von } x}}_{=-\alpha^2}+\underbrace{\underbrace{\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)}_{=\text{Function of } y\,/\,\text{Funktion von } y}}_{=-\beta^2}=0$$

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=\underbrace{\left(-\alpha^2\right)+\left(-\beta^2\right)}_{=0}$$

Separation Condition / Separationsbedingung $\quad \alpha^2+\beta^2=0$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) = -\alpha^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y) = -\beta^2 Y(y)$$

Separation Condition /
Separationsbedingung

$$\alpha^2 + \beta^2 = 0$$

With / Mit $\alpha^2 = -\beta^2 = k^2$

We Obtain Two ODE /
Wir erhalten zwei GDG

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) = -k^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y) = k^2 Y(y)$$

Solutions of these Equations are /
Lösungen dieser Gleichungen sind

$X(x) \sim \cos(kx)$ or / oder $\sim \sin(kx)$

$Y(y) \sim \cosh(ky)$ or / oder $\sim \sinh(ky)$

For $k=0$ these Solutions Degenerate to /
Für $k=0$ diese Lösungen degenerieren zu

$X(x) \sim \text{const.}$ or / oder $\sim x$

$Y(y) \sim \text{const.}$ or / oder $\sim y$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

We Obtain Two ODE /
Wir erhalten zwei GDG

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = -k^2 X(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = \; k^2 Y(y)$$

$$X(x) = \cos(kx)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) = -k \frac{\mathrm{d}}{\mathrm{d}x}\sin(kx) = -k^2 \underbrace{\cos(kx)}_{=X(x)}$$

$$Y(y) = \; \cosh(ky)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} Y(y) = \frac{\mathrm{d}^2}{\mathrm{d}y^2}\cosh(ky) = \; k \frac{\mathrm{d}}{\mathrm{d}y}\sinh(ky) = \; k^2 \underbrace{\cosh(y)}_{=Y(y)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(kx) = -k\sin(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(kx) = -k^2\cos(kx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh(kx) = k\sinh(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cosh(kx) = k^2\cosh(kx)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(kx) = k\cos(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(kx) = -k^2\sin(kx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh(kx) = k\cosh(kx) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sinh(kx) = k^2\sinh(kx)$$

Electrostatic (ES) Fields – Separation of Variables / Elektrostatische (ES) Felder – Separation der Variablen

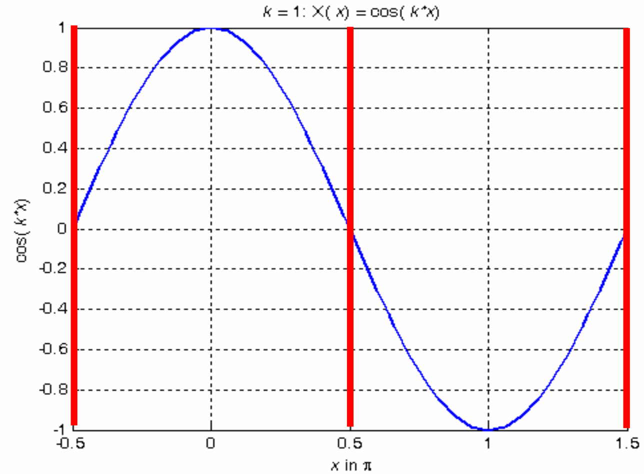
Solutions of the 2-D Laplace Equation in the Cartesian Coordinate System /
Lösungen der 2D-Laplace-Gleichung im Kartesischen Koordinatensystem

$$\Phi_{\mathrm{e}}(x,y) = X(x)\,Y(y) =$$

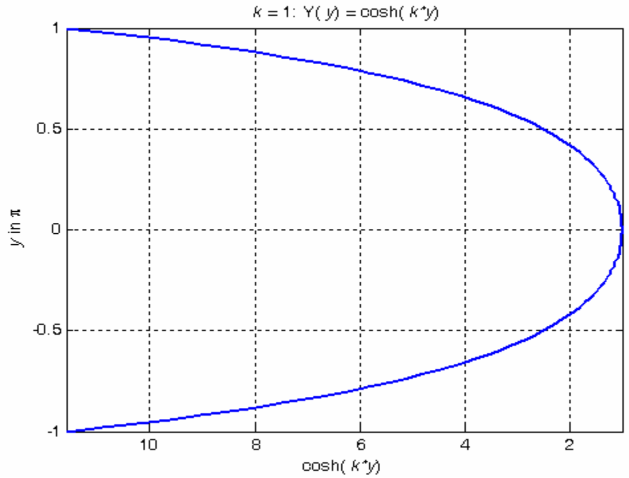
$k=0$	$k^2 \geq 0$	$k^2 \leq 0\ (k \to jk')$
const.	$\cos(kx)\cosh(ky)$	$\cosh(k'x)\cos(k'y)$
y	$\cos(kx)\sinh(ky)$	$\cosh(k'x)\sin(k'y)$
x	$\sin(kx)\cosh(ky)$	$\sinh(k'x)\cos(k'y)$
xy	$\sin(kx)\sinh(ky)$	$\sinh(k'x)\sin(k'y)$
	$\cos(kx)\,\mathrm{e}^{ky}$	$\mathrm{e}^{k'x}\cos(k'y)$
	$\cos(kx)\,\mathrm{e}^{-ky}$	$\mathrm{e}^{-k'x}\cos(k'y)$
	$\sin(kx)\,\mathrm{e}^{ky}$	$\mathrm{e}^{k'x}\sin(k'y)$
	$\sin(kx)\,\mathrm{e}^{-ky}$	$\mathrm{e}^{-k'x}\sin(k'y)$

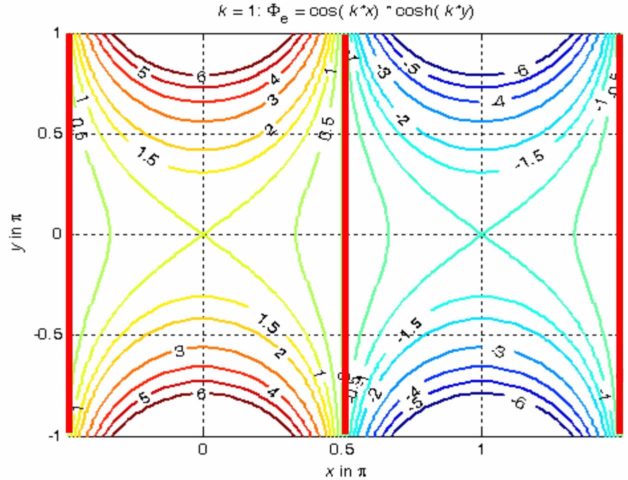
ES Fields – Separation of Variables / ES Felder – Separation der Variablen (...)

$$\Phi_e(x,y) = X(x)Y(y) = \cos(kx)\cosh(ky)$$



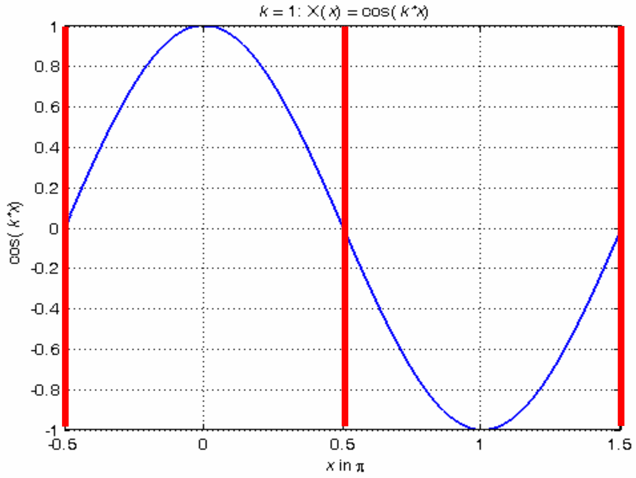
$$\Phi_e(x,y) = 0$$

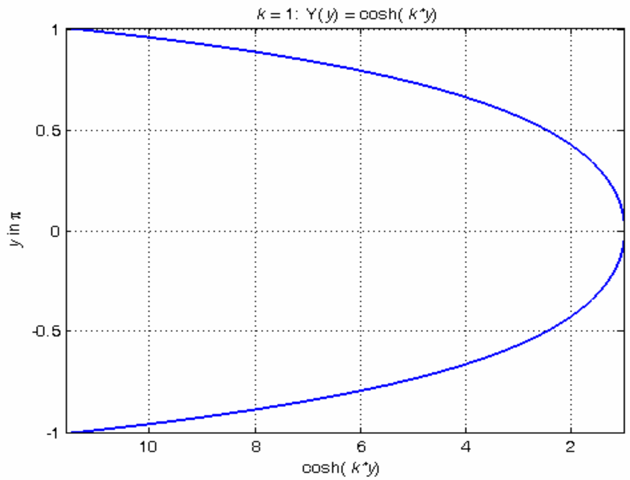


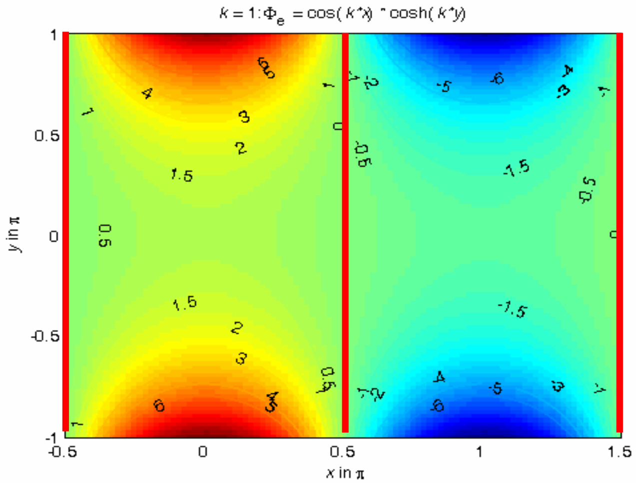


ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_\mathrm{e}(x,y) = X(x)Y(y) = \cos(kx)\cosh(ky)$$

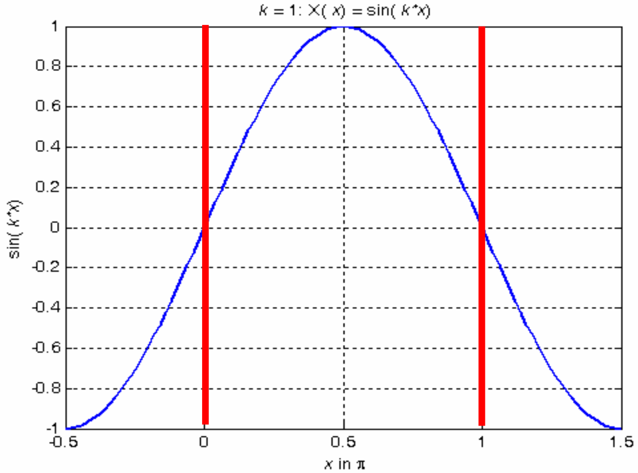




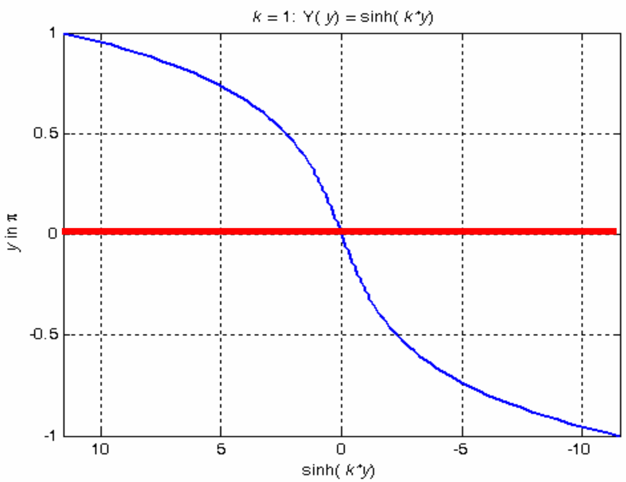


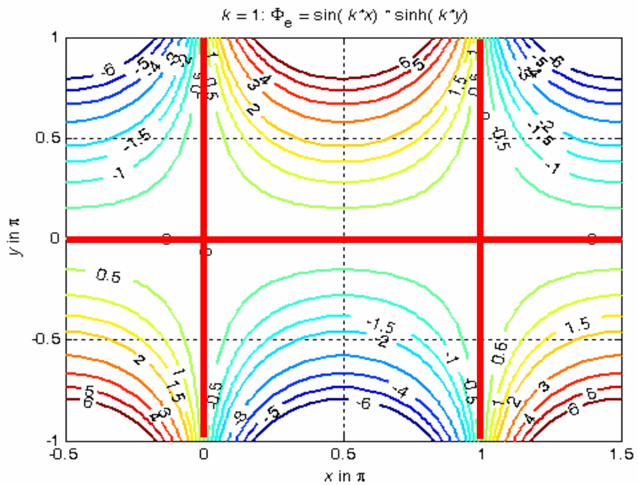
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_\mathrm{e}(x,y) = X(x)Y(y) = \sin(kx)\sinh(ky)$$



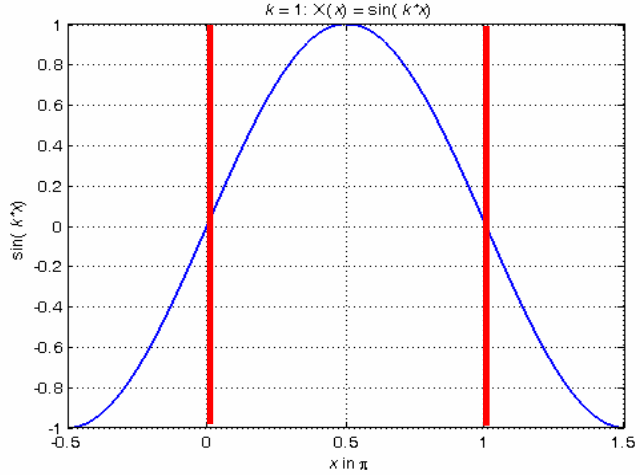
$$\Phi_\mathrm{e}(x,y) = 0$$



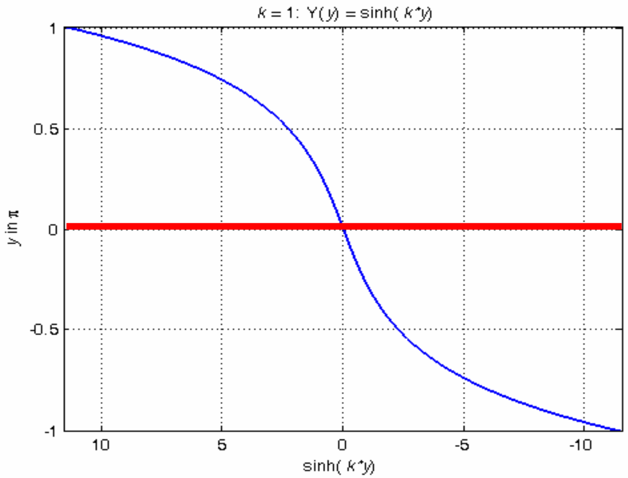


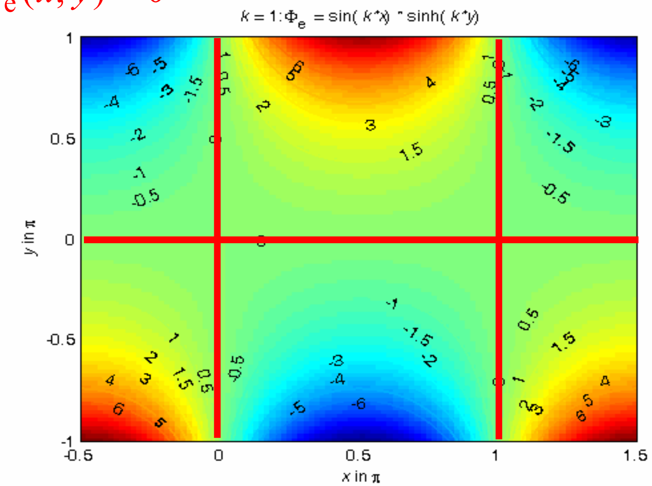
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\Phi_{\mathrm{e}}(x, y) = X(x)Y(y) = \sin(kx)\sinh(ky)$$

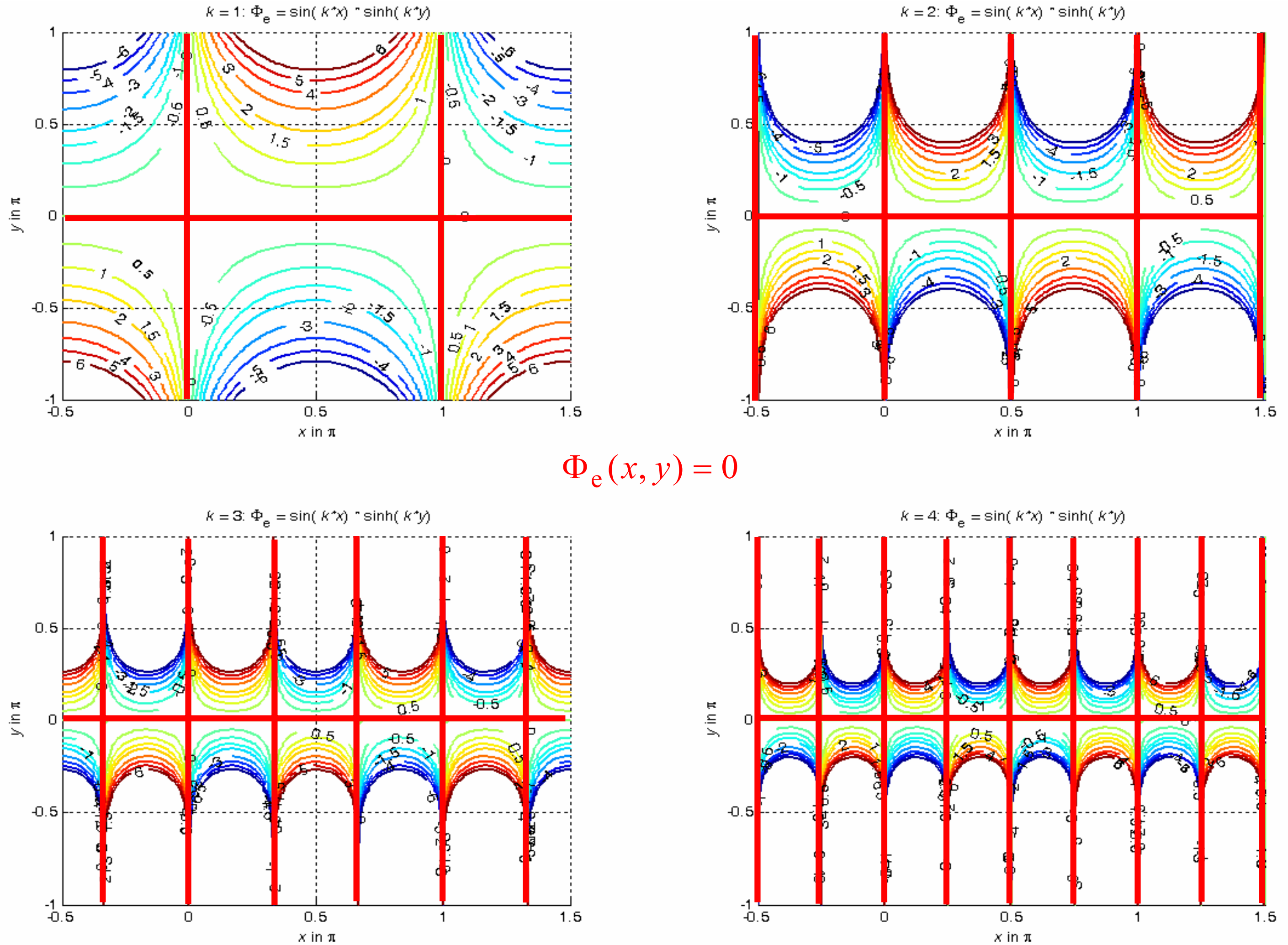


$$\Phi_{\mathrm{e}}(x, y) = 0$$



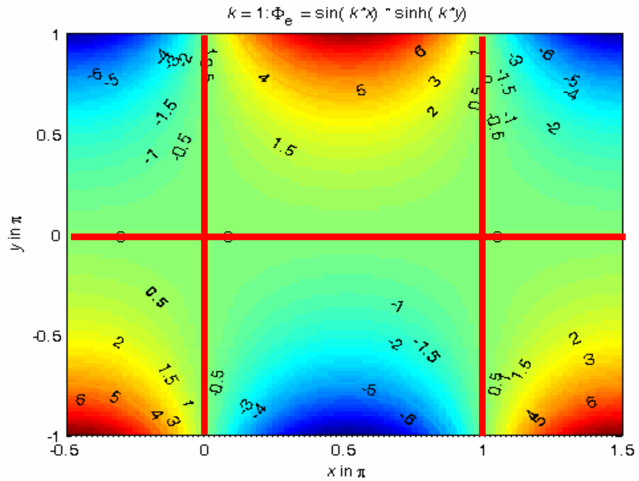


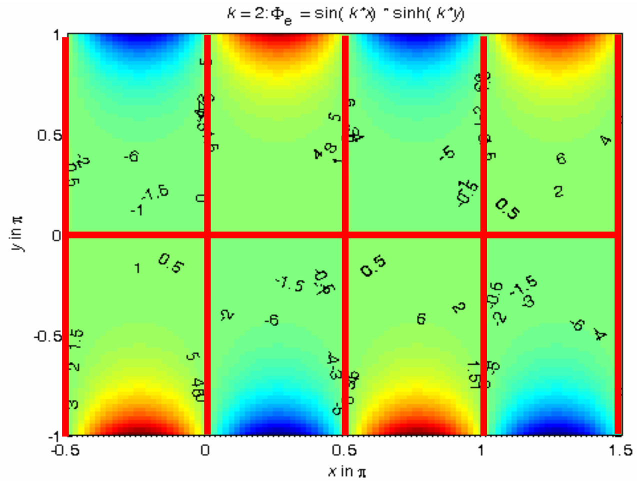
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



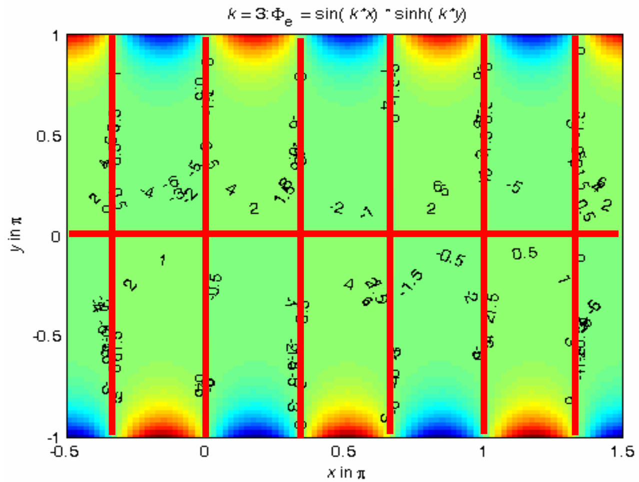
$\Phi_{\mathrm{e}}(x, y) = 0$

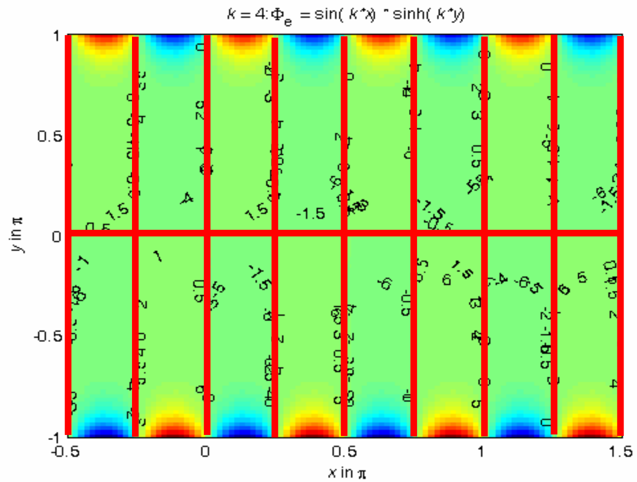
ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)





$$\Phi_\mathrm{e}(x, y) = 0$$





ES Fields – Separation of Variables – Superposition of Modes / ES Felder – Separation der Variablen – Superposition von Moden (...)

Superposition of Modes to Ensure Boundary Conditions /
Superposition von Moden zur Erfüllung von Randbedingungen:

Each solution of the Laplace equation – eigen solution, mode – obtained by the separation of variables displays lines (surfaces) of vanishing potential. At these lines (surfaces) we could place a Dirichlet boundary with $\Phi_e(x,y) = 0$ V /
Jede Lösung der Laplace-Gleichung – Eigenlösung, Mode –, die man über die Methode der Separation bestimmt, weist Linien (Flächen) mit dem Null-Potential auf.
Auf diesen Linien (Flächen) kann man eine Dirichlet-Rand mit $\Phi_e(x,y) = 0$ V platzieren.

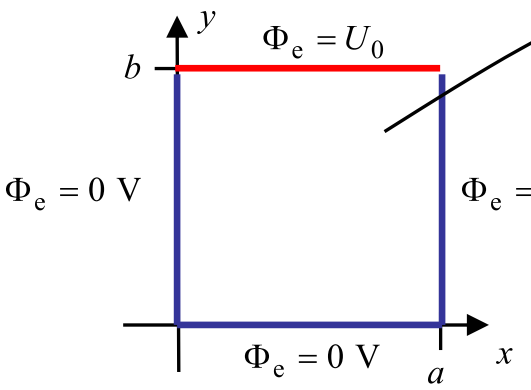
For Example, Consider the Solution / Betrachte beispielsweise die Lösung

$$\Phi_\mathrm{e}(x,y) = \Phi_{\mathrm{e}0} \sin(kx) \sinh(ky)$$

This Functions is Zero for / Diese Funktion ist gleich null für

$$\Phi_\mathrm{e}(x,y) = 0 \quad \begin{cases} y = 0 & & \text{because / weil} & \sinh(ky) = \sinh(0) = 0 \\ x = \dfrac{n\pi}{k} & n = -\infty, \ldots -1, 0, 1, \ldots \infty & \text{because / weil} & \sin(kx) = \sin(n\pi) = 0 \end{cases}$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x,y) = 0 \quad \begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$$

We Set / Wir setzen:

$$\Phi_e(x,y) = \Phi_{e0}\sin(kx)\sinh(ky) \qquad k = \frac{n\pi}{a} \qquad \Phi_e(x,y) = \Phi_{e0}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

then it follows / dann folgt

$$\Phi_e(x,y) = 0 \qquad \begin{matrix} x = 0 \\ x = a \\ y = 0 \end{matrix}$$

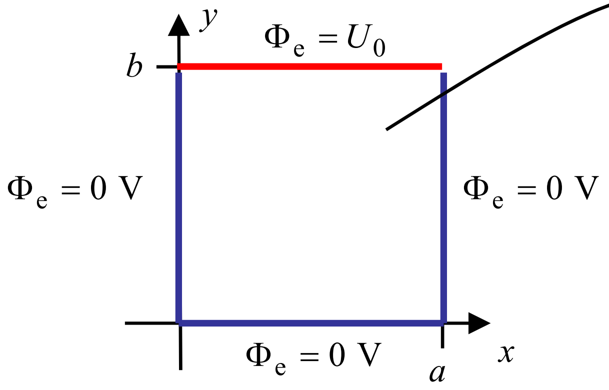
$$y = b : \Phi_e = U_0$$

$$\Phi_e(x, y = b) = \Phi_{e0}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right) \neq U_0$$

ES Fields – Separation of Variables – Superposition of Modes / ES Felder – Separation der Variablen – Superposition von Moden (...)

$$\Rightarrow \quad \Phi_{\mathrm{e}}(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right)$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_e(x, y) = 0 \quad \begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$$

$$\Phi_e(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right)$$

A_n **?!**

Adjust die Coefficients A_n, $n = 1,2,\ldots,\infty$ in Order to Ensure the Inhomogeneous Dirichlet Boundary Condition $\Phi_e(x,b) = U_0$ at the Top Boundary. / Die Koeffizienten A_n, $n = 1,2,\ldots,\infty$ sind so zu bestimmen, dass die inhomogene Dirichlet-Randbedingung $\Phi_e(x,b) = U_0$ am oberen Rand erfüllt wird.

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Adjust die Coefficients A_n, $n = 1,2,\ldots,\infty$ in Order to Ensure the Inhomogeneous Dirichlet Boundary Condition $\Phi_e(x,b) = U_0$ at the Top Boundary. / Die Koeffizienten A_n, $n = 1,2,\ldots,\infty$ sind so zu bestimmen, dass die inhomogene Dirichlet-Randbedingung $\Phi_e(x,b) = U_0$ am oberen Rand erfüllt wird.

$$\Phi_e(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

1. Determine / Bestimme $\left.\Phi_e(x,y)\right|_{y=b}$

$$\Phi_e(x,y=b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

2. Multiply Both Sides with / Multipliziere beide Seiten mit $\sin\left(\frac{m\pi}{a}x\right)$

$$\int_{x=0}^{a} \Phi_e(x,b)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \int_{x=0}^{a}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right)\int_{x=0}^{a}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$\int\limits_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \begin{cases} \dfrac{a}{2} & n = m \\ 0 & n \neq m \end{cases}$$

$$= \frac{a}{2}\delta_{nm}$$

Kronecker Delta / Kronecker-Delta

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\int\limits_{x=0}^{a} \Phi_{\mathrm{e}}(x,b)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \underbrace{\int\limits_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\mathrm{d}x}_{=\frac{a}{2}\delta_{nm}}$$

3. It Follows for $m = n$ / Es folgt für $m = n$

$$\int\limits_{x=0}^{a} \Phi_{\mathrm{e}}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \frac{a}{2}\sum_{n=m} A_n \sinh\left(\frac{n\pi}{a}b\right)$$

$$= \frac{a}{2} A_n \sinh\left(\frac{n\pi}{a}b\right)$$

$$A_n = \frac{2}{a\sinh\left(\dfrac{n\pi}{a}b\right)} \int\limits_{x=0}^{a} \Phi_{\mathrm{e}}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)} \int_{x=0}^{a} \Phi_\mathrm{e}(x,b)\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x$$

$$\Phi_\mathrm{e}(x,b) = U_0$$

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)} \int_{x=0}^{a} U_0 \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x$$

$$= \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x$$

$$\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = -\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\Bigg|_{x=0}^{a}$$

$$= -\frac{a}{n\pi}\left[\cos\left(\frac{n\pi}{a}a\right) - \underbrace{\cos(0)}_{=1}\right]$$

$$= \frac{a}{n\pi}\left[1 - \cos(n\pi)\right]$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$$A_n = \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \underbrace{\int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x} \quad \rightarrow \quad \int_{x=0}^{a} \sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x = \frac{a}{n\pi}\left[1-\cos\left(n\pi\right)\right]$$

$$= \frac{2U_0}{a\sinh\left(\frac{n\pi}{a}b\right)} \frac{a}{n\pi}\left[1-\cos\left(n\pi\right)\right]$$

$$= \frac{2U_0}{n\pi\sinh\left(\frac{n\pi}{a}b\right)} \underbrace{\left[1-\cos\left(n\pi\right)\right]}$$

$$\cos\left(n\pi\right) = \begin{cases} -1 & n = 1,3,5,\ldots \\ 1 & n = 2,4,6 \end{cases}$$

$$\rightarrow \quad 1-\cos\left(n\pi\right) = \begin{cases} 2 & n = 1,3,5,\ldots \\ 0 & n = 2,4,6 \end{cases}$$

$$\Rightarrow \quad A_n = \begin{cases} \frac{4U_0}{n\pi} \frac{1}{\sinh\left(\frac{n\pi}{a}b\right)} & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots \end{cases}$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

Solution / Lösung

Infinite Series / Unendliche Reihe

$$\Phi_{\mathrm{e}}(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

Coefficients / Koeffizienten

with / mit

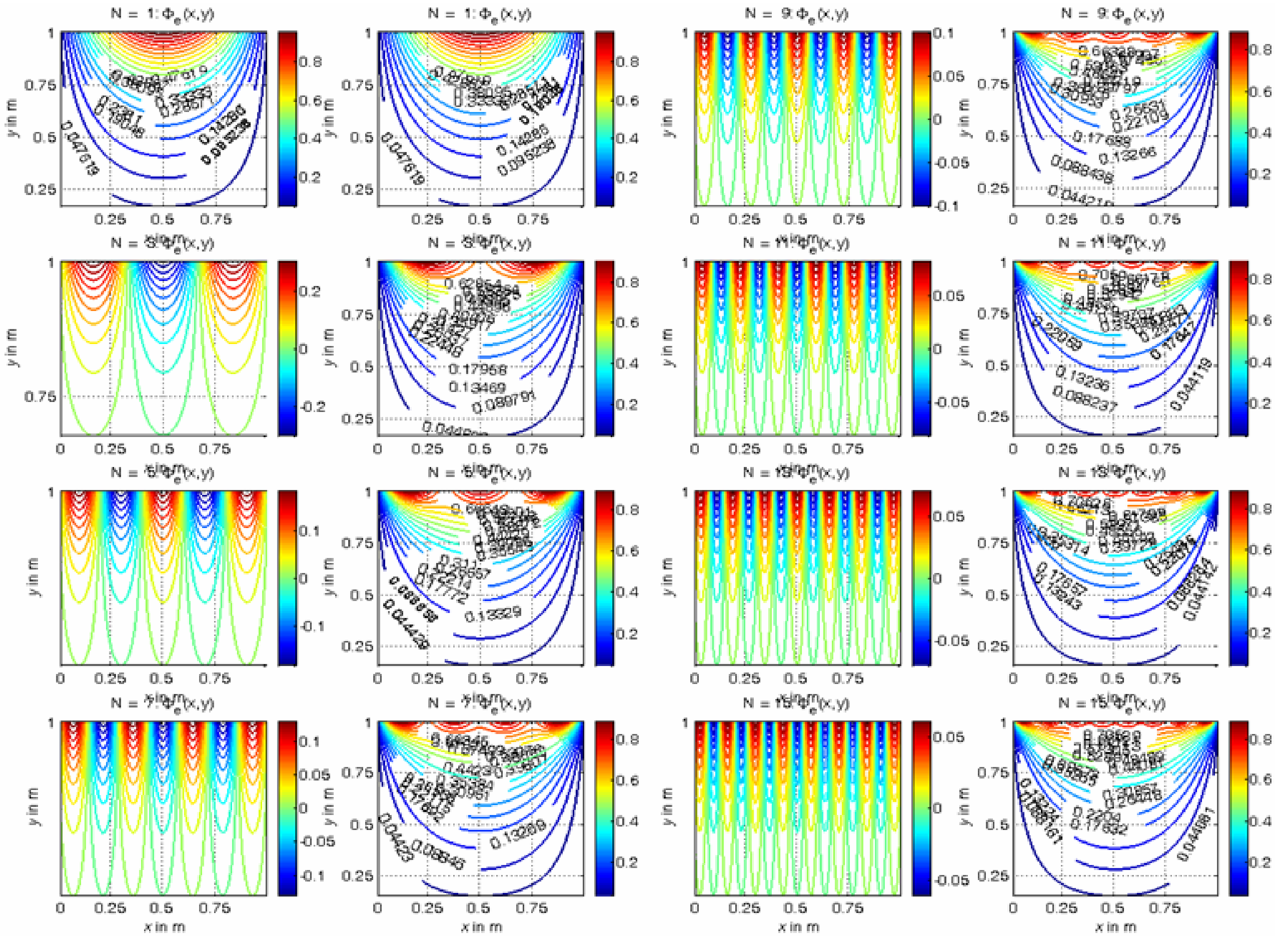
$$A_n = \begin{cases} \dfrac{4U_0}{n\pi}\dfrac{1}{\sinh\left(\dfrac{n\pi}{a}b\right)} & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots \end{cases}$$

Complete Solution / Komplette Lösung

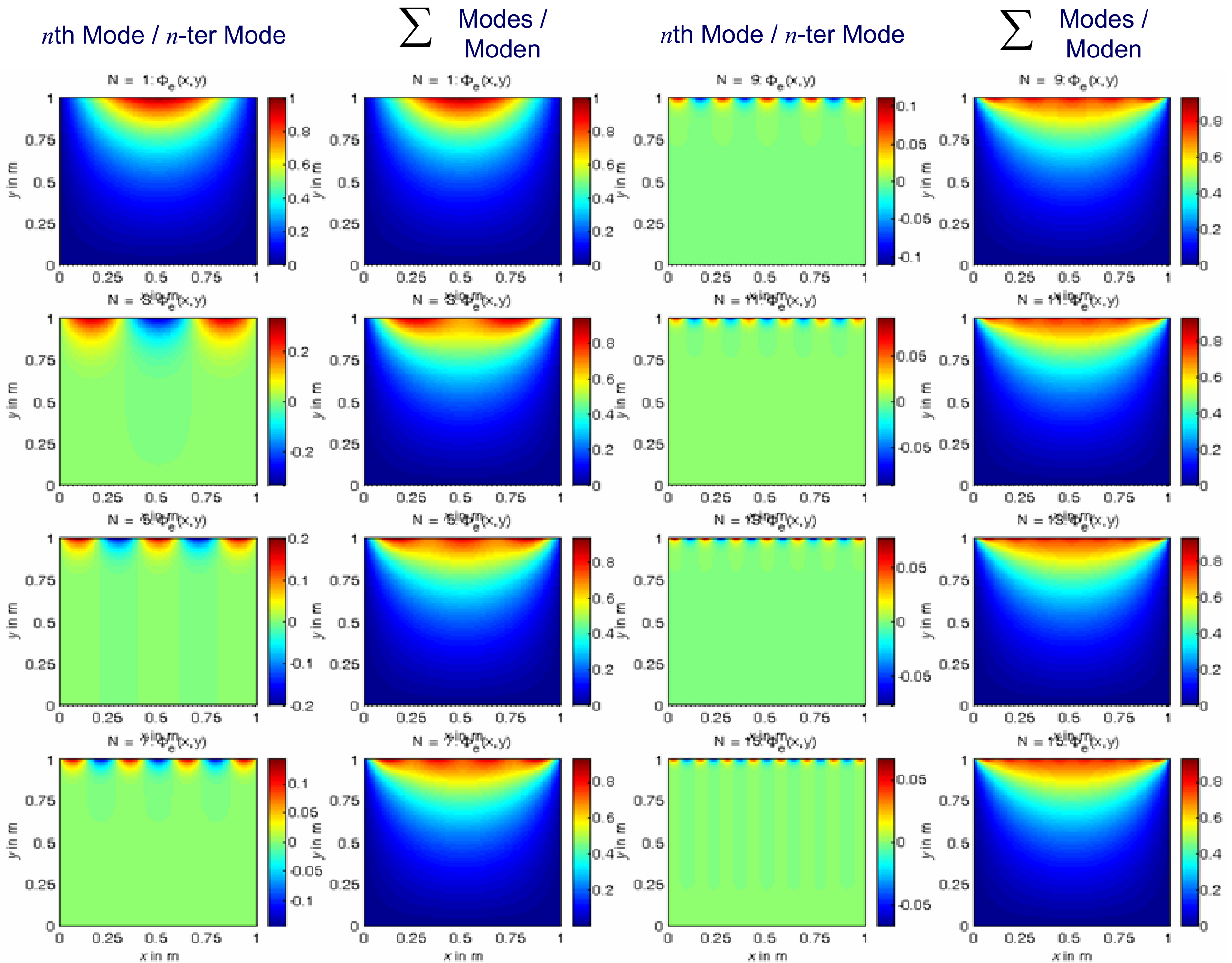
$$\Rightarrow \quad \Phi_{\mathrm{e}}(x,y) = \frac{4U_0}{\pi} \sum_{\substack{n=1 \\ \text{odd / ungerade}}}^{\infty} \frac{1}{n} \frac{\sinh\left(\dfrac{n\pi}{a}y\right)}{\sinh\left(\dfrac{n\pi}{a}b\right)} \sin\left(\frac{n\pi}{a}x\right)$$

ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

nth Mode / n-ter Mode	$\sum$ Modes / Moden	nth Mode / n-ter Mode	$\sum$ Modes / Moden

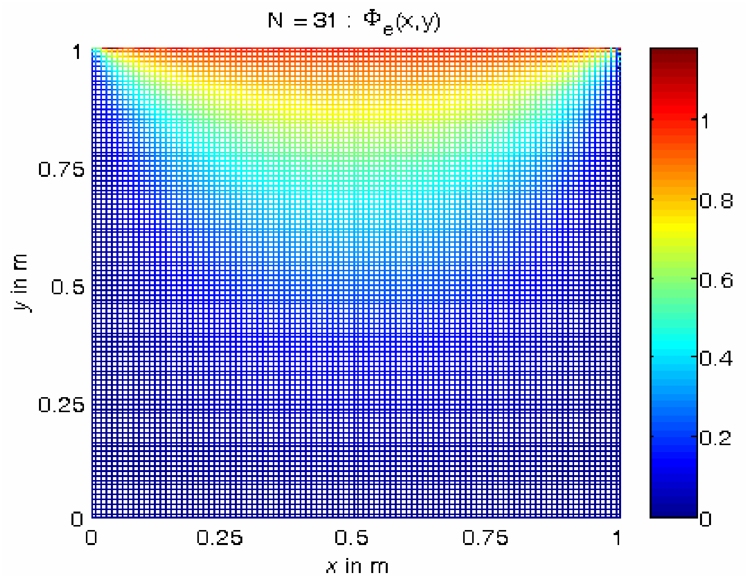


ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

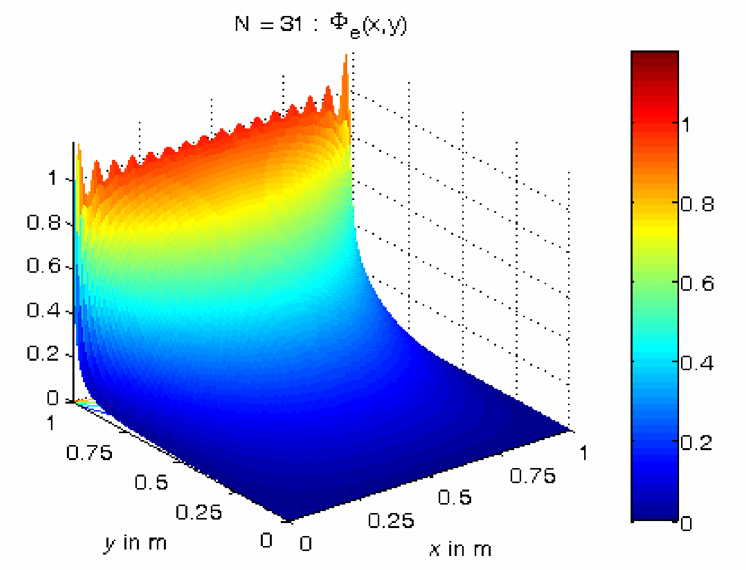


ES Fields – Separation of Variables – Example / ES Felder – Separation der Variablen – Beispiel (...)

$\sum_{n=1}^{31}$ Modes / Moden



$\sum_{n=1}^{31}$ Modes / Moden

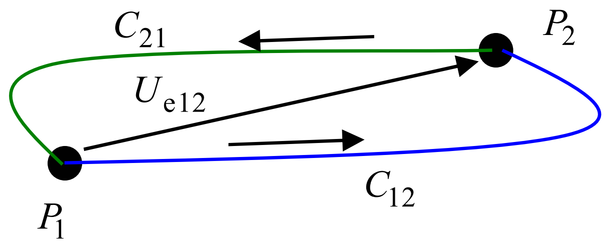


ES Fields – Electric Voltage / ES Felder – Elektrische Spannung

$$\nabla \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$$

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R = 0$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla \Phi_\mathrm{e}(\underline{\mathbf{R}})$$



$$C = C_{12} + C_{21}$$

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R = \int_{C_{12}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R + \int_{C_{21}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R = 0$$

$$\int_{C_{12}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R = -\int_{C_{21}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R$$

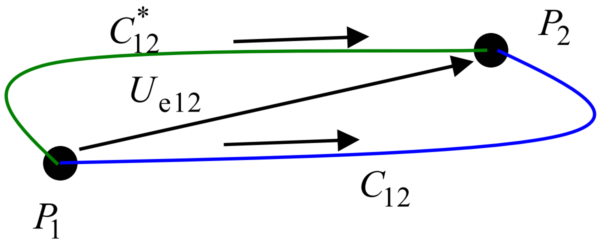
$$\begin{aligned}
\int_{C_{12}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R &= \int_{P_1}^{P_2} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R \\
&= -\int_{P_1}^{P_2} \nabla \Phi_\mathrm{e}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R \\
&= -\int_{P_1}^{P_2} \underline{\mathbf{s}} \bullet \nabla \Phi_\mathrm{e}(\underline{\mathbf{R}})\mathrm{d}R \\
&= -\int_{P_1}^{P_2} \frac{\mathrm{d}}{\mathrm{d}R} \Phi_\mathrm{e}(\underline{\mathbf{R}})\mathrm{d}R \\
&= -\int_{P_1}^{P_2} \mathrm{d}\Phi_\mathrm{e}(\underline{\mathbf{R}}) \\
&= -\Phi_\mathrm{e}(\underline{\mathbf{R}})\Big|_{P_1}^{P_2} \\
&= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}(P_2)) - \Phi_\mathrm{e}(\underline{\mathbf{R}}(P_1))\right] \\
&= \Phi_\mathrm{e}(\underline{\mathbf{R}}(P_1)) - \Phi_\mathrm{e}(\underline{\mathbf{R}}(P_2)) \\
&= U_{e12}
\end{aligned}$$

ES Fields – Electric Voltage / ES Felder – Elektrische Spannung

$$\nabla \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}}$$

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R = 0$$

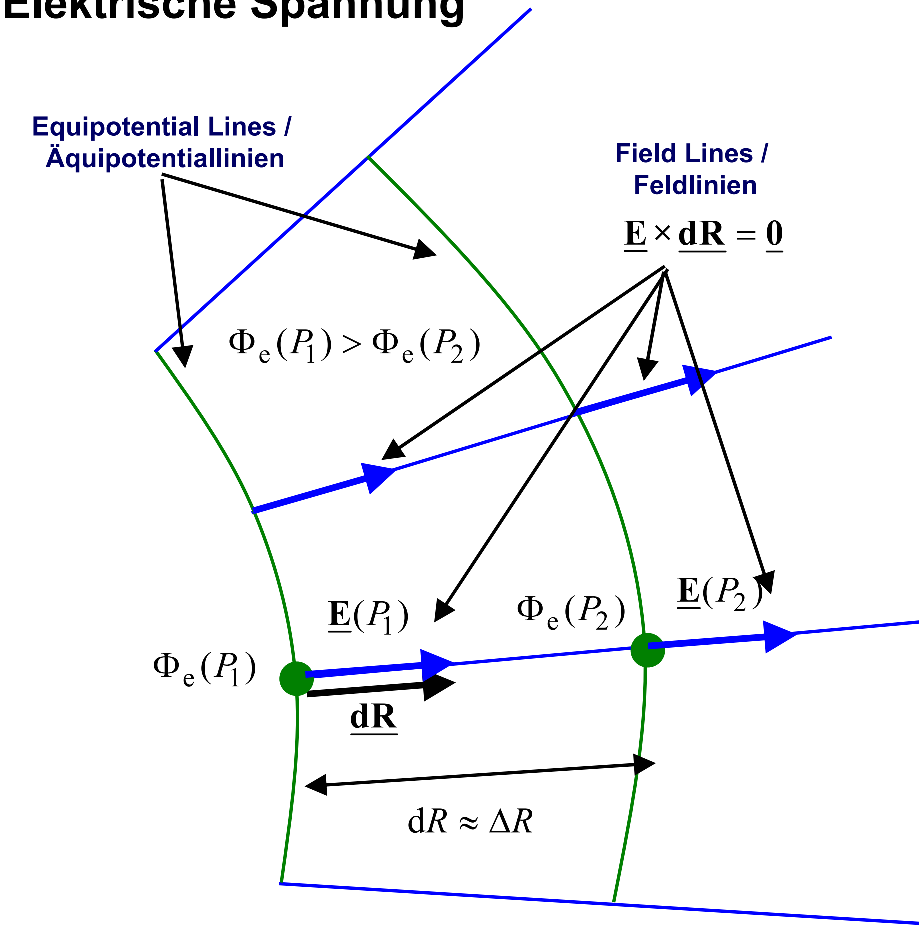
$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}})$$



$$U_{\mathrm{e}12} = \int_{C_{12}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R$$

$$= \int_{C_{12}^{*}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R$$

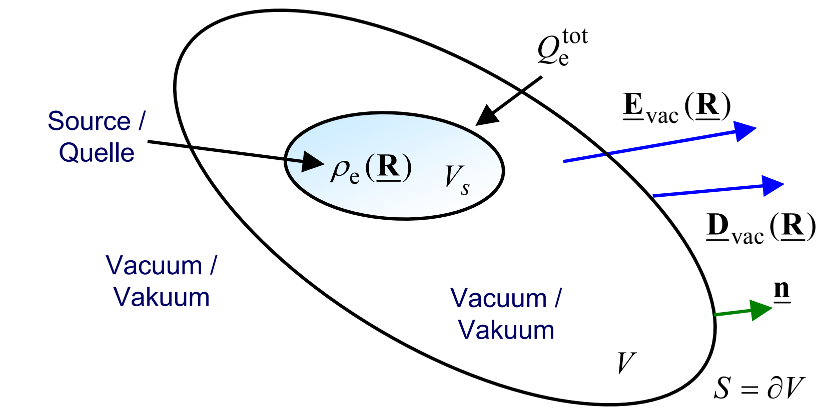
In Electrostatics is the Electric Voltage Independent of the Integration Path. /
In der Elektrostatik ist die elektrische Spannung unabhängig vom Integrationsweg.



Because / Weil

$$\nabla \times \underline{\mathbf{E}}(\underline{\mathbf{R}}) = \underline{\mathbf{0}} \qquad \oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{s}}\, \mathrm{d}R = 0 \qquad \underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}}) \approx -\frac{\Phi_{\mathrm{e}}(P_2) - \Phi_{\mathrm{e}}(P_1)}{\Delta R}\, \underline{\mathbf{s}}$$

ES Fields – Electric Polarization of Materials / ES Felder – Elektrische Polarisation von Materialien



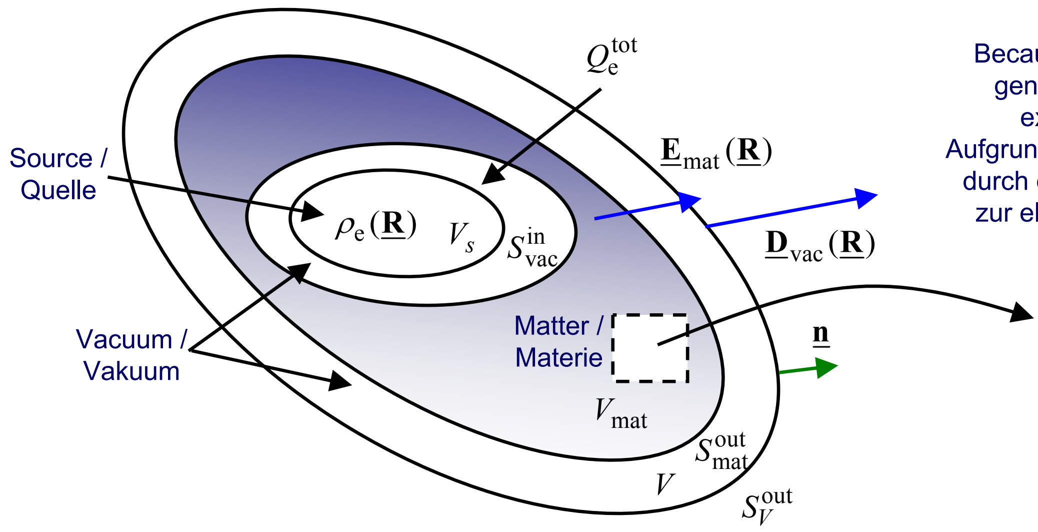
$$\oiint_{S_s=\partial V_s} \underline{\mathbf{D}}_{vac}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{n}} \, dS = Q_e^{tot}$$

$$= \oiint_{S=\partial V} \underline{\mathbf{D}}_{vac}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{n}} \, dS$$

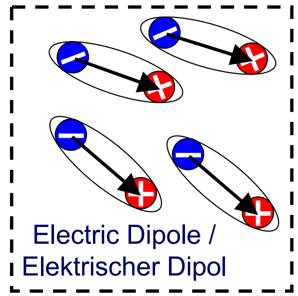
$$\underline{\mathbf{D}}_{vac}(\underline{\mathbf{R}}) = \varepsilon_0 \underline{\mathbf{E}}_{vac}(\underline{\mathbf{R}}) = \varepsilon_0 \underbrace{\varepsilon_{r,vac}}_{=1} \underline{\mathbf{E}}_{vac}(\underline{\mathbf{R}})$$

Vacuum / Vakuum

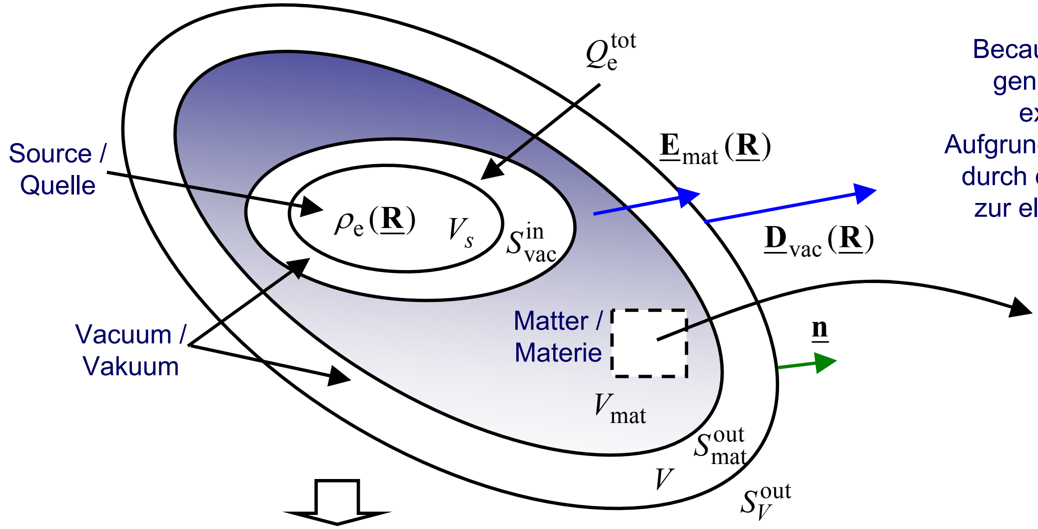
$$\varepsilon_{r,vac} = 1$$



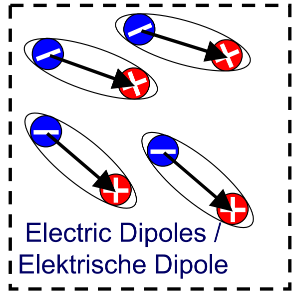
Because of the applied E Field, which is generated by the source, the matter exhibits an electric polarization. /
Aufgrund des anliegenden E-Feldes, welches durch die Quelle generiert wird, kommt es zur elektrischen Polarisation der Materie



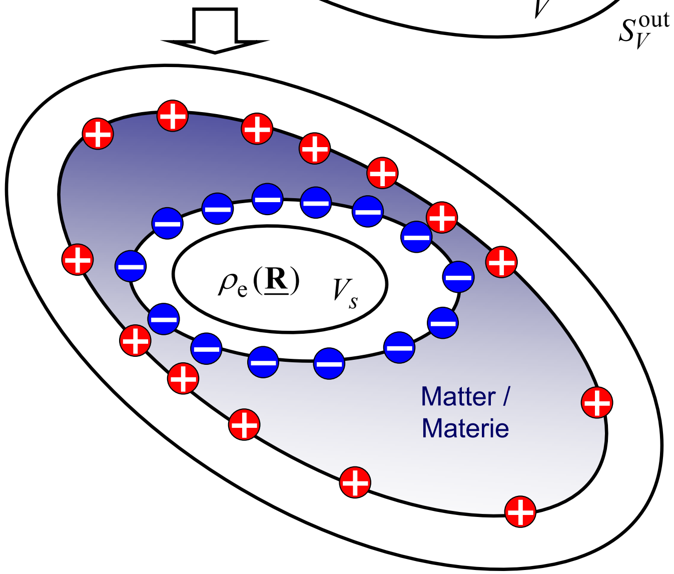
ES Fields – Electric Polarization of Materials /
ES Felder – Elektrische Polarisation von Materialien



Because of the applied E Field, which is generated by the source, the matter exhibits an electric polarization. /
Aufgrund des anliegenden E-Feldes, welches durch die Quelle generiert wird, kommt es zur elektrischen Polarisation der Materie



$\underline{\mathbf{p}}_e^{(i)}, i = 1, 2, 3, 4$



Total Electric Dipole Density / Elektrisches Gesamtdipolmoment

$$\iiint_{V_M} \underline{\mathbf{P}}_e(\underline{\mathbf{R}})\,\mathrm{d}V = \sum_{i=1}^{N} \underline{\mathbf{p}}_e^{(i)}$$

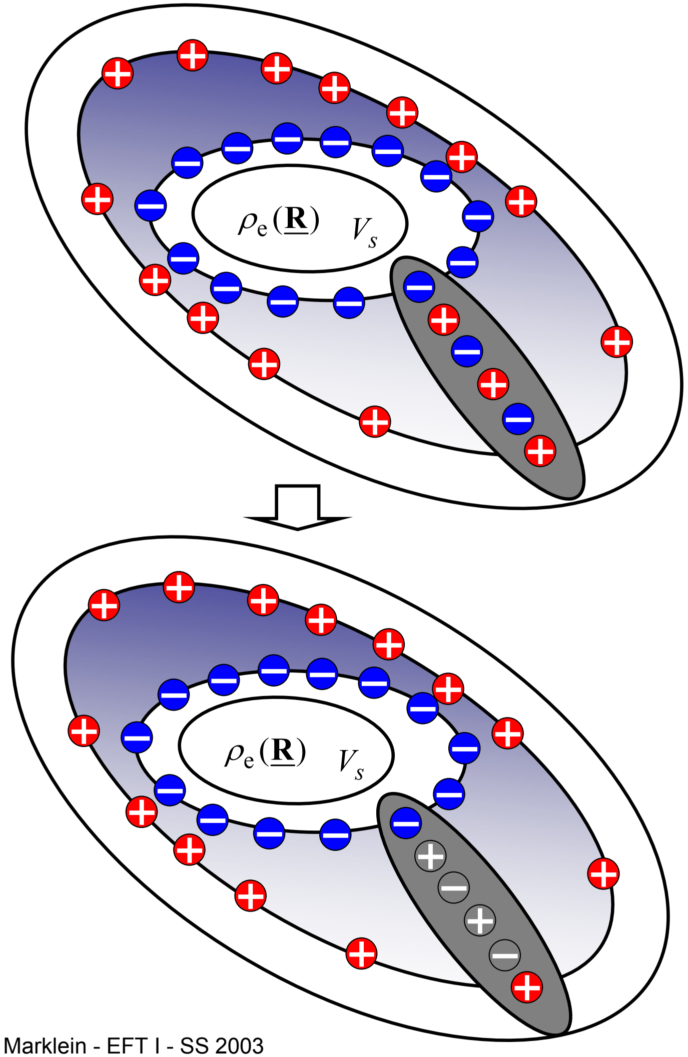
Electric Dipole Moment Density / Elektrische Dipolmomentendichte

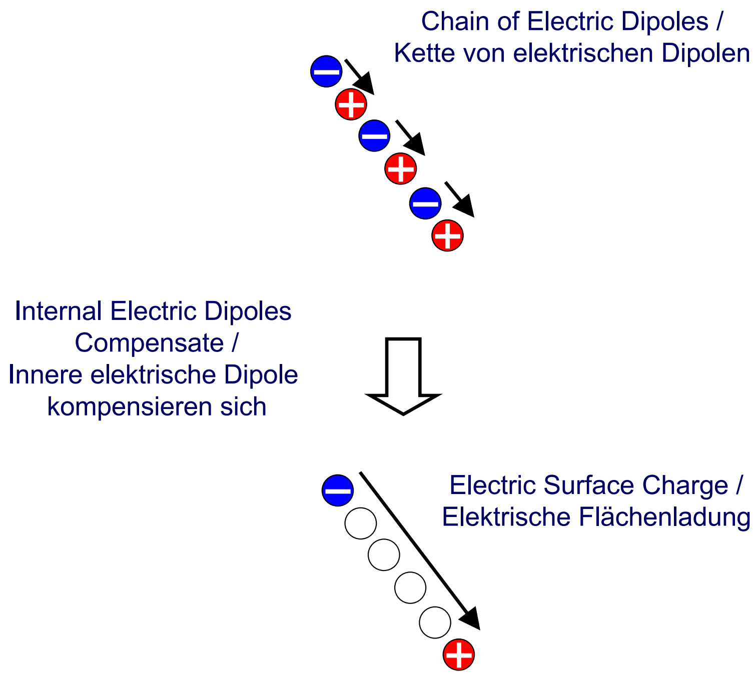
$$\underline{\boldsymbol{\pi}}_e^{(i)}(\underline{\mathbf{R}}) = \underline{\mathbf{p}}_e^{(i)}\delta\left(\underline{\mathbf{R}} - \underline{\mathbf{R}}^{(i)}\right)$$

Total Electric Dipole Moment Density /
Elektrische Gesamtdipolmomentendichte

$$\underline{\mathbf{P}}_e(\underline{\mathbf{R}}) = \sum_{i=1}^{N} \underline{\boldsymbol{\pi}}_e^{(i)}(\underline{\mathbf{R}})$$

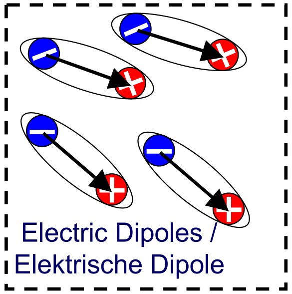
ES Fields – Electric Polarization of Materials / ES Felder – Elektrische Polarisation von Materialien





ES Fields – Electric Polarization of Materials / ES Felder – Elektrische Polarisation von Materialien

$$\iiint_{V_M} \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}})\,\mathrm{d}V = \sum_{i=1}^{N} \underline{\mathbf{p}}_{\mathrm{e}}^{(i)}$$



$\underline{\mathbf{p}}_{\mathrm{e}}^{(i)}, i = 1, 2, 3, 4$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \begin{cases} \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) & \underline{\mathbf{R}} \in V_{\mathrm{mat}} \\ \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) & \underline{\mathbf{R}} \in V_{\mathrm{vac}} \end{cases}$$

$$\begin{aligned} \underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \varepsilon_{\mathrm{r}}(\underline{\mathbf{R}}) \underline{\mathbf{E}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underbrace{\varepsilon_0 \left[\varepsilon_{\mathrm{r}}(\underline{\mathbf{R}}) - 1\right] \underline{\mathbf{E}}(\underline{\mathbf{R}})}_{= \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}})} \end{aligned}$$

$\varepsilon_{\mathrm{r}}(\underline{\mathbf{R}})$ Relative Permittivity / Relative Permittivität

$$\begin{aligned} \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) &= \varepsilon_0 \underbrace{\left[\varepsilon_{\mathrm{r}}(\underline{\mathbf{R}}) - 1\right]}_{= \chi_{\mathrm{e}}(\underline{\mathbf{R}})} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \chi_{\mathrm{e}}(\underline{\mathbf{R}}) \underline{\mathbf{E}}(\underline{\mathbf{R}}) \end{aligned}$$

$\chi_{\mathrm{e}}(\underline{\mathbf{R}})$ Electric Susceptibility / Elektrische Suszeptibilität

ES Fields – Electric Polarization of Materials / ES Felder – Elektrische Polarisation von Materialien

General Case / Allgemeiner Fall

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}})$$

Isotropic Case / Isotroper Fall

$$\underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) = \varepsilon_0 \chi_{\mathrm{e}}(\underline{\mathbf{R}})\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \varepsilon_0 \chi_{\mathrm{e}}(\underline{\mathbf{R}})\underline{\mathbf{E}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \varepsilon_0 \left[\varepsilon_{\mathrm{r}}(\underline{\mathbf{R}}) - 1\right]\underline{\mathbf{E}}(\underline{\mathbf{R}})\end{aligned}$$

$$\chi_{\mathrm{e}}(\underline{\mathbf{R}}) = \varepsilon_{\mathrm{r}}(\underline{\mathbf{R}}) - 1$$

$$\varepsilon_{\mathrm{r}}(\underline{\mathbf{R}}) = \chi_{\mathrm{e}}(\underline{\mathbf{R}}) + 1$$

Anisotropic Case / Anisotroper Fall

$$\underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) = \varepsilon_0 \underline{\underline{\boldsymbol{\chi}}}_{\mathrm{e}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})$$

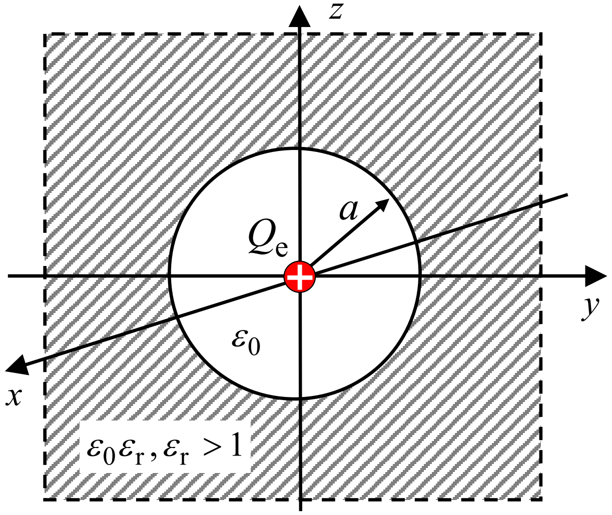
$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_{\mathrm{e}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \varepsilon_0 \underline{\underline{\boldsymbol{\chi}}}_{\mathrm{e}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}}) \\ &= \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \varepsilon_0 \left[\underline{\underline{\boldsymbol{\varepsilon}}}_{\mathrm{r}}(\underline{\mathbf{R}}) - \underline{\underline{\mathbf{I}}}\right] \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\end{aligned}$$

$$\underline{\underline{\boldsymbol{\chi}}}_{\mathrm{e}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\varepsilon}}}_{\mathrm{r}}(\underline{\mathbf{R}}) - \underline{\underline{\mathbf{I}}}$$

$$\underline{\underline{\boldsymbol{\varepsilon}}}_{\mathrm{r}}(\underline{\mathbf{R}}) = \underline{\underline{\boldsymbol{\chi}}}_{\mathrm{e}}(\underline{\mathbf{R}}) + \underline{\underline{\mathbf{I}}}$$

ES Fields – Electric Polarization of Materials – Example / ES Felder – Elektrische Polarisation von Materialien – Beispiel

Electric Point Charge Embedded in a Sphere Filled with Vacuum, which is Embedded in a Dielectric Material / Elektrische Punktladung eingebettet in einer mit Vakuum gefüllter Kugel, die in ein dielektrisches Material eingebettet ist.



$$\varepsilon(R) = \begin{cases} \varepsilon_0 & R < a \\ \varepsilon_0\varepsilon_\mathrm{r} & R > a \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi}\frac{1}{R^2}\hat{\underline{\mathbf{R}}} = \frac{Q_\mathrm{e}}{4\pi}\frac{1}{R^2}\underline{\mathbf{e}}_R$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{\underline{\mathbf{D}}(\underline{\mathbf{R}})}{\varepsilon_0} & R < a \\ \dfrac{\underline{\mathbf{D}}(\underline{\mathbf{R}})}{\varepsilon_0\varepsilon_\mathrm{r}} & R > a \end{cases}$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \begin{cases} \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}}) & R < a \\ \varepsilon_0\underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_\mathrm{e}(\underline{\mathbf{R}}) & R > a \end{cases}$$

$$\underline{\mathbf{P}}_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \underline{\mathbf{0}} & R < a \\ \varepsilon_0(\varepsilon_\mathrm{r} - 1)\underline{\mathbf{E}}(\underline{\mathbf{R}}) & R > a \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{\underline{\mathbf{D}}(\underline{\mathbf{R}})}{\varepsilon_0} & R < a \\ \dfrac{\underline{\mathbf{D}}(\underline{\mathbf{R}})}{\varepsilon_0} + \dfrac{\underline{\mathbf{P}}_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon_0} & R > a \end{cases} = \begin{cases} \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\dfrac{1}{R^2}\underline{\mathbf{e}}_R & R < a \\ \dfrac{Q_\mathrm{e}}{4\pi\varepsilon_0}\dfrac{1}{R^2}\underline{\mathbf{e}}_R - \dfrac{\underline{\mathbf{P}}_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon_0} & R > a \end{cases}$$

ES Fields – Electric Polarization of Materials – Example / ES Felder – Elektrische Polarisation von Materialien – Beispiel (...)

$$\underline{\mathbf{P}}_\mathrm{e}(\underline{\mathbf{R}}) = \begin{cases} \underline{\mathbf{0}} & R < a \\ \varepsilon_0(\varepsilon_\mathrm{r}-1)\underline{\mathbf{E}}(\underline{\mathbf{R}}) & R > a \end{cases}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \begin{cases} \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R & R < a \\ \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R - \frac{\underline{\mathbf{P}}_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon_0} & R > a \end{cases}$$

$$= \begin{cases} \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R & R < a \\ \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R - \frac{\varepsilon_0(\varepsilon_\mathrm{r}-1)\underline{\mathbf{E}}(\underline{\mathbf{R}})}{\varepsilon_0} & R > a \end{cases}$$

$$= \begin{cases} \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R & R < a \\ \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R - (\varepsilon_\mathrm{r}-1)\underline{\mathbf{E}}(\underline{\mathbf{R}}) & R > a \end{cases}$$

$R > a$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R - (\varepsilon_\mathrm{r}-1)\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) + (\varepsilon_\mathrm{r}-1)\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R$$

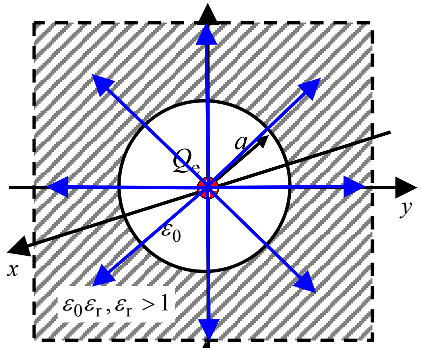
$$\varepsilon_\mathrm{r}\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \frac{Q_\mathrm{e}}{4\pi\varepsilon_0\varepsilon_\mathrm{r}}\frac{1}{R^2}\underline{\mathbf{e}}_R$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \begin{cases} \frac{Q_\mathrm{e}}{4\pi\varepsilon_0}\frac{1}{R^2}\underline{\mathbf{e}}_R & R < a \\ \frac{Q_\mathrm{e}}{4\pi\varepsilon_0\varepsilon_\mathrm{r}}\frac{1}{R^2}\underline{\mathbf{e}}_R & R > a \end{cases}$$

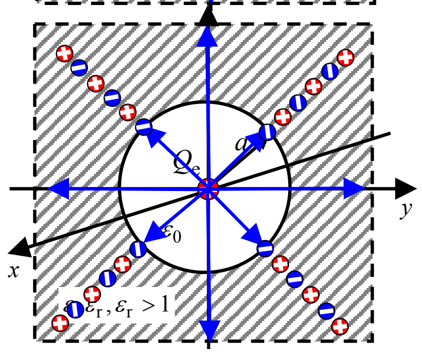
ES Fields – Electric Polarization of Materials – Example / ES Felder – Elektrische Polarisation von Materialien – Beispiel (...)

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \frac{Q_e}{4\pi}\frac{1}{R^2}\underline{\mathbf{e}}_R$$

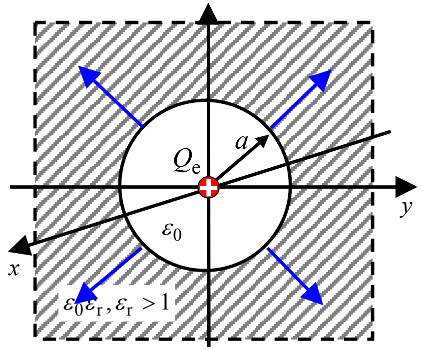


$$\varepsilon(R) = \begin{cases} \varepsilon_0 & R < a \\ \varepsilon_0\varepsilon_r & R > a \end{cases}$$

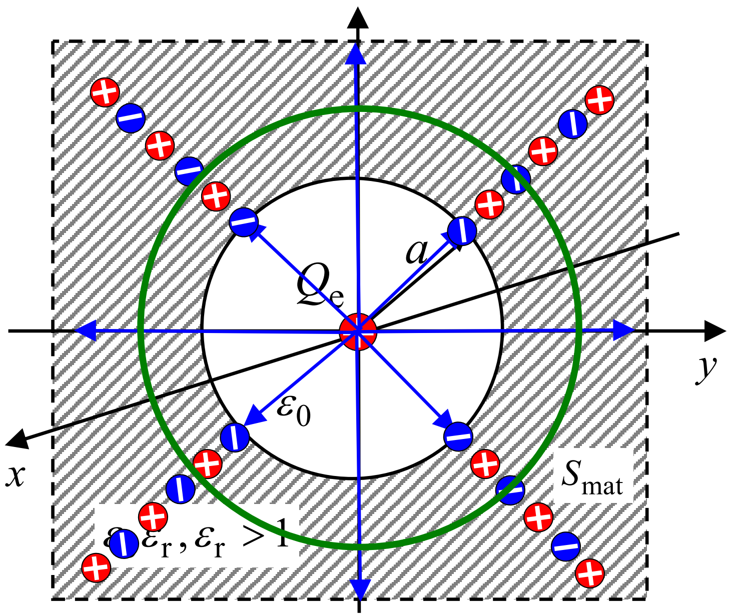
$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = \begin{cases} \dfrac{Q_e}{4\pi\varepsilon_0}\dfrac{1}{R^2}\underline{\mathbf{e}}_R & R < a \\ \dfrac{Q_e}{4\pi\varepsilon_0\varepsilon_r}\dfrac{1}{R^2}\underline{\mathbf{e}}_R & R > a \end{cases}$$



$$\underline{\mathbf{P}}_e(\underline{\mathbf{R}}) = \begin{cases} \underline{\mathbf{0}} & R < a \\ \dfrac{Q_e}{4\pi}\dfrac{\varepsilon_r - 1}{\varepsilon_r}\dfrac{1}{R^2}\underline{\mathbf{e}}_R & R > a \end{cases}$$



ES Fields – Electric Polarization of Materials – Example / ES Felder – Elektrische Polarisation von Materialien – Beispiel (...)



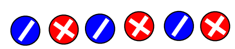
$$\oiint_{S_{mat}} \underline{\mathbf{D}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dS}} = Q_e$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}}) = \varepsilon_0 \underline{\mathbf{E}}(\underline{\mathbf{R}}) + \underline{\mathbf{P}}_e(\underline{\mathbf{R}})$$

$$\oiint_{S_{mat}} \underline{\mathbf{D}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dS}} = \varepsilon_0 \oiint_{S_{mat}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dS}} + \oiint_{S_{mat}} \underline{\mathbf{P}}_e(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dS}}$$
$$= Q_e$$

$$\varepsilon_0 \oiint_{S_{mat}} \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dS}} = Q_e - \underbrace{\oiint_{S_{mat}} \underline{\mathbf{P}}_e(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dS}}}_{=-Q_e^{pol}}$$
$$= Q_e + Q_e^{pol}$$

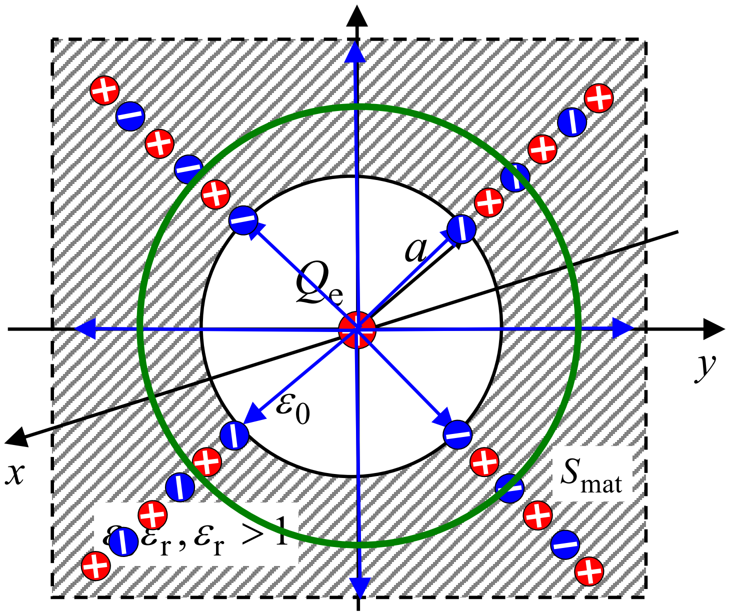
$$Q_e = Q_e^{unpaired} \qquad Q_e^{pol} = Q_e^{paired}$$



Unpaired Electric Charge / Ungepaarte elektrische Ladungen

Paired Electric Charge / Gepaarte elektrische Ladungen

ES Fields – Electric Polarization of ES Felder – Elektrische Polarisation von Materialien – Beispiel (...)



$$Q_e = \oiint_{S_{mat}} \underline{\mathbf{D}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dS}} = \iiint_{V_{mat}} \nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}})\, dV$$

$$\nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) = \rho_e(\underline{\mathbf{R}}) = \rho_e^{\text{unpaired}}(\underline{\mathbf{R}})$$

$$Q_e^{\text{pol}} = -\oiint_{S_{mat}} \underline{\mathbf{P}}_e(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dS}} = -\iiint_{V_{mat}} \nabla \bullet \underline{\mathbf{P}}_e(\underline{\mathbf{R}})\, dV$$

$$\nabla \bullet \underline{\mathbf{P}}_e(\underline{\mathbf{R}}) = -\rho_e^{\text{pol}}(\underline{\mathbf{R}}) = -\rho_e^{\text{paired}}(\underline{\mathbf{R}})$$

End of the 10th Lecture /
Ende der 10. Vorlesung