Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

10th Lecture / 10. Vorlesung

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FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

FIT

Maxwell's equations in integral form / Maxwellsche Gleichungen in Integralform

Maxwell's grid equations / Maxwellsche Gittergleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$[\mathbf{S}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$\left.\begin{aligned}
&\oint\!\!\oint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV \\
&\oint\!\!\oint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV
\end{aligned}\right\}\ ?$$

FIT Discretization of the 3rd Maxwell Equation / FIT-Diskretisierung der 3. Maxwellschen Gleichung

Integral form / Integralform

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\, dV$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

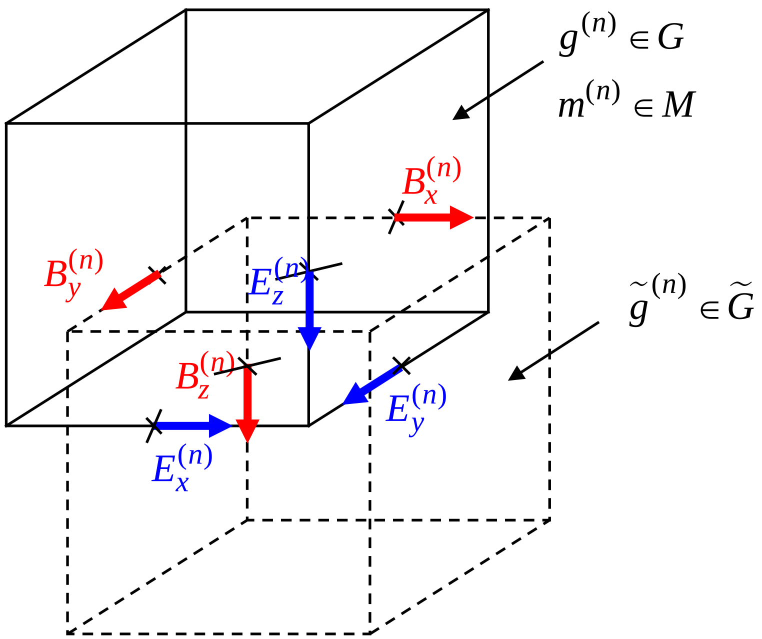
$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] \bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\, dV$$

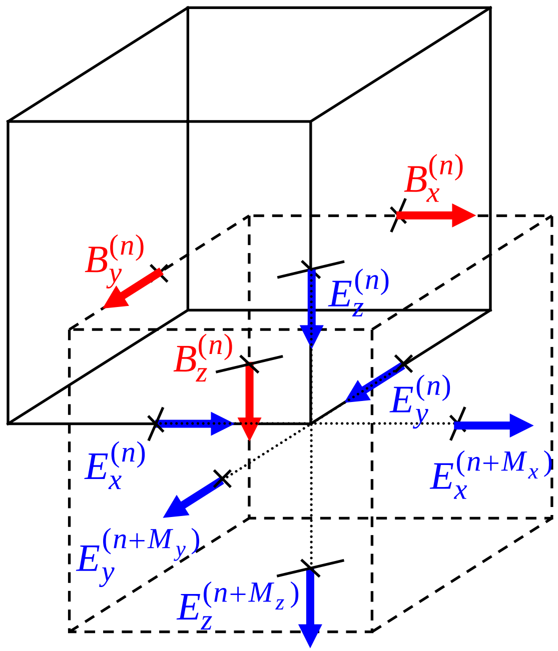
Differential form / Differentialform

$$\nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)$$

$$\nabla \bullet \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] = \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

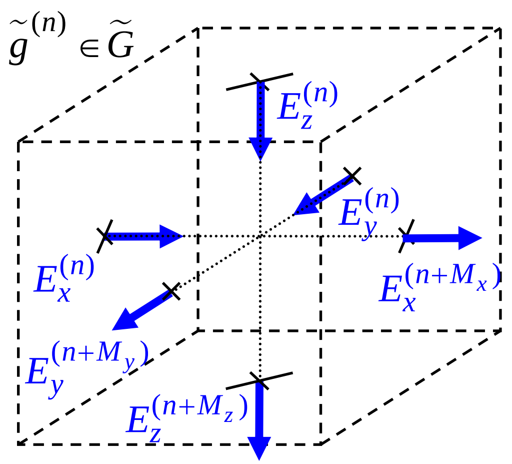


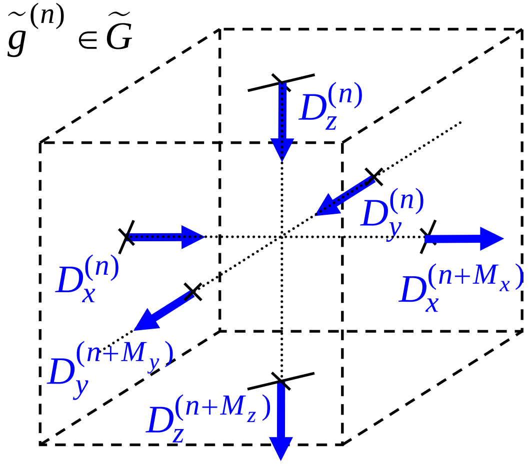


FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\, dV$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] \bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\, dV$$





$$D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

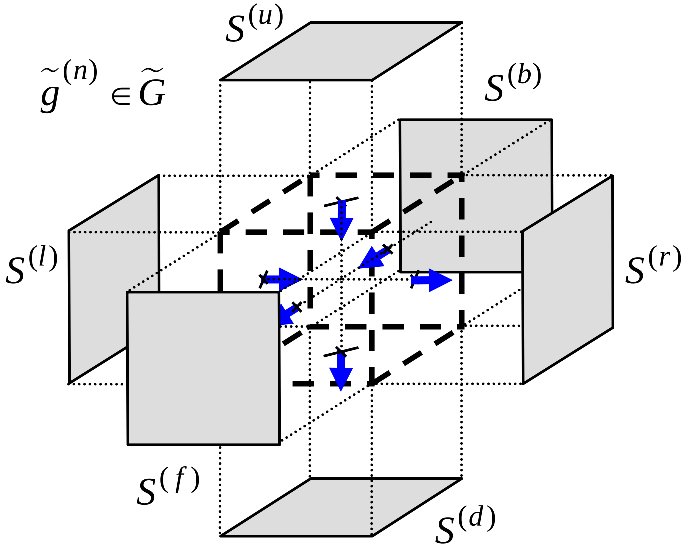
$$D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$D_y^{(n)} = \tilde{\varepsilon}_{yy}^{(n)} E_x^{(n)}$$

$$D_y^{(n+M_y)} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}$$

$$D_z^{(n)} = \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}$$

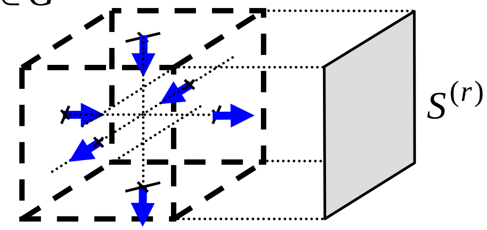
$$D_z^{(n+M_z)} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}$$



$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} + \iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$
$$+ \iint_{S^{(f)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} + \iint_{S^{(b)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$
$$+ \iint_{S^{(d)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} + \iint_{S^{(u)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$

FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$\tilde{g}^{(n)} \in \tilde{G}$



$$\mathrm{d}\underline{\mathbf{S}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

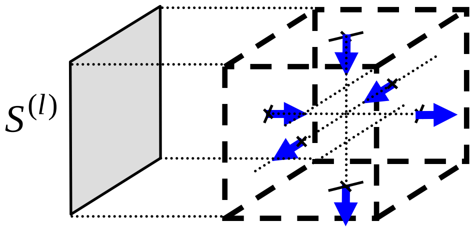
$$S^{(r)}: \quad D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

$$\begin{aligned}
\iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} &= \iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathrm{e}}_x\,\mathrm{d}y\mathrm{d}z \\
&= \iint_{S^{(r)}} D_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= \iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}}) E_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= E_x^{(n+M_x)}(t) \underbrace{\iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}y\mathrm{d}z}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\Delta y \Delta z} + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right] \\
&= \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right] \\
&= D_x^{(n+M_x)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]
\end{aligned}$$

$$\begin{aligned}
&\iint_{S^{(r)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S \\
&= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n+M_x)} + \varepsilon_{xx}^{(n+M_x+M_y)} + \varepsilon_{xx}^{(n+M_x+M_z)} + \varepsilon_{xx}^{(n+M_x+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}} \Delta y \Delta z \\
&= \tilde{\varepsilon}_{xx}^{(n+M_x)} \Delta y \Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$$\tilde{g}^{(n)} \in \widetilde{G}$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$

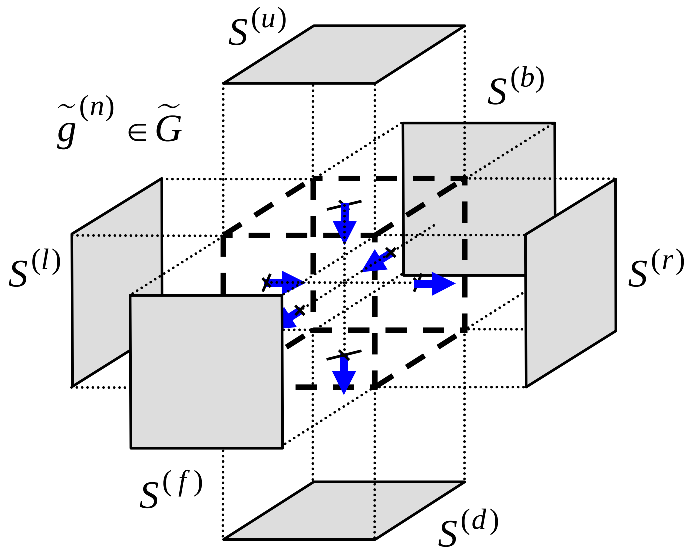
$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = -\underline{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

$$S^{(l)}: \quad D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

$$\begin{aligned}
\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} &= -\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathrm{e}}_x\,\mathrm{d}y\mathrm{d}z \\
&= -\iint_{S^{(l)}} D_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= -\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}}) E_x(\underline{\mathbf{R}},t)\,\mathrm{d}y\mathrm{d}z \\
&= -E_x^{(n)}(t) \underbrace{\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}y\mathrm{d}z}_{=\tilde{\varepsilon}_{xx}^{(n)}\Delta y \Delta z} + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right] \\
&= -\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t) \Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right] \\
&= -D_x^{(n)}(t) \Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]
\end{aligned}$$

$$\begin{aligned}
&\iint_{S^{(l)}} \varepsilon_{xx}(\underline{\mathbf{R}})\,\mathrm{d}S \\
&= \underbrace{\frac{1}{4}\left[\varepsilon_{xx}^{(n)} + \varepsilon_{xx}^{(n+M_y)} + \varepsilon_{xx}^{(n+M_z)} + \varepsilon_{xx}^{(n+M_y+M_z)}\right]}_{=\tilde{\varepsilon}_{xx}^{(n)}} \Delta y \Delta z \\
&= \tilde{\varepsilon}_{xx}^{(n)} \Delta y \Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$$S^{(l)}: \quad D_x^{(n)} = \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}$$

$$S^{(r)}: \quad D_x^{(n+M_x)} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}$$

$$S^{(b)}: \quad D_y^{(n)} = \tilde{\varepsilon}_{yy}^{(n)} E_x^{(n)}$$

$$S^{(f)}: \quad D_y^{(n+M_y)} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}$$

$$S^{(u)}: \quad D_z^{(n)} = \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}$$

$$S^{(d)}: \quad D_z^{(n+M_z)} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \sum_{i=1}^{6} \iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}}$$

$$\iint_{S^{(r)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]$$
$$= D_x^{(n+M_x)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]$$

$$\iint_{S^{(l)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]$$
$$= -D_x^{(n)}(t)\Delta y \Delta z + \mathrm{O}\left[(\Delta y)^3 \Delta z + \Delta y (\Delta z)^3\right]$$

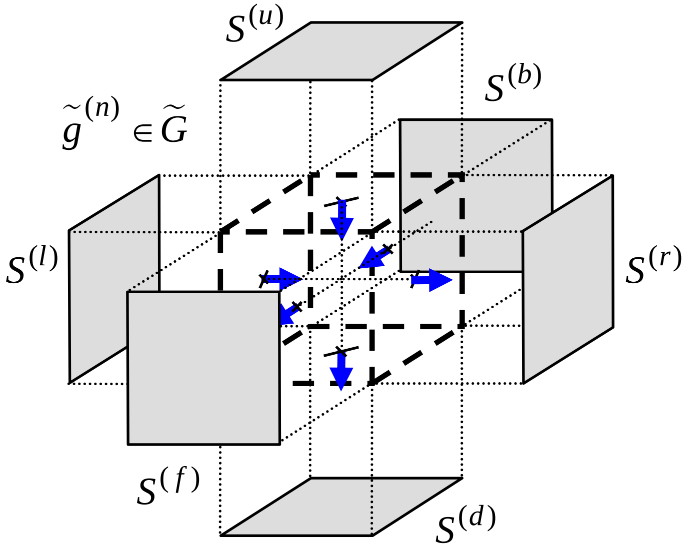
$$\iint_{S^{(f)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t)\Delta x \Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x (\Delta z)^3\right]$$
$$= D_y^{(n+M_y)}(t)\Delta x \Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x (\Delta z)^3\right]$$

$$\iint_{S^{(b)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x \Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x (\Delta z)^3\right]$$
$$= -D_y^{(n)}(t)\Delta x \Delta z + \mathrm{O}\left[(\Delta x)^3 \Delta z + \Delta x (\Delta z)^3\right]$$

$$\iint_{S^{(d)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t)\Delta x \Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x (\Delta y)^3\right]$$
$$= D_z^{(n+M_z)}(t)\Delta x \Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x (\Delta y)^3\right]$$

$$\iint_{S^{(u)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = -\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x \Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x (\Delta y)^3\right]$$
$$= -D_z^{(n)}(t)\Delta x \Delta y + \mathrm{O}\left[(\Delta x)^3 \Delta y + \Delta x (\Delta y)^3\right]$$

FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)



$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$= \tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t)\Delta y\Delta z - \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z$$
$$+\tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t)\Delta x\Delta z - \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z$$
$$+\tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t)\Delta x\Delta y - \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$
$$= D_x^{(n+M_x)}(t)\Delta y\Delta z - D_x^{(n)}(t)\Delta y\Delta z$$
$$+D_y^{(n+M_y)}(t)\Delta x\Delta z - D_y^{(n)}(t)\Delta x\Delta z$$
$$+D_z^{(n+M_z)}(t)\Delta x\Delta y - D_z^{(n)}(t)\Delta x\Delta y$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$= \left[\tilde{\varepsilon}_{xx}^{(n+M_x)} E_x^{(n+M_x)}(t) - \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\right]\Delta y\Delta z$$
$$+\left[\tilde{\varepsilon}_{yy}^{(n+M_y)} E_y^{(n+M_y)}(t) - \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\right]\Delta x\Delta z$$
$$+\left[\tilde{\varepsilon}_{zz}^{(n+M_z)} E_z^{(n+M_z)}(t) - \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\right]\Delta x\Delta y$$
$$= \left[D_x^{(n+M_x)}(t) - D_x^{(n)}(t)\right]\Delta y\Delta z$$
$$+\left[D_y^{(n+M_y)}(t) - D_y^{(n)}(t)\right]\Delta x\Delta z$$
$$+\left[D_z^{(n+M_z)}(t) - D_z^{(n)}(t)\right]\Delta x\Delta y$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \sum_{i=1}^{6}\iint_{S^{(i)}} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$
$$= \left(S_{M_x} - I\right)\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z$$
$$+\left(S_{M_y} - I\right)\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z$$
$$+\left(S_{M_z} - I\right)\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$
$$= \left(S_{M_x} - I\right) D_x^{(n)}(t)\Delta y\Delta z$$
$$+\left(S_{M_y} - I\right) D_y^{(n)}(t)\Delta x\Delta z$$
$$+\left(S_{M_z} - I\right) D_z^{(n)}(t)\Delta x\Delta y$$

FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\cdot \underline{\mathbf{dS}} = \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x\Delta y$$

$$= \underbrace{\begin{bmatrix} S_{M_x}-I & S_{M_y}-I & S_{M_z}-I \end{bmatrix}}_{=[\widetilde{\mathrm{div}}]} \underbrace{\begin{bmatrix} \tilde{\varepsilon}_{zz}^{(n)} & & \\ & \tilde{\varepsilon}_{yy}^{(n)} & \\ & & \tilde{\varepsilon}_{zz}^{(n)} \end{bmatrix}}_{=[\tilde{\varepsilon}]^{(n)}} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta z \end{bmatrix}}_{=[\tilde{S}]} \underbrace{\begin{Bmatrix} E_x^{(n)}(t) \\ E_y^{(n)}(t) \\ E_z^{(n)}(t) \end{Bmatrix}}_{=\{E\}^{(n)}(t)}$$

$$= [\widetilde{\mathrm{div}}][\tilde{\varepsilon}]^{(n)}[\tilde{S}]\{E\}^{(n)}(t)$$

$$= \left(S_{M_x}-I\right) D_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right) D_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right) D_z^{(n)}(t)\Delta x\Delta y$$

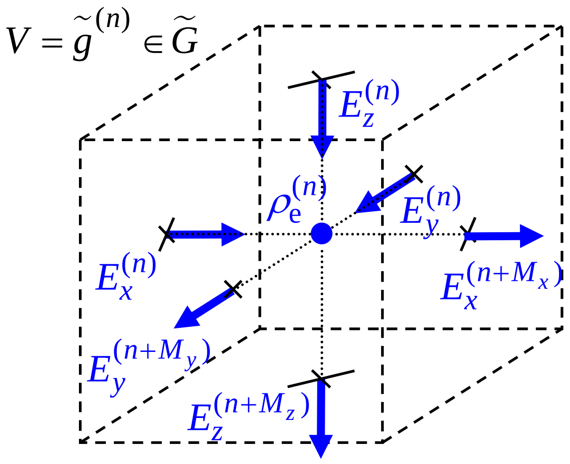
$$= \underbrace{\begin{bmatrix} S_{M_x}-I & S_{M_y}-I & S_{M_z}-I \end{bmatrix}}_{=[\widetilde{\mathrm{div}}]} \underbrace{\begin{bmatrix} \Delta y\Delta z & & \\ & \Delta x\Delta z & \\ & & \Delta x\Delta z \end{bmatrix}}_{=[S]} \underbrace{\begin{Bmatrix} D_x^{(n)}(t) \\ D_y^{(n)}(t) \\ D_z^{(n)}(t) \end{Bmatrix}}_{=\{D\}^{(n)}(t)}$$

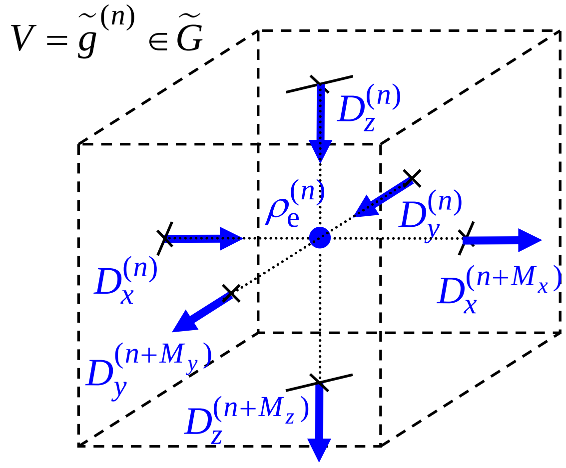
$$= [\widetilde{\mathrm{div}}][\tilde{S}]\{D\}^{(n)}(t)$$

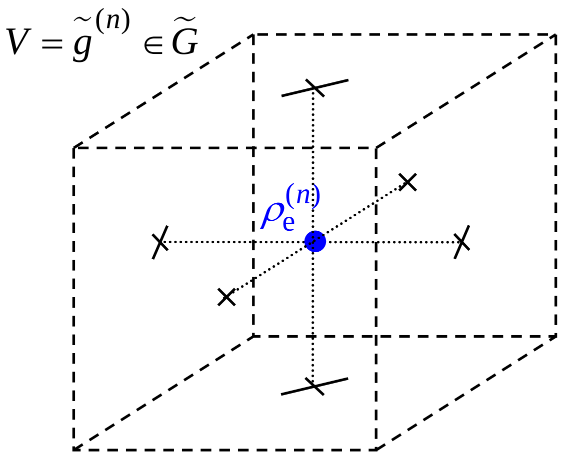
FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$







$$\begin{aligned}
\iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV &= \rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z + \mathcal{O}\left[(\Delta x)^3\Delta y\Delta z + \Delta x(\Delta y)^3\Delta z + \Delta x\Delta y(\Delta z)^3\right]\\
&= Q_{\mathrm{e}}^{(n)}(t) + \mathcal{O}\left[(\Delta x)^3\Delta y\Delta z + \Delta x(\Delta y)^3\Delta z + \Delta x\Delta y(\Delta z)^3\right]\\
&= \rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z + \mathcal{O}\left[(\Delta x)^5\right] \qquad \text{if} \quad \Delta x \approx \Delta y \approx \Delta z\\
&= Q_{\mathrm{e}}^{(n)}(t) + \mathcal{O}\left[(\Delta x)^5\right] \qquad \text{if} \quad \Delta x \approx \Delta y \approx \Delta z
\end{aligned}$$

FIT Discretization of the 3rd Maxwell Equation (...) / FIT-Diskretisierung der 3. Maxwellschen Gleichung (...)

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = Q_\mathrm{e}(t)$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = Q_\mathrm{e}(t)$$

Discrete grid equations in local matrix form /
Diskrete Gittergleichungen in lokaler Matrixform

$$\left[\widetilde{\mathrm{div}}\right]\left[\widetilde{\varepsilon}\right]^{(n)}\left[\widetilde{S}\right]\left\{E\right\}^{(n)}(t) = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z = Q_\mathrm{e}^{(n)}(t)$$

$$\left[\widetilde{\mathrm{div}}\right]\left[\widetilde{S}\right]\left\{D\right\}^{(n)}(t) = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z = Q_\mathrm{e}^{(n)}(t)$$

Discrete grid equations in global matrix form /
Diskrete Gittergleichungen in globaler Matrixform

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{E}\right\}(t) = \left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_\mathrm{e}\right\}(t) = \left\{\mathbf{Q}_\mathrm{e}\right\}(t)$$

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{D}\right\}(t) = \left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_\mathrm{e}\right\}(t) = \left\{\mathbf{Q}_\mathrm{e}\right\}(t)$$

with / mit $\left[\widetilde{\mathbf{div}}\right] := \left[\left[\mathbf{P}_x\right],\left[\mathbf{P}_y\right],\left[\mathbf{P}_z\right]\right]_{N\times 3N}$

Discrete Local and Global Gradient, Divergence, and Curl Operators / Diskrete lokale und globale Gradienten-, Divergenz- und Rotationsoperatoren

Discrete gradient operator / Diskreter Gradientenoperator

$$[\mathbf{grad}] = \begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}_{3N\times N}$$

$$[\widetilde{\mathbf{grad}}] = \begin{bmatrix} [\mathbf{P}_x] \\ [\mathbf{P}_y] \\ [\mathbf{P}_z] \end{bmatrix}_{3N\times N}$$

Discrete curl operator / Diskreter Rotationsoperator

$$[\mathbf{curl}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{P}_z]^{\mathrm{T}} & -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} & [\mathbf{0}] & [\mathbf{P}_x]^{\mathrm{T}} \\ [\mathbf{P}_y]^{\mathrm{T}} & -[\mathbf{P}_x]^{\mathrm{T}} & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

$$[\widetilde{\mathbf{curl}}] = \begin{bmatrix} [\mathbf{0}] & -[\mathbf{P}_z] & [\mathbf{P}_y] \\ [\mathbf{P}_z] & [\mathbf{0}] & -[\mathbf{P}_x] \\ -[\mathbf{P}_y] & [\mathbf{P}_x] & [\mathbf{0}] \end{bmatrix}_{3N\times 3N}$$

Discrete divergence operator / Diskreter Divergenzoperator

$$[\mathbf{div}] := \left[-[\mathbf{P}_x]^{\mathrm{T}}, -[\mathbf{P}_y]^{\mathrm{T}}, -[\mathbf{P}_z]^{\mathrm{T}} \right]_{N\times 3N}$$

$$[\widetilde{\mathbf{div}}] := \left[[\mathbf{P}_x], [\mathbf{P}_y], [\mathbf{P}_z] \right]_{N\times 3N}$$

$$\mathrm{curl\ grad} = \nabla\times\nabla = \underline{\mathbf{0}}$$

$$\mathrm{div\ curl} = \nabla\bullet\nabla = 0$$

$$-[\widetilde{\mathbf{div}}] = [\mathbf{grad}]^{\mathrm{T}}$$

$$[\widetilde{\mathbf{grad}}]^{\mathrm{T}} = [\mathbf{div}]$$

$$[\mathbf{curl}] = [\widetilde{\mathbf{curl}}]^{\mathrm{T}}$$

$$[\mathbf{curl}][\mathbf{grad}] = [\mathbf{0}]$$

$$[\widetilde{\mathbf{curl}}][\widetilde{\mathbf{grad}}] = [\mathbf{0}]$$

$$[\mathbf{div}][\mathbf{curl}] = [\mathbf{0}]$$

$$[\widetilde{\mathbf{div}}][\widetilde{\mathbf{curl}}] = [\mathbf{0}]$$

3-D FIT – … Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form / Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{E}\right\}(t) = \left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}(t) = \left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$$

$\left[\widetilde{\mathbf{div}}\right]$	$\in \mathbb{R}^{N\times 3N}$	Topological divergence operator in matrix form on the grid $\widetilde{G}$ / Topologischer Divergenzoperator in Matrixform auf dem Gitter $\widetilde{G}$
$\left[\widetilde{\boldsymbol{\varepsilon}}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of permittivities on the grid $\widetilde{G}$ / Diagonalmatrix der Permittivitäten auf dem Gitter $\widetilde{G}$
$\left[\widetilde{\mathbf{S}}\right]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarflächen auf dem Gitter $\widetilde{G}$
$\left\{\mathbf{E}\right\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic electric field strength vector / Algebraischer elektrischer Feldstärkevektor
$\left[\widetilde{\mathbf{V}}\right]$	$\in \mathbb{R}^{N\times N}$	Diagonal matrix of elementary volumes on the grid $\widetilde{G}$ / Diagonalmatrix der Elementarvolumina auf dem Gitter $\widetilde{G}$
$\left\{\rho_{\mathrm{e}}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic electric charge density vector / Algebraischer elektrischer Ladungsdichtevektor
$\left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$	$\in \mathbb{R}^{N}$	Algebraic electric charge vector / Algebraischer elektrischer Ladungsvektor

3-D FIT – ... Discrete Grid Equations in Local Matrix Form / 3D-FIT – ... diskreten Gittergleichungen in lokaler Matrixform

Magnetic Gauss' grid equation – 4th Maxwell's grid equation in global matrix form / Magnetische Gaußsche Gittergleichung – 4. Maxwellsche Gittergleichung in globaler Matrixform

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\boldsymbol{\rho}_\mathrm{m}\}(t) = \{\mathbf{Q}_\mathrm{m}\}(t)$$

$[\mathbf{div}]$	$\in \mathbb{R}^{N\times 3N}$	Topological divergence operator in matrix form on the grid G / Topologischer Divergenzoperator in Matrixform auf dem Gitter G
$[\mathbf{S}]$	$\in \mathbb{R}^{3N\times 3N}$	Diagonal matrix of elementary surfaces on the grid G / Diagonalmatrix der Elementarflächen auf dem Gitter G
$\{\mathbf{B}\}(t)$	$\in \mathbb{R}^{3N}$	Algebraic magnetic flux density vector / Algebraischer magnetischer Flussdichtevektor
$[\mathbf{V}]$	$\in \mathbb{R}^{N\times N}$	Diagonal matrix of elementary volumes on the grid G / Diagonalmatrix der Elementarvolumina auf dem Gitter G
$\{\rho_\mathrm{m}\}(t)$	$\in \mathbb{R}^{N}$	Algebraic magnetic charge density vector / Algebraischer magnetischer Ladungsdichtevektor
$\{\mathbf{Q}_\mathrm{m}\}(t)$	$\in \mathbb{R}^{N}$	Algebraic magnetic charge vector / Algebraischer magnetischer Ladungsvektor

FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Governing Analytic Equations

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

FIT Grid Equations

Maxwell's grid equations /
Maxwellsche Gittergleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$[\mathbf{S}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}(t) = \{\mathbf{Q}_{\mathrm{e}}\}(t)$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\boldsymbol{\rho}_{\mathrm{m}}\}(t) = \{\mathbf{Q}_{\mathrm{m}}\}(t)$$

End of Lecture 10 /
Ende der 10. Vorlesung