Numerical Methods in Electromagnetic Field Theory I (NFT I) / Numerische Methoden in der Elektromagnetischen Feldtheorie I (NFT I)

2nd Lecture / 2. Vorlesung

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One-Dimensional Electromagnetic Wave Propagation / Eindimensionale elektromagnetische Wellenausbreitung

(Homogeneous) 1-D Wave Equation for $E_x(z,t)$ /
Homogene 1-D Wellengleichung für $E_x(z,t)$

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

The 1-D Wave Equation is a Partial Differential Equation of Second Order/
Die 1-D Wellengleichung ist eine partielle Differentialgleichung zweiter Ordnung

Solution of the homogeneous 1-D wave equation is a plane wave of the form /
Lösung der homogenen 1-D Wellengleichung ist eine ebene Welle der Form

$$E_x(z,t) = E_0\left(t \mp \frac{z}{c_0}\right)$$

$$E_0\left(t \mp \frac{z}{c_0}\right)$$

This is an electric field strength of arbitrary time dependence, which is time retarded by the factor $\pm z/c_0$. /
Dies ist eine elektrische Feldstärke beliebiger Zeitabhängigkeit, die um den Faktor $\pm z/c_0$ zeitverzögert wird.

One–Dimensional Electromagnetic Wave Propagation / Eindimensionale elektromagnetische Wellenausbreitung

(Homogeneous) 1–D wave equation for $E_x(z,t)$ / Homogene 1–D Wellengleichung für $E_x(z,t)$

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

Solution / Lösung

$$E_x(z,t) = E_0\left(t \mp \frac{z}{c_0}\right)$$

Proof / Beweis

$$\frac{\partial^2}{\partial z^2} E_x(z,t) = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} E_0\left(t \mp \frac{z}{c_0}\right)\right] = \frac{\partial}{\partial z}\left[-\frac{1}{c_0} E_0'\left(t \mp \frac{z}{c_0}\right)\right] = \frac{1}{c_0^2} E_0''\left(t \mp \frac{z}{c_0}\right)$$

$$\frac{\partial^2}{\partial t^2} E_x(z,t) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} E_0\left(t \mp \frac{z}{c_0}\right)\right] = \frac{\partial}{\partial t}\left[E_0'\left(t \mp \frac{z}{c_0}\right)\right] = E_0''\left(t \mp \frac{z}{c_0}\right)$$

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = \frac{1}{c_0^2} E_0''\left(t \mp \frac{z}{c_0}\right) - \frac{1}{c_0^2} E_0''\left(t \mp \frac{z}{c_0}\right) = 0 \quad \blacksquare$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Common definitions of the first-order derivative of a 1-D function *f(x)* with respect to *x* /
Gebräuchliche Definitionen der ersten Ableitung von einer 1D Funktion *f(x)* nach *x*

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\mathrm{d}x \to 0} \frac{f(x) - f(x - \mathrm{d}x)}{\mathrm{d}x}$$

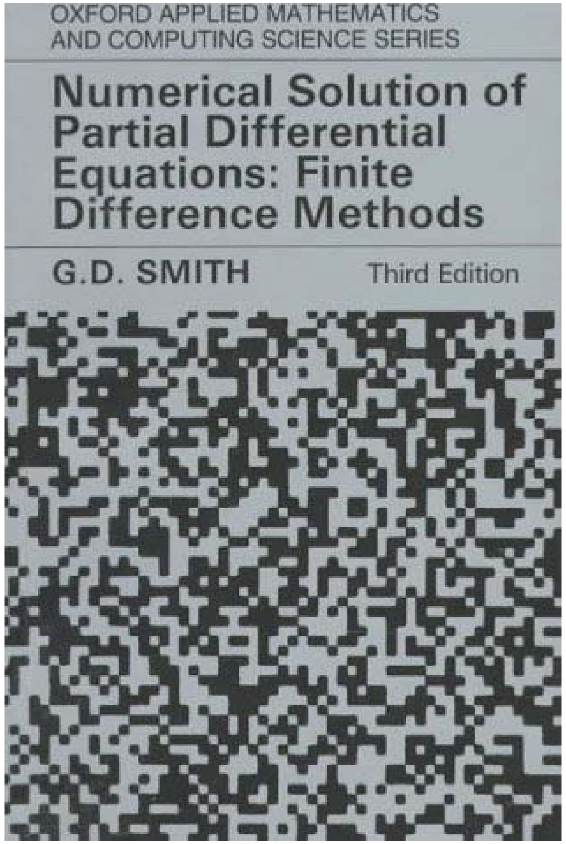
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\mathrm{d}x \to 0} \frac{f(x + \mathrm{d}x) - f(x)}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\mathrm{d}x \to 0} \frac{f(x + \mathrm{d}x) - f(x - \mathrm{d}x)}{2\mathrm{d}x}$$

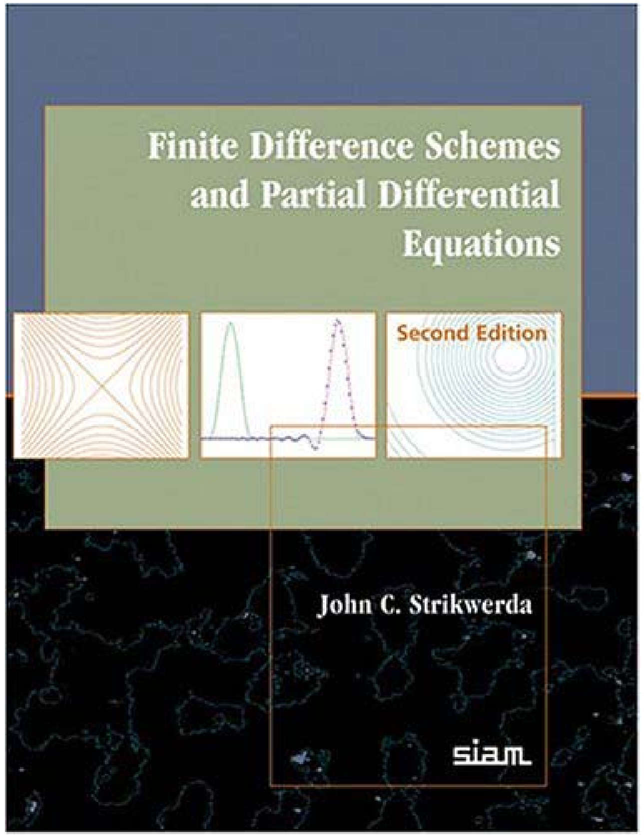
These are all Correct Definitions in the Limit $\mathrm{d}x \to 0$ /
Diese sind alle korrekte Definitionen im Grenzübergang $\mathrm{d}x \to 0$

But we want $\mathrm{d}x$ to remain FINITE: $\mathrm{d}x \to \Delta x$ /
Aber wir wollen, dass $\mathrm{d}x$ ENDLICH bleibt: $\mathrm{d}x \to \Delta x$

General Books on the Finite Difference (FD) Method / Allgemeine Bücher über die Finite Differenzen (FD) Methode



G. D. Smith:
Numerical Solution of Partial Differential Equations: Finite Difference Methods.
Oxford Applied Mathematics & Computing Science Series, **3rd. ed., 350 p.** Oxford University Press, Oxford, 1986.



John C. Strikwerda:
Finite Difference Schemes and Partial Differential Equations.
2nd ed., p. 446, SIAM Society for Industrial & Applied Mathematics, Nov. 2004.

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1–D FD Operators / 1D–FD–Operatoren

Computational Molecule / Berechnungsmolekül

Backward FD Operator / Rückwärts–FD–Operator

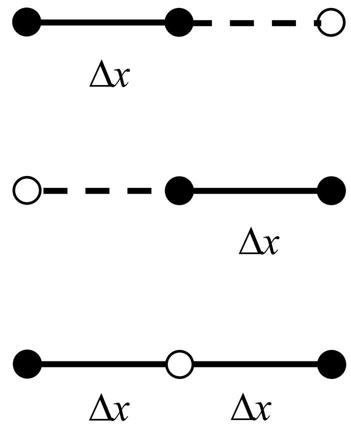
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

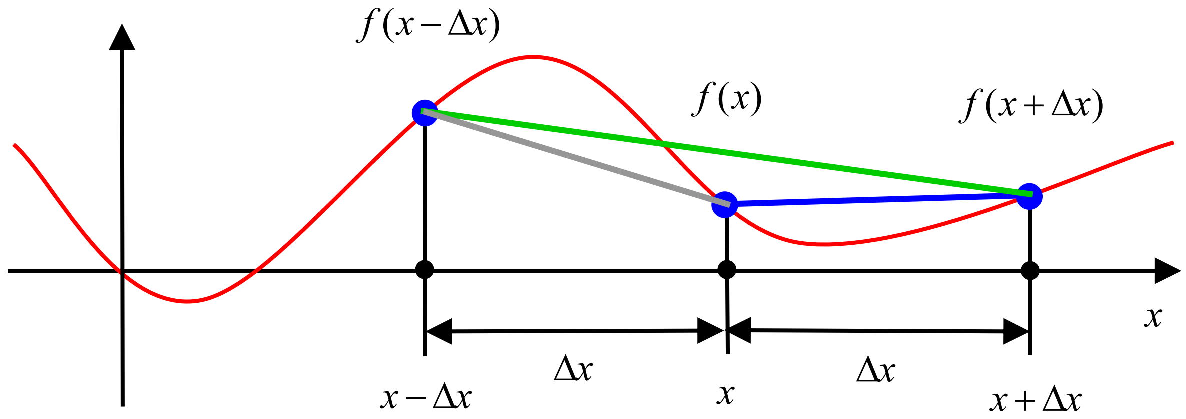
Forward FD Operator / Vorwärts–FD–Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Central FD Operator / Zentraler FD–Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$





Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators of Higher Order / 1D-FD-Operatoren höherer Ordnung

Backward FD operator / Rückwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-}(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x} \quad (1)$$

Forward FD operator / Vorwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{+}(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (2)$$

Using (1) and (2) it follows for the derivative of second order / Mit (1) und (2) folgt für die Ableitung zweiter Ordnung

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \approx \frac{\frac{\mathrm{d}}{\mathrm{d}x} f^{+}(x) - \frac{\mathrm{d}}{\mathrm{d}x} f^{-}(x)}{\Delta x} \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

The big question is now: how good are the FD approximations? / Die große Frage ist nun: Wie gut sind die FD-Approximationen?

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators – Taylor Series / 1D-FD-Operatoren – Taylor-Reihe

Taylor series are expansions of a function $f(x)$ in a finite distance Δx: $f(x+\Delta x)$ /
Taylor-Reihen sind Entwicklungen einer Funktion $f(x)$ in einer endlichen Distanz Δx: $f(x+\Delta x)$

$$f(x \pm \Delta x) = \sum_{n=0}^{\infty} \frac{1}{n!} (\pm \Delta x)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)$$

HOT: higher order terms / Terme höherer Ordnung

$$f(x \pm \Delta x) = f(x) \pm \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \pm \frac{(\Delta x)^3}{3!} \frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!} \frac{\mathrm{d}^4 f(x)}{dx^4} + HOT$$

$$= f(x) \pm \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!} \frac{d^2 f(x)}{\mathrm{d}x^2} \pm \frac{(\Delta x)^3}{3!} \frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!} \frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4} + \bigcirc[(\Delta x)^5]$$

Landau symbol "O", big "oh" / Landau-Symbol „O", großes „oh"

What results, if we use the Taylor series expansion for the following term / Was resultiert, wenn wir die Taylor-Reihenentwicklung auf den folgenden Term anwenden

$$f(x + \Delta x)$$

The Taylor series expansion reads / Die Taylor-Reihenentwicklung lautet

$$f(x + \Delta x) = f(x) + \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \frac{(\Delta x)^3}{3!} \frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!} \frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4} + \bigcirc[(\Delta x)^5]$$

Landau Symbols “Big oh” and “Small oh ”/ Landau-Symbole „großes oh“ und „kleines oh“

“Big oh” / „großes oh“

$F(\alpha) = O[G(\alpha)]$ as/als $\alpha \to 0$

$\left|\frac{F(\alpha)}{G(\alpha)}\right| \le C$ C: constant and / konstant und α sufficiently small / ausreichend klein

“Small oh” / „kleines oh“

$F(\alpha) = o[G(\alpha)]$ as/als $\alpha \to 0$

$\left|\frac{F(\alpha)}{G(\alpha)}\right| \to 0$ für / for $\alpha \to 0$

$$f(x \pm \Delta x) = f(x) \pm \Delta x \frac{df(x)}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} \pm \frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} \underbrace{\pm \frac{(\Delta x)^5}{5!}\frac{d^5 f(x)}{dx^5}}_{=O[(\Delta x)^5]} + \frac{(\Delta x)^6}{6!}\frac{d^6 f(x)}{dx^5}$$

$$\underbrace{\frac{(\Delta x)^5}{5!}\frac{d^5 f(x)}{dx^5}}_{=F(\Delta x)} = O[\underbrace{(\Delta x)^5}_{=G(\Delta x)}]$$

$$F(\Delta x) = \frac{(\Delta x)^5}{5!}\frac{d^5 f(x)}{dx^5}$$

$$G(\Delta x) = (\Delta x)^5$$

$$\left|\frac{F(\Delta x)}{G(\Delta x)}\right| = \left|\frac{\frac{(\Delta x)^5}{5!}\frac{d^5 f(x)}{dx^5}}{(\Delta x)^5}\right| = \left|\frac{1}{5!}\frac{d^5 f(x)}{dx^5}\right| \le C$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators – Taylor Series / 1D-FD-Operatoren – Taylor-Reihe

$$f(x+\Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} + \frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} + O[(\Delta x)^5] \quad (1)$$

$$f(x) = f(x) \quad (2)$$

Compute (1) minus (2) and subsequently divide by Δx /
Berechne (1) minus (2) und dividiere nachfolgend durch Δx

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df(x)}{dx} + \frac{\Delta x}{2!}\frac{d^2 f(x)}{dx^2} + \frac{(\Delta x)^2}{3!}\frac{d^3 f(x)}{dx^3} + \frac{(\Delta x)^3}{4!}\frac{d^4 f(x)}{dx^4} + O[(\Delta x)^4]$$

$$\frac{d}{dx} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \underbrace{- \frac{\Delta x}{2!}\frac{d^2 f(x)}{dx^2} - \frac{(\Delta x)^2}{3!}\frac{d^3 f(x)}{dx^3} - \frac{(\Delta x)^3}{4!}\frac{d^4 f(x)}{dx^4} + O[(\Delta x)^4]}_{=O(\Delta x)}$$

$$O(\Delta x) = -\frac{\Delta x}{2!}\frac{d^2 f(x)}{dx^2} \underbrace{- \frac{(\Delta x)^2}{3!}\frac{d^3 f(x)}{dx^3} - \frac{(\Delta x)^3}{4!}\frac{d^4 f(x)}{dx^4} + O[(\Delta x)^4]}_{=O[(\Delta x)^2]}$$

$$= -\frac{\Delta x}{2}\frac{d^2 f(x)}{dx^2} + O[(\Delta x)^2]$$

$$\frac{d}{dx} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

Approximation error /
Approximationsfehler

Landau symbol “O”, big “oh” /
Landau-Symbol „O“, großes „oh“

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Backward FD Operator / Rückwärts-FD-Operator

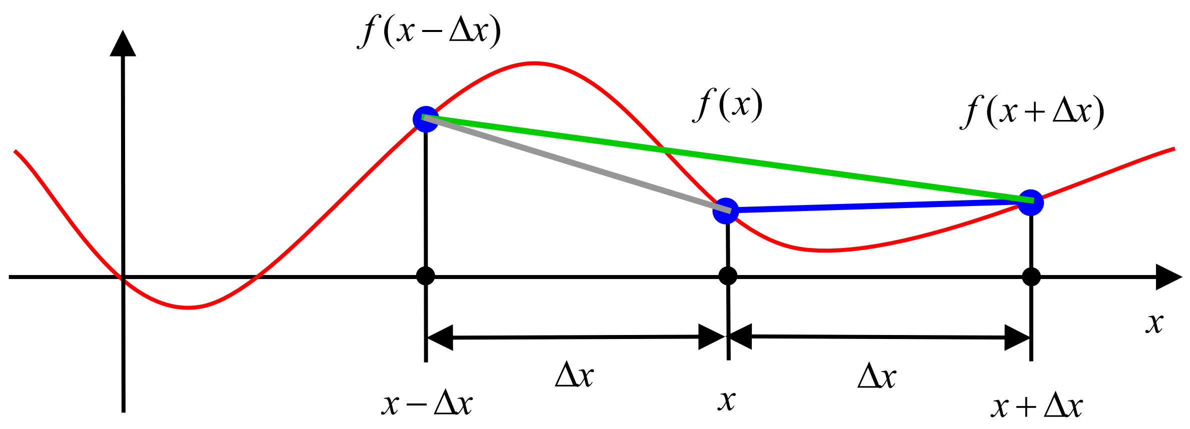
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Forward FD Operator / Vorwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Central FD Operator / Zentraler FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\,\Delta x} + \mathrm{O}[(\Delta x)^2] \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\,\Delta x}$$



End of Lecture 2 /
Ende der 2. Vorlesung