Numerical Methods in Electromagnetic Field Theory I (NFT I) / Numerische Methoden in der Elektromagnetischen Feldtheorie I (NFT I)

2nd Lecture / 2. Vorlesung

Dr.-Ing. René Marklein

marklein@uni-kassel.de
http://www.tet.e-technik.uni-kassel.de
http://www.uni-kassel.de/fb16/tet/marklein/index.html
Tel.: ++49 (0)561 804 6426
Fax: ++49 (0)561 804 6489

Universität Kassel
Fachbereich Elektrotechnik / Informatik (FB 16)
Fachgebiet Theoretische Elektrotechnik (FG TET)
Wilhelmshöher Allee 71
Büro: Raum 2113 / 2115
D-34121 Kassel

University of Kassel
Dept. Electrical Engineering / Computer Science (FB 16)
Electromagnetic Field Theory (FG TET)
Wilhelmshöher Allee 71
Office: Room 2113 / 2115
D-34121 Kassel

One-Dimensional Electromagnetic Wave Propagation / Eindimensionale elektromagnetische Wellenausbreitung

(Homogeneous) 1-D Wave Equation for $E_x(z,t)$ / Homogene 1-D Wellengleichung für $E_x(z,t)$

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

The 1-D Wave Equation is a Partial Differential Equation of Second Order/ Die 1-D Wellengleichung ist eine partielle Differentialgleichung zweiter Ordnung

Solution of the homogeneous 1-D wave equation is a plane wave of the form / Lösung der homogenen 1-D Wellengleichung ist eine ebene Welle der Form

$$E_x(z,t) = E_0\left(t \mp \frac{z}{c_0}\right)$$

$$E_0\left(t \mp \frac{z}{c_0}\right)$$

This is an electric field strength of arbitrary time dependence, which is time retarded by the factor $\pm z/c_0$. / Dies ist eine elektrische Feldstärke beliebiger Zeitabhängigkeit, die um den Faktor $\pm z/c_0$ zeitverzögert wird.

One-Dimensional Electromagnetic Wave Propagation / Eindimensionale elektromagnetische Wellenausbreitung

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$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

Solution / Lösung

$$E_x(z,t) = E_0\left(t \mp \frac{z}{c_0}\right)$$

Proof / Beweis

$$\frac{\partial^2}{\partial z^2} E_x(z,t) = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} E_0\left(t \mp \frac{z}{c_0}\right)\right] = \frac{\partial}{\partial z}\left[-\frac{1}{c_0} E_0'\left(t \mp \frac{z}{c_0}\right)\right] = \frac{1}{c_0^2} E_0''\left(t \mp \frac{z}{c_0}\right)$$

$$\frac{\partial^2}{\partial t^2} E_x(z,t) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} E_0\left(t \mp \frac{z}{c_0}\right)\right] = \frac{\partial}{\partial t}\left[E_0'\left(t \mp \frac{z}{c_0}\right)\right] = E_0''\left(t \mp \frac{z}{c_0}\right)$$

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = \frac{1}{c_0^2} E_0''\left(t \mp \frac{z}{c_0}\right) - \frac{1}{c_0^2} E_0''\left(t \mp \frac{z}{c_0}\right) = 0 \quad \blacksquare$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Common definitions of the first-order derivative of a 1-D function *f(x)* with respect to *x* /
Gebräuchliche Definitionen der ersten Ableitung von einer 1D Funktion *f(x)* nach *x*

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\mathrm{d}x \to 0} \frac{f(x) - f(x - \mathrm{d}x)}{\mathrm{d}x}$$

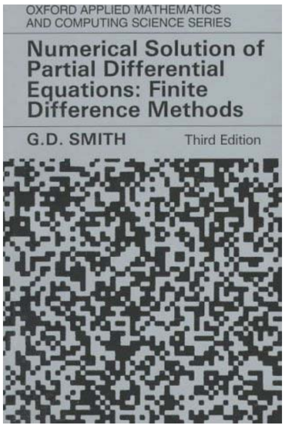
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\mathrm{d}x \to 0} \frac{f(x + \mathrm{d}x) - f(x)}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\mathrm{d}x \to 0} \frac{f(x + \mathrm{d}x) - f(x - \mathrm{d}x)}{2\mathrm{d}x}$$

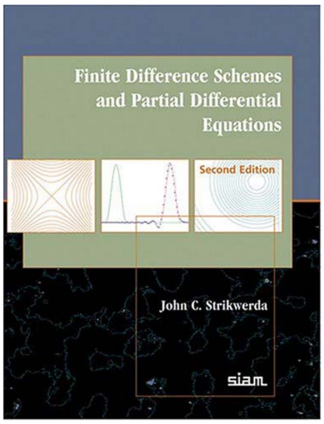
These are all Correct Definitions in the Limit $\mathrm{d}x \to 0$ /
Diese sind alle korrekte Definitionen im Grenzübergang $\mathrm{d}x \to 0$

But we want d*x* to remain FINITE: $\mathrm{d}x \to \Delta x$ /
Aber wir wollen, dass dx ENDLICH bleibt: $\mathrm{d}x \to \Delta x$

General Books on the Finite Difference (FD) Method / Allgemeine Bücher über die Finite Differenzen (FD) Methode



G. D. Smith:
Numerical Solution of Partial Differential Equations: Finite Difference Methods.
Oxford Applied Mathematics & Computing Science Series, 3rd. ed., 350 p. Oxford University Press, Oxford, 1986.



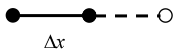
John C. Strikwerda:
Finite Difference Schemes and Partial Differential Equations.
2nd ed., p. 446, SIAM Society for Industrial & Applied Mathematics, Nov. 2004.

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Computational Molecule / Berechnungsmolekül

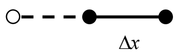
Backward FD Operator / Rückwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$



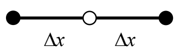
Forward FD Operator / Vorwärts-FD-Operator

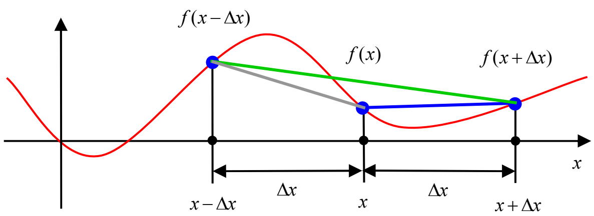
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Central FD Operator / Zentraler FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$





Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators of Higher Order / 1D-FD-Operatoren höherer Ordnung

Backward FD operator / Rückwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f^-(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x} \quad (1)$$

Forward FD operator / Vorwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f^+(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (2)$$

Using (1) and (2) it follows for the derivative of second order / Mit (1) und (2) folgt für die Ableitung zweiter Ordnung

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \approx \frac{\frac{\mathrm{d}}{\mathrm{d}x} f^+(x) - \frac{\mathrm{d}}{\mathrm{d}x} f^-(x)}{\Delta x} \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

The big question is now: how good are the FD approximations? /
Die große Frage ist nun: Wie gut sind die FD-Approximationen?

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators - Taylor Series / 1D-FD-Operatoren - Taylor-Reihe

Taylor series are expansions of a function $f(x)$ in a finite distance Δx: $f(x+\Delta x)$ /
Taylor-Reihen sind Entwicklungen einer Funktion $f(x)$ in einer endlichen Distanz Δx: $f(x+\Delta x)$

$$f(x \pm \Delta x) = \sum_{n=0}^{\infty} \frac{1}{n!} (\pm \Delta x)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)$$

HOT: higher order terms / Terme höherer Ordnung

$$f(x \pm \Delta x) = f(x) \pm \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \pm \frac{(\Delta x)^3}{3!} \frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!} \frac{\mathrm{d}^4 f(x)}{dx^4} + HOT$$

$$= f(x) \pm \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!} \frac{d^2 f(x)}{\mathrm{d}x^2} \pm \frac{(\Delta x)^3}{3!} \frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!} \frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4} + \mathcal{O}[(\Delta x)^5]$$

Landau symbol "O", big "oh" / Landau-Symbol „O", großes „oh"

What results, if we use the Taylor series expansion for the following term / Was resultiert, wenn wir die Taylor-Reihenentwicklung auf den folgenden Term anwenden

$$f(x+\Delta x)$$

The Taylor series expansion reads / Die Taylor-Reihenentwicklung lautet

$$f(x+\Delta x) = f(x) + \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \frac{(\Delta x)^3}{3!} \frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!} \frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4} + \mathcal{O}[(\Delta x)^5]$$

Landau Symbols "Big oh" and "Small oh "/ Landau-Symbole „großes oh" und „kleines oh"

"Big oh" / „großes oh"

$F(\alpha)=\mathcal{O}[G(\alpha)]$ as/als $\alpha\to 0$

$\left|\frac{F(\alpha)}{G(\alpha)}\right|\le C$ $C:$ constant and / konstant und α sufficiently small / ausreichend klein

"Small oh" / „kleines oh"

$F(\alpha)=o[G(\alpha)]$ as/als $\alpha\to 0$

$\left|\frac{F(\alpha)}{G(\alpha)}\right|\to 0$ für / for $\alpha\to 0$

$$f(x\pm\Delta x)=f(x)\pm\Delta x\frac{df(x)}{dx}+\frac{(\Delta x)^2}{2!}\frac{d^2f(x)}{dx^2}\pm\frac{(\Delta x)^3}{3!}\frac{d^3f(x)}{dx^3}+\frac{(\Delta x)^4}{4!}\frac{d^4f(x)}{dx^4}\pm\underbrace{\frac{(\Delta x)^5}{5!}\frac{d^5f(x)}{dx^5}}_{=\mathcal{O}[(\Delta x)^5]}+\frac{(\Delta x)^6}{6!}\frac{d^6f(x)}{dx^5}$$

$$\underbrace{\frac{(\Delta x)^5}{5!}\frac{d^5f(x)}{dx^5}}_{=F(\Delta x)}=\mathcal{O}[\underbrace{(\Delta x)^5}_{=G(\Delta x)}]$$

$$F(\Delta x)=\frac{(\Delta x)^5}{5!}\frac{d^5f(x)}{dx^5}$$

$$G(\Delta x)=(\Delta x)^5$$

$$\left|\frac{F(\Delta x)}{G(\Delta x)}\right|=\left|\frac{\frac{(\Delta x)^5}{5!}\frac{d^5f(x)}{dx^5}}{(\Delta x)^5}\right|=\left|\frac{1}{5!}\frac{d^5f(x)}{dx^5}\right|\le C$$

Finite Difference (FD) Method / Finite Differenzen (FD) Methode 1-D FD Operators - Taylor Series / 1D-FD-Operatoren - Taylor-Reihe

$$f(x+\Delta x)=f(x)+\Delta x\frac{df(x)}{dx}+\frac{(\Delta x)^2}{2!}\frac{d^2f(x)}{dx^2}+\frac{(\Delta x)^3}{3!}\frac{d^3f(x)}{dx^3}+\frac{(\Delta x)^4}{4!}\frac{d^4f(x)}{dx^4}+\mathcal{O}[(\Delta x)^5] \quad (1)$$

$$f(x)=f(x) \quad (2)$$

Compute (1) minus (2) and subsequently divide by Δx / Berechne (1) minus (2) und dividiere nachfolgend durch Δx

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{df(x)}{dx}+\frac{\Delta x}{2!}\frac{d^2f(x)}{dx^2}+\frac{(\Delta x)^2}{3!}\frac{d^3f(x)}{dx^3}+\frac{(\Delta x)^3}{4!}\frac{d^4f(x)}{dx^4}+\mathcal{O}[(\Delta x)^4]$$

$$\frac{d}{dx}f(x)=\frac{f(x+\Delta x)-f(x)}{\Delta x}\underbrace{-\frac{\Delta x}{2!}\frac{d^2f(x)}{dx^2}-\frac{(\Delta x)^2}{3!}\frac{d^3f(x)}{dx^3}-\frac{(\Delta x)^3}{4!}\frac{d^4f(x)}{dx^4}+\mathcal{O}[(\Delta x)^4]}_{=\mathcal{O}(\Delta x)}$$

$$\mathcal{O}(\Delta x)=-\frac{\Delta x}{2!}\frac{d^2f(x)}{dx^2}\underbrace{-\frac{(\Delta x)^2}{3!}\frac{d^3f(x)}{dx^3}-\frac{(\Delta x)^3}{4!}\frac{d^4f(x)}{dx^4}+\mathcal{O}[(\Delta x)^4]}_{=\mathcal{O}[(\Delta x)^2]}$$

$$=-\frac{\Delta x}{2}\frac{d^2f(x)}{dx^2}+\mathcal{O}[(\Delta x)^2]$$

Approximation error / Approximationsfehler

$$\frac{d}{dx}f(x)=\frac{f(x+\Delta x)-f(x)}{\Delta x}+\mathcal{O}(\Delta x)$$

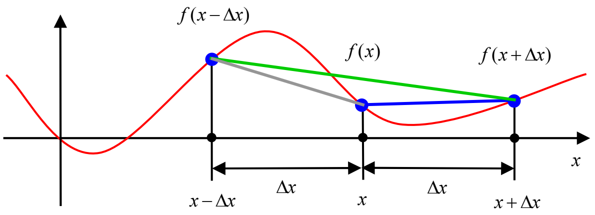
Landau symbol "O", big "oh" / Landau-Symbol „O", großes „oh"

Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Backward FD Operator / Rückwärts-FD-Operator
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

Forward FD Operator / Vorwärts-FD-Operator
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Central FD Operator / Zentraler FD-Operator
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + \mathrm{O}[(\Delta x)^2] \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$



End of Lecture 2 / Ende der 2. Vorlesung