Some notes concerning bounds for m-cycles in the Collatz (3x+1)-problem

In this note I discuss a method to estimate bounds for cycles of a certain form under the Collatz (3x+1)-transformation. The notion is adapted to the 1-cycle and m-cycle-concept as named by Ray Steiner in 1978 and Simons and DeWeger/Simons 2000 and 2003.
I developed this type of computation in the previous years independently ("primitive loop") but found then articles covering the same subject and partly with the same tools. However I didn't see the complete computation anwhere yet, so it may be new , or at least somehow useful in detailing.
Comments are much appreciated, please mail to helms@uni-kassel.de

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Denote a collatz-transformation, where a and b are odd numbers, and A is chosen appropriately:

(1) $$b = T(a;A) \qquad := b = \frac{3*a+1}{2^A}$$

and let a transformation of more steps simply be written as

(2) $$a_N = T(a_0; A_0, A_1, A_2, \ldots, A_{N-1})$$

then a cycle would occur, if $a_k = a_0$

(3) $$a_0 = T(a_0; A_0, A_1, A_2, \ldots, A_{N-1})$$

A transformation, which ascends only and descends in one transformation at the end looks like

(4) $$a_N = T(a_0; 1,1,1,1,\ldots,1,A_{N-1})$$
define also PT(a; N:A) := T(a;1,1,1...1,A) where N= number-of-steps

and a cycle, which ascends only and descends in one transformation at the end looks like

(5) $$a = T(a;1,1,1,1,\ldots,1,A) \qquad = PT(a;N:A)$$

with N steps. Let's call such transformations "1-peak-transformation" and such cycles "1-peak-cycle".

We have for a 1-peak-cycle of the type (*N* steps or *N-1* times the coefficent *1*)

$$(6) \qquad a = T(a;1,1,1,\dots,1,A) = PT(a;N:A)$$

then *a* must satisfy two congruences with a free parameter *k*

$$(7) \qquad a = \frac{k*3^N - 1}{2^{A-1}} = k*2^N - 1$$

Proof:

expand a series of steps, here *b = T(a;1,1,A) = PT(a;3:A)*

$$(8) \qquad b = \frac{\frac{\frac{a*3+1}{2}*3+1}{2}*3+1}{2^A} = \frac{\frac{a*3^2+3^1+2}{2^2}*3+1}{2^A} = \frac{a*3^3+3^2+3^1*2+2^2}{2^{2+A}} = a\frac{3^3}{2^{3+A-1}} + \frac{3^3-2^3}{2^{3+A-1}}$$

Generally:

$$(9) \qquad b = a*\frac{3^N}{2^{N+A-1}} + \frac{3^N-2^N}{2^{N+A-1}} \quad = PT(a;N:A)$$

From here follows:

$$(10) \qquad \begin{aligned} &b*2^{N+A-1} = (a+1)*3^N - 2^N \\ &(b*2^{A-1}+1)*2^N = (a+1)*3^N \end{aligned}$$

Rearrange to see the required congruence-classes:

$$(11) \qquad \frac{b*2^{A-1}+1}{3^N} = \frac{a+1}{2^N}$$

and explicitely:

$$(12) \qquad \begin{aligned} &a = k*2^N - 1 \\ &b = \frac{k*3^N - 1}{2^{A-1}} \qquad\qquad since: b*2^{A-1} = k*3^N - 1 \end{aligned}$$

To form a cycle, *a* equals *b* and a has to satisfy both conditions simultaneously.
End of proof.

Note, that since *a* and *b* must be odd by the definition of the transformation, *k* must be odd as well, and also, by the collatz-property, that the result of a transformation cannot be divisible by *3*, in a cycle ***no*** element can be divisible by *3*, so *k* needs not only to be odd, but is also required to keep the results *a* and *b* congruent *1* or *-1 (mod 6)*. But that further restrictive requirements are not discussed here.

In the following these congruence-conditions are used to derive bounds for the exponents of an m-peak-cycle and the length N in relation to the number m of concatenations.

From (12) follows the proportion of b/a:

$$(13)\qquad \frac{b}{a}=\frac{1}{2^{A-1}}*\frac{k*3^{N}-1}{k*2^{N}-1}$$

For a concatenation of 1-peak-transformations, finally forming a cycle:

$$(14)\qquad \begin{aligned} &b_0=PT(a_0;N_0:A_0) && use\, b_0\, as\, a_1;then\\ &b_1=PT(a_1;N_1:A_1) && make\, it\, a\, cycle: a_0=b_1\\ &a_0=PT(a_0;N_0:A_0,N_1:A_1) \end{aligned}$$

we can write a table of conditions for the intermediate values of a_j , say $m=4$ and $j=0$ *to* $m-1$ according to each 1-peak-transformation:

$$(15)\qquad a_0=PT(a_0;N_0:A_0,N_1:A_1,N_2:A_2,N_3:A_3)$$

which gives, expressing the congruences for the 4 intermediate values,

$$(16)\qquad \begin{aligned} a_0&=k_0*2^{N_0}-1 &\Rightarrow\quad b_0&=\frac{k_0*3^{N_0}-1}{2^{A_0-1}}\\ a_1&=k_1*2^{N_1}-1 &\Rightarrow\quad b_1&=\frac{k_1*3^{N_1}-1}{2^{A_1-1}}\\ a_2&=k_2*2^{N_2}-1 &\Rightarrow\quad b_2&=\frac{k_2*3^{N_2}-1}{2^{A_2-1}}\\ a_3&=k_3*2^{N_3}-1 &\Rightarrow\quad b_3&=\frac{k_3*3^{N_3}-1}{2^{A_3-1}} \end{aligned}$$

If such transformations are concatenated, then this means, that $a_1 = b_0$, $a_2 = b_1$, $a_3 = b_2$, and if it is finally a cycle, then also $a_0 = b_3$, and the four a-values must each satisfy two congruences.
Also dividing b_0 by a_1, b_2 by a_1 etc is always *1*, and the product of all that fractions is also *1*.

$$(17)\qquad \frac{b_0}{a_1}*\frac{b_1}{a_2}*\frac{b_2}{a_3}*\frac{b_3}{a_0}=\frac{1}{2^{A_0-1}}\frac{k_0 3^{N_0}-1}{k_1 2^{N_1}-1}*\frac{1}{2^{A_1-1}}\frac{k_1 3^{N_1}-1}{k_2 2^{N_2}-1}*\frac{1}{2^{A_2-1}}\frac{k_2 3^{N_2}-1}{k_3 2^{N_3}-1}*\frac{1}{2^{A_3-1}}\frac{k_3 3^{N_3}-1}{k_0 2^{N_0}-1}$$

Rearranging gives

$$(18)\qquad 2^{A_0-1}*2^{A_1-1}2^{A_2-1}2^{A_3-1}=\frac{k_0 3^{N_0}-1}{k_1 2^{N_1}-1}*\frac{k_1 3^{N_1}-1}{k_2 2^{N_2}-1}*\frac{k_2 3^{N_2}-1}{k_3 2^{N_3}-1}*\frac{k_3 3^{N_3}-1}{k_0 2^{N_0}-1}$$

and with a general number of m 1-peak-transformations forming the cycle:

$$(19)\qquad 2^{A_0+A_1+A_2+A_3+\dots A_{m-1}-m}=\frac{k_0 3^{N_0}-1}{k_1 2^{N_1}-1}*\frac{k_1 3^{N_1}-1}{k_2 2^{N_2}-1}*\frac{k_2 3^{N_2}-1}{k_3 2^{N_3}-1}*\dots*\frac{k_{m-1} 3^{N_{m-1}}-1}{k_0 2^{N_0}-1}$$

Now the denominators can be rotated:

$$(20)\qquad 2^{A_0+A_1+A_2+A_3...+A_{m-1}-m} = \frac{k_0 3^{N_0}-1}{k_0 2^{N_0}-1} * \frac{k_1 3^{N_1}-1}{k_1 2^{N_1}-1} * \frac{k_2 3^{N_2}-1}{k_2 2^{N_2}-1} * \ldots * \frac{k_{m-1} 3^{N_{m-1}}-1}{k_{m-1} 2^{N_{m-1}}-1}$$

From this equation some sharp bounds for the possibility of a cycle can be derived.

For a term

$$(21)\qquad t = \frac{k*3^N-1}{k*2^N-1}$$

one can write

$$(22)\qquad \begin{aligned} t &= \frac{k*3^N}{k*2^N-1} - \frac{1}{k*2^N-1} \\ &= k*3^N\left(\frac{1}{k*2^N} + \frac{1}{k*2^N}^2 + \frac{1}{k*2^N}^3 + \ldots\right) - \left(\frac{1}{k*2^N} + \frac{1}{k*2^N}^2 + \frac{1}{k*2^N}^3 + \ldots\right) \\ &= \frac{3^N}{2^N} + \left(\frac{3^N}{2^N}-1\right)\frac{1}{k*2^N-1} \\ &= \frac{3^N}{2^N} + \frac{3^N-2^N}{k*4^N-2^N} \\ &= \frac{3^N}{2^N} + \delta_{N,k} \qquad ;0<\delta_{N,k}<1 \end{aligned}$$

Which leads to

$$(23)\qquad 2^{\Sigma(A)-m} = \left(\frac{3^{N_0}}{2^{N_0}} + \delta_{N_0,k_0}\right) * \left(\frac{3^{N_1}}{2^{N_1}} + \delta_{N_1,k_1}\right) * \left(\frac{3^{N_2}}{2^{N_2}} + \delta_{N_2,k_2}\right) * \ldots * \left(\frac{3^{N_{m-1}}}{2^{N_{m-1}}} + \delta_{N_{m-1},k_{m-1}}\right)$$

Setting

$$(24)\qquad \begin{aligned} \varepsilon_{N,k} &= \frac{2^N}{3^N} * \delta_{N,k} && 0<\varepsilon_{N,k}<\left(\frac{2}{3}\right)^N \\ \varepsilon_{N,k} &= \frac{2^N}{3^N} * \frac{1}{2^N} * \frac{3^N-2^N}{k*2^N-1} = \frac{1-\frac{2}{3}^N}{k*2^N-1} = \frac{1-\frac{2}{3}^N}{a} && 0<\varepsilon_{N,k}<\frac{1}{a} \end{aligned}$$

factoring for *3/2* and multiplying through by 2^N gives

$$(25)\qquad \begin{aligned} 2^{\Sigma(A)-m} &= \frac{3^N}{2^N}\left(1+\varepsilon_{N_0,k_0}\right)\left(1+\varepsilon_{N_1,k_1}\right)\left(1+\varepsilon_{N_2,k_2}\right)\cdots\left(1+\varepsilon_{N_{m-1},k_{m-1}}\right) \\ 2^{N+\Sigma(A-1)} &= 3^N\left(1+\varepsilon_{N_0,k_0}\right)\left(1+\varepsilon_{N_1,k_1}\right)\left(1+\varepsilon_{N_2,k_2}\right)\cdots\left(1+\varepsilon_{N_{m-1},k_{m-1}}\right) \end{aligned}$$

which means, that the rhs must be a perfect power of *2* greater than 3^N but smaller than 3^N*2^m .

Let *ß = log(3)/log(2)* and let *e* denote the exponent on the lhs, then *e* must be in the interval

$$(26)\qquad N*\beta\,. < e < N*\beta + m$$

(which is much overestimated, since all ε in(25) are smaller than 1), and the sum of the *m* coefficients $A_j - 1$ must be in the interval:

$$(27)\qquad N^*\beta\, . < N-m + \Sigma(A_j) < N^*\beta + m$$

$$(28)\qquad N^*(\beta-1) < \Sigma(A_j - 1) < N^*(\beta-1) + m$$

which is aproximately

$$(29)\qquad ceil(N^*\, 0.584) <= \Sigma(A_j - 1) <= floor(N^*\, 0.585) + m$$

One can see, that the concatenation of many 1-peak-transformations adds degrees of freedom for a solution because of widening the bounds in the above inequality.

The upper bound *m* for the interval is of little meaning; a better upper bound is (see eq.(24))

$$(30)\qquad log(1+\varepsilon_{N,k}) < \varepsilon_{N,k} < 1/(k^*2^N-1)$$

so

$$(31)\qquad ceil(N^*(\beta-1)) <= \Sigma(A_k-1) \quad <= floor(N^*(\beta-1) + \Sigma\, 1/(k_j{}^*2^{Nj}-1)/log(2)$$
$$<= floor(N^*(\beta-1) + \Sigma\, 1/a_j/log(2)$$

from which, given a certain overall length *N* of the m-peak-cycle, bounds for the least possible number of concatenations *m* can be computed, and conversely, given a number *m* of concatenations, a lower bound for the overall length *N* of the cycle can be computed as well.

Note also that $\varepsilon_{N,k}$ contains the value of the element *a* in its denominator such that

$$(32)\qquad \varepsilon_{N_j,k_j} = \frac{1-\frac{2}{3}^{N_j}}{a_j} \leq\approx \frac{1}{2^{N_j}}$$

and, following an empirical result of Eric Roosendaal, all $1<a_j<2^{50}$ are known not to be a member of a cycle, we have already by empirical results, that all $\varepsilon_j<2^{-50}$ and in (31) we have an upper bound of

$$(33)\qquad ceil(N^*(\beta-1)) \quad \leq \quad \Sigma(A_j - 1) \quad \leq\approx \quad floor(N^*(\beta-1) + \Sigma 1/a_j/log(2))$$
$$\leq\approx \quad floor(N^*(\beta-1) + m\, 2^{-50})$$

The term $c = N^*(\beta-1)$ needs to be near below an integer number *r*, such that $c + m\, 2^{-50}$ can be greater than *r*. About the approximation of *frac(N*ß)* only the order is known; it seems, using the locally best approximations (which also occur as convergents in the continued fraction representation) are decreasing with *N*, so to find an approximation of $frac(N^*\beta) < 1/s$ then *N* must be greater than *s*. So to find an *N* such that $N^*(\beta-1)$ is less than $m^*1/2^{50}$ apart from $ceil(N^*(\beta-1)$ it seems, that *N* must itself be of the order $2^{50}/m$. With a m-peak-cycle with $m>1$ the rhs- " $+ m^*2^{-50}$ " -term in (33) increases and *N* can decrease accordingly to the weaker requirement of approximation of $N^*(\beta-1)$ to $ceil(N^*(\beta-1))$.

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