Two-Scale Simulation of a Disc With a Crack by X-FEM and Cell Model

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The two-scale simulation of a linear-elastic orthotropic disc with a central crack under mode-I loading may be used to verify the extended finite element method implementation of orthotropic enrichment functions into finite element codes such as FEAP. The stress distribution on the finer scale is simultaneously resolved by the high fidelity generalized method of cells called at each integration point of the macro elements.

1 Extended Finite Element Method

The extended finite element method is a numerical technique [1] enabling the mesh-independent representation of cracks by a local extension of the displacement field. The displacement approximation $\boldsymbol{u}^{\mathrm{h}}$ at a point $\boldsymbol{x}$ for a plane elastic body with a crack consists of the standard and the enriched part:

$$\boldsymbol{u}^{\mathrm{h}}(\boldsymbol{x}) = \sum_{i\in\mathcal{S}} N_i(\boldsymbol{x})\boldsymbol{u}_i + \sum_{l\in\mathcal{S}_{\mathrm{CE}}} N_l(\boldsymbol{x})\Big(H(\boldsymbol{x}) - H(\boldsymbol{x}_l)\Big)\boldsymbol{a}_l + \sum_{j\in\mathcal{S}_{\mathrm{CT}}} N_j(\boldsymbol{x})\Big[\sum_{k=1}^{4}\Big(F^k(\boldsymbol{x}) - F^k(\boldsymbol{x}_j)\Big)\boldsymbol{b}_j^k\Big] , \qquad (1)$$

see [2], where $N_{(.)}$ denotes the isoparametric shape functions; H the HEAVISIDE function to represent the jump in the displacement field across a crack; $F^{(\cdot)}$ the enrichment functions for orthotropic material behavior to take the stress singularity at the crack tip into account; $\boldsymbol{u}$ the vector of unknown nodal displacements; $\boldsymbol{a}$ and $\boldsymbol{b}$ vectors with the unknowns of the enrichment part; and i, j, k, and l are indices of summation. The set $\mathcal{S}$ in Eq. (1) contains all nodes, $\mathcal{S}_{\mathrm{CE}}$ comprises only the ones for the HEAVISIDE enrichment, and $\mathcal{S}_{\mathrm{CT}}$ those for the near tip enrichment. The latter enrichment is addressed to all nodes within a defined radius R around the crack tip, which is called "fixed area enrichment," see [5]. Hence, some nodes of the blending elements are outside the radius. In these elements the partition of unity of the enrichment functions is violated. To overcome this deficiency, the nodes outside the radius of the blending elements are also enriched with the near tip solution, see Fig. 1 and the corrected X-FEM approach proposed in [4]. The four enrichment functions F^k with $k = 1, \dots, 4$ in Eq. (1) are specified in Eq. (2) by means of polar coordinates (r, θ) at the crack tip and the functions α_1, α_2, g_1, and g_2 given in [3]. They depend on the angle θ and the entries C_{pqrs} of the orthotropic elastic stiffness tensor.

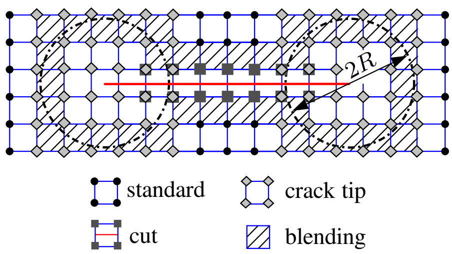

Fig. 1: Differently enriched nodes and elements

$$F^k(r,\theta) = \sqrt{r}\cos\left(\frac{\alpha_k(\theta, C_{pqrs})}{2}\right)\sqrt{g_k(\theta, C_{pqrs})}\ , \ F^{k+2}(r,\theta) = \sqrt{r}\sin\left(\frac{\alpha_k(\theta, C_{pqrs})}{2}\right)\sqrt{g_k(\theta, C_{pqrs})} \quad k = 1, 2 \quad (2)$$

2 High Fidelity Generalized Method of Cells

The high fidelity generalized method of cells (HFGMC), see [6], is a displacement-based, numerical method to solve boundary value problems, which has been developed specifically for the analysis of periodic microstructures. The displacement and traction continuity as well as periodicity along a face shared by two subcells, which are the "elements" of a discretized repeating unit cell (RUC), see Fig. 2 right, need to be satisfied in the HFGMC. The static equilibrium must also hold true for each subcell. All mentioned conditions are fulfilled in an average sense, which leads to a system of linear equations for the linear-elastic material phases and small deformations considered here. The strain concentration tensor $\mathbf{G}^{(\beta)}$ is computed for each subcell $\Omega^{(\beta)}$ and utilized to calculate the effective stress $\langle\boldsymbol{\sigma}\rangle$ and elastic stiffness $\mathbf{C}^*$ with a summation over all subcells N:

$$\langle\boldsymbol{\sigma}\rangle = \sum_{\beta=1}^{N} v^{(\beta)}\langle\boldsymbol{\sigma}^{(\beta)}\rangle = \sum_{\beta=1}^{N} v^{(\beta)}\mathbf{C}^{(\beta)}\mathbf{G}^{(\beta)}\boldsymbol{\varepsilon}^0 , \qquad \mathbf{C}^* = \sum_{\beta=1}^{N} v^{(\beta)}\mathbf{C}^{(\beta)}\mathbf{G}^{(\beta)} , \qquad (3)$$

where the "volume" fraction is $v^{(\beta)} = A^{(\beta)}/A$, A the total area of the RUC , $A^{(\beta)}$ the area of a subcell, $\langle\boldsymbol{\sigma}^{(\beta)}\rangle$ the subcell-averaged stress, $\mathbf{C}^{(\beta)}$ the elastic stiffness of a subcell, and $\boldsymbol{\varepsilon}^0$ the macroscopic strain.

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3 Two-Scale Model and Verification

The two-scale model of a disc with a central crack under a far field load σ_{22}^{∞} is shown in Fig. 2. Therein, (x_1, x_2) denotes the global coordinates of the disc and (y_2, y_3) the material coordinates of the single-fiber RUC. Isoparametric, bilinear, plane strain elements are used. The disc consists of a heterogeneous long fiber reinforced composite with the assumption that the fibers are arranged in a quadratic array. The parameters of the transversely isotropic fiber and isotropic matrix are listed in Fig. 2 left. The homogenization of the single-fiber RUC with the HFGMC provides an effective stiffness with orthotropic properties in each Gaussian point. For the applied far field load and prescribed crack length, the analytical stress intensity factor (SIF) of $K_{\mathrm{I}}^{\mathrm{anal}} = \sigma_{22}^{\infty}\sqrt{\pi a} = 1.77\,\mathrm{MPa}\sqrt{\mathrm{mm}}$ is fairly close to the numerical one of $K_{\mathrm{I}}^{\mathrm{num}} = 1.75\,\mathrm{MPa}\sqrt{\mathrm{mm}}$ obtained via the J_1-integral and its relation to the SIF K_1 for orthotropic materials, see [7]. Furthermore, a comparison of the analytical and numerical stress distribution along chosen paths around the crack tip leads to the same result, see Fig. 3 center and right. The analytical solution is presented in [8, p. 139]. Numerical stress values are provided by the integration points lying along the evaluated paths in the crack tip element, see Fig. 3 left. Simultaneously, the solution to the microscopic boundary value problem with the HFGMC shows the microscopic stress distribution at each of these integration points when it is assumed that an RUC exists near the crack tip, see Fig. 3 center and right.

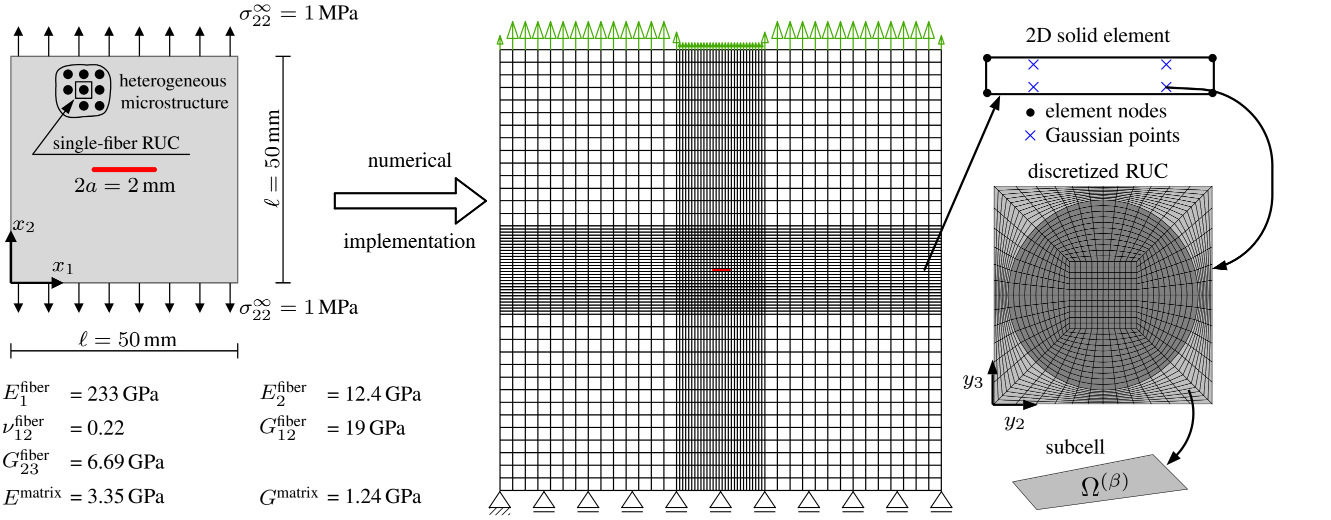

Fig. 2: left - macroscopic boundary value problem of the disc, center - FE-discretization of the disc, right - spatial discretization of the RUC

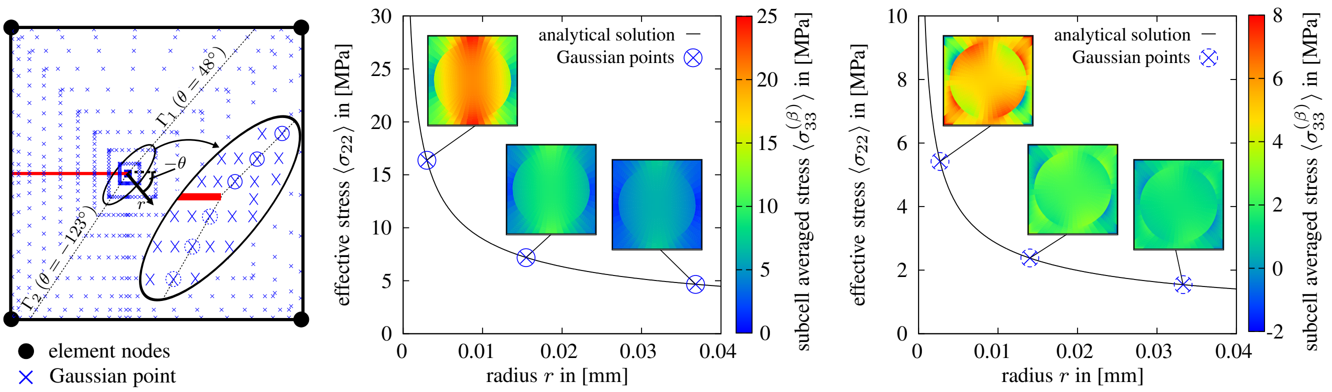

Fig. 3: left - paths Γ_1 and Γ_2 in the crack tip element (dimension: $0.34483\,\mathrm{mm} \times 0.34483\,\mathrm{mm}$), element integrated with “almost polar integration method,” see [5], center - two-scale plot along Γ_1, right - two-scale plot along Γ_2

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