Macroscopic Coulomb force vs. Maxwell stress theory

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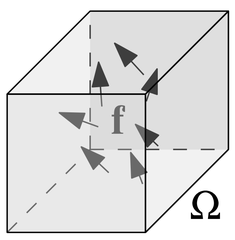
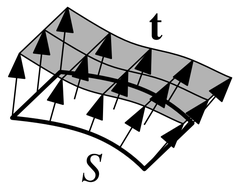
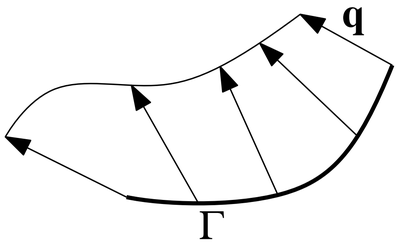
Motivation

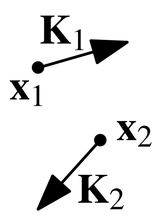
In macroscopic electrostatics, free charges q may be distributed continuously on volumes (q_V), surfaces (q_S), lines (q_L) or at discrete points (q_P^i). There are two rivaling concepts describing the electrostatic action of force on these charge distributions: the continuous formulation of Coulomb's force and the Maxwell stress theory, which both provide individual volume forces $\mathbf{f}$, surface forces $\mathbf{t}$, line forces $\mathbf{q}$ and point forces $\mathbf{K}_i$, summing up to the resultant force

$$\mathbf{F} = \int_\Omega \mathbf{f}\,\mathrm{d}V + \int_S \mathbf{t}\,\mathrm{d}A + \int_\Gamma \mathbf{q}\,\mathrm{d}s + \sum_{i=1}^{m} \mathbf{K}_i$$









Coulomb's law: $q\mathbf{E}^\mathrm{other}$

$$\mathbf{f}^\mathrm{C} = q_\mathrm{V}\mathbf{E}$$

$$\mathbf{t}^\mathrm{C} = q_\mathrm{S}\langle\mathbf{E}\rangle$$

$$\mathbf{q}^\mathrm{C} = q_\mathrm{L}\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_{\partial D_\Gamma(\mathbf{x},\varepsilon)}\mathbf{E}(\mathbf{x}')\,\mathrm{d}s(\mathbf{x}')$$

$$\mathbf{K}_i^\mathrm{C} = q_\mathrm{P}^i\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon^2}\int_{\partial\mathbb{B}^3(\mathbf{x}_i,\varepsilon)}\mathbf{E}\,\mathrm{d}A$$

Maxwell stress theory: $\int_{\partial\Omega}\mathbf{e}^\mathrm{n}\cdot\mathbf{T}\,\mathrm{d}A$

$$\mathbf{f}^\mathrm{M} = \operatorname{div}\mathbf{T}$$

$$\mathbf{t}^\mathrm{M} = \mathbf{e}^\mathrm{n}\cdot[\![\mathbf{T}]\!]$$

$$\mathbf{q}^\mathrm{M} = \lim_{\varepsilon\to 0}\int_{\partial D_\Gamma(\mathbf{x},\varepsilon)}(\mathbf{e}^\mathrm{n}\cdot\mathbf{T})(\mathbf{x}')\,\mathrm{d}s(\mathbf{x}')$$

$$\mathbf{K}_i^\mathrm{M} = \lim_{\varepsilon\to 0}\int_{\partial\mathbb{B}^3(\mathbf{x}_i,\varepsilon)}\mathbf{e}^\mathrm{n}\cdot\mathbf{T}\,\mathrm{d}A$$

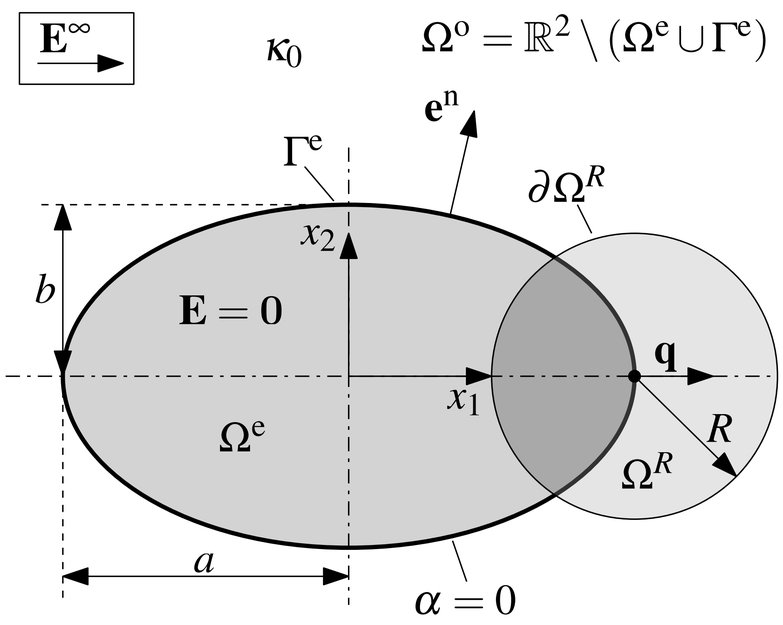
$\mathbb{B}^3(\mathbf{x}_0,\varepsilon), D_\Gamma(\mathbf{x}_0,\varepsilon)$: ball and disc of radius ε with center at $\mathbf{x}_0$, where the latter's normal lies tangentially at Γ

$\langle\cdot\rangle$: average, $[\![\cdot]\!]$: jump across a surface

Maxwell stress tensor: $\mathbf{T} = \kappa_0\mathbf{E}\otimes\mathbf{E} - \frac{1}{2}\kappa_0(\mathbf{E}\cdot\mathbf{E})\mathbf{I}$

Their mutual consistency is put to the test using the following example.

Infinite conducting elliptic cylinder in external electric field



Parameterization:
$\boldsymbol{\gamma}\colon[-\pi,\pi]\to\Gamma^\mathrm{e},\ \boldsymbol{\gamma}(\vartheta) = a\cos(\vartheta)\mathbf{e}_1 + b\sin(\vartheta)\mathbf{e}_2$

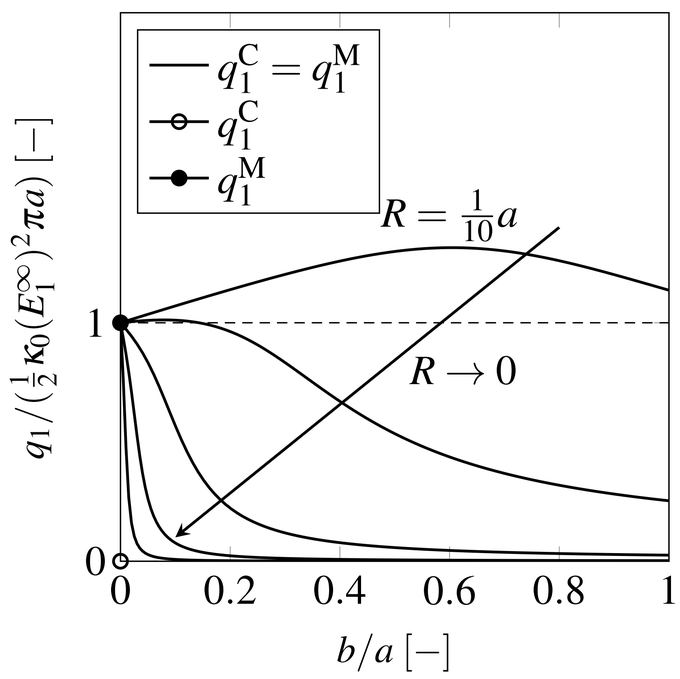
Electrostatic potential: $\mathbf{E} = -\operatorname{grad}\alpha$ in $\Omega^\mathrm{e}\cup\Omega^\mathrm{o}$
Boundary value problem:
Find $\alpha\in C^2(\Omega^\mathrm{e}\cup\Omega^\mathrm{o},\mathbb{R})\cap C(\mathbb{R}^2,\mathbb{R})$ satisfying

$$\begin{aligned}\Delta\alpha &= 0 && \text{in } \Omega^\mathrm{o},\\ \alpha &= 0 && \text{on } \Gamma^\mathrm{e},\\ \operatorname{grad}\alpha &\to -\mathbf{E}^\infty && \text{for } \|\mathbf{x}\|\to\infty.\end{aligned}$$

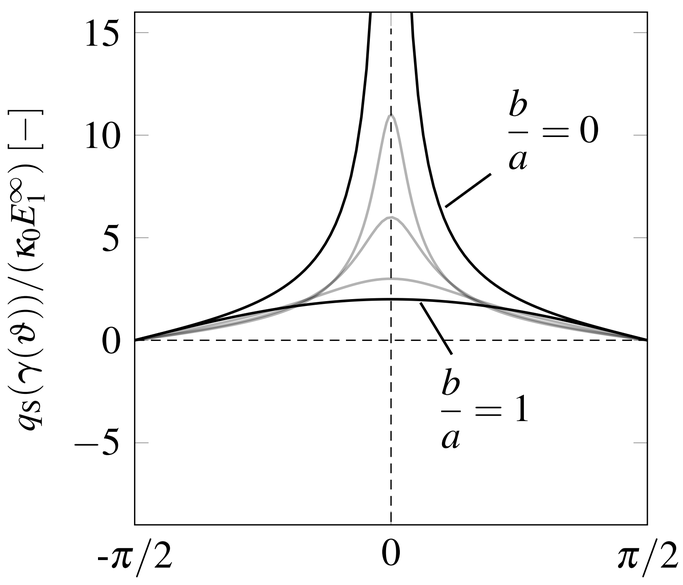
Surface charges: $q_\mathrm{S} = \kappa_0\mathbf{e}^\mathrm{n}\cdot[\![\mathbf{E}]\!]$ on Γ^e
Force on $\Omega^R\times[0,B]$**:** $B\mathbf{q} = (F_1\mathbf{e}_1 + F_2\mathbf{e}_2)$

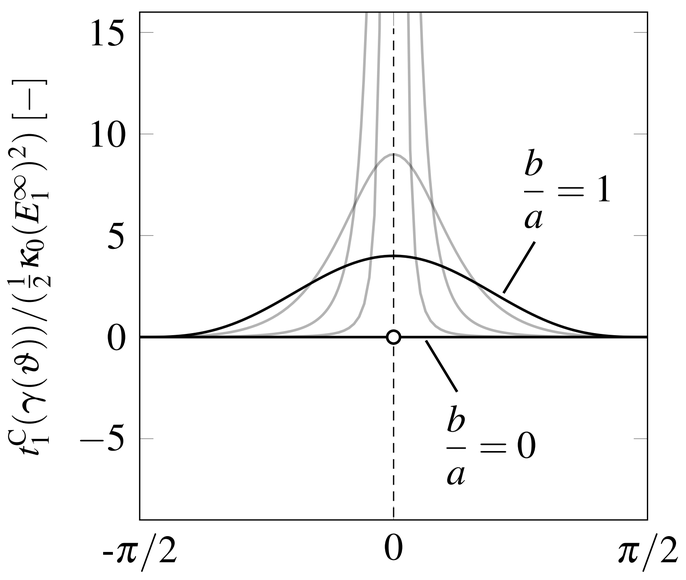
Thickness: B

$$\mathbf{q}^\mathrm{C} = \int_{\partial\Omega^\mathrm{e}\cap\Omega^R} q_\mathrm{S}\langle\mathbf{E}\rangle\,\mathrm{d}s, \quad \mathbf{q}^\mathrm{M} = \int_{\partial\Omega^R}\mathbf{e}^\mathrm{n}\cdot\mathbf{T}\,\mathrm{d}s$$



For the sheet $(b = 0)$, Coulomb's law is **discontinuous**: $q_1^\mathrm{C} = 0 \quad\longleftrightarrow\quad q_1^\mathrm{M} = \frac{1}{2}\kappa_0(E_1^\infty)^2\pi a$





The traction t_1^C converges to a Dirac distribution on $\Gamma^{\mathrm{e,r}} = \boldsymbol{\gamma}([-\pi/2,\pi/2])$ for $b\to 0$, i.e.,

$$\lim_{b\to 0}\int_{\Gamma^{\mathrm{e,r}}} t_1^\mathrm{C}\varphi\,\mathrm{d}s = \frac{1}{2}\kappa_0(E_1^\infty)^2\pi a\varphi(a,0)$$

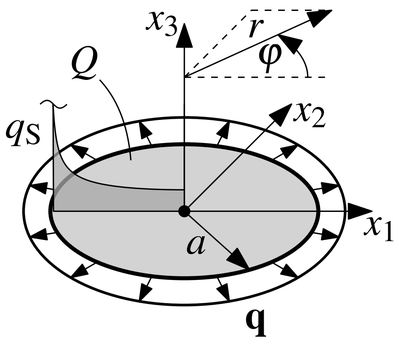
for all test functions $\varphi\in C(\Gamma^{\mathrm{e,r}},\mathbb{R})$.

Conclusions

- electrostatic line forces $\mathbf{q}$ act at conducting edges
- → predicted by the Maxwell stress theory ($\mathbf{q}^\mathrm{M} = \mathbf{q}$), but not by Coulomb's law ($\mathbf{q}^\mathrm{C} = \mathbf{0}$)
- Maxwell stress theory is the superior concept

Prevalent examples of hidden electrostatic line forces

Charged conducting disc



Q : total charge on the disc

Surface charge distribution

$$q_\mathrm{S} = \frac{Q}{2\pi a\sqrt{a^2 - r^2}}$$

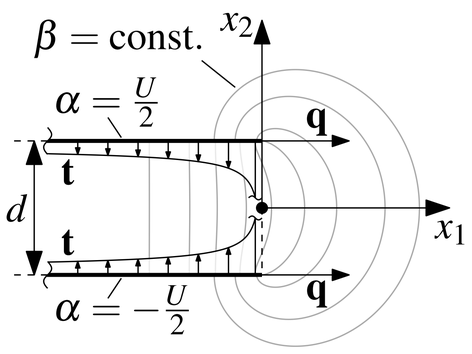
Electric intensity factor

$$K_E = \lim_{r\nearrow a}\sqrt{\frac{\pi(a-r)}{2}}\frac{q_\mathrm{S}}{\kappa_0} = \frac{Q}{4a\sqrt{\pi a}\kappa_0}$$

Line force at the edge

$$\mathbf{q} = \frac{Q^2}{32a^3\pi\kappa_0}\mathbf{e}_r$$

Semi-infinite plate capacitor



β : stream function

Implicit solution

$$x_1 + \mathrm{i}x_2 = \frac{d}{2\pi}\left(1 + \frac{2\pi}{U}(-\beta + \mathrm{i}\alpha) + e^{2\pi(-\beta+\mathrm{i}\alpha)/U}\right)$$

Electric intensity factor

$$K_E = -\lim_{x_1\searrow 0}\sqrt{\frac{\pi}{2x_1}}\left(\alpha\left(x_1,\pm\frac{d}{2}\right)\mp\frac{U}{2}\right) = \pm\frac{U}{\sqrt{2d}}$$

Line force at the edge

$$\mathbf{q} = \frac{1}{4}\kappa_0\frac{U^2}{d}\mathbf{e}_1$$

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